Math 220, Linear Algebra II — Spring 2024
`https://sites.aub.edu.lb/kmakdisi/`
Problem set 9, due Thursday, March 28 at the beginning of class

Exercises from Artin:

Chapter 8, exercises 4.12, 6.2 (see remark), 6.7, 6.18, 8.3, M.8.

Remark for 6.2: give one proof using the spectral theorem, and another without assuming that V is finite-dimensional.

Additional Exercises (also required):

Exercise A9.1: A proof of the spectral theorem for commuting self-adjoint linear transformations, and similarly for commuting normal linear transformations (compare to Artin, Theorems 8.6.5 and 8.6.6 for a different treatment). Throughout this exercise assume that V is a finite-dimensional complex vector space.

a) If $T, U : V \to V$ are commuting linear transformations (so $T \circ U = U \circ T$), show that T and U have a common eigenvector. (Hint: let λ be an eigenvalue of T [remember, the field of scalars is $\mathbf{C}$] and show that the nonzero eigenspace $V^{(\lambda)} = \ker(\lambda I - T)$ is invariant under U, and hence [why?] contains an eigenvector for U.)

b) If, furthermore, T and U are self-adjoint, show that V has an orthonormal basis of simultaneous eigenvectors for T and U. Generalize to an arbitrary number of commuting self-adjoint linear transformations. Bonus: show that the result still holds over $\mathbf{R}$, and, whether over $\mathbf{R}$ or over $\mathbf{C}$, that it still holds even if one has an infinite set of commuting self-adjoint linear transformations.

c) Back to the case of $\mathbf{C}$, Assume that T is normal; this means that T and T^* commute. In this case, show that there exists an orthonormal basis of eigenvectors for T. (Hint: show that $T + T^*$ and $i(T - T^*)$ are commuting self-adjoint linear transformations.) Bonus: generalize to an arbitrary number of commuting normal linear transformations.

Exercise A9.2: Let V be a finite-dimensional inner product space over $\mathbf{R}$ (or $\mathbf{C}$), and let $T : V \to V$ be a linear transformation.

(Note: parts (c–e) of this problem are an introduction to the "singular value decomposition", which I encourage you to look up. The problem works just as well if $T : V \to W$ with $W \neq V$, but you can stick to the case $W = V$.)

a) Show that $\vec{\mathbf{v}}$ is an eigenvector of T^* (the adjoint of T) if and only iff its orthogonal complement $\vec{\mathbf{v}}^\perp$ is stable under T. (Recall that this means $T(\vec{\mathbf{v}}^\perp) \subset \vec{\mathbf{v}}^\perp$.) Hint: show first that $(\vec{\mathbf{v}}^\perp)^\perp$ is the set of multiples of $\vec{\mathbf{v}}$.

b) Show that $\vec{\mathbf{v}}$ is an eigenvector of T^*T if and only if $T(\vec{\mathbf{v}}^\perp) \subset (T\vec{\mathbf{v}})^\perp$.

c) Show that there exists an orthonormal basis $\{\vec{\mathbf{v}}_i\}$ of V such that the $T(\vec{\mathbf{v}}_i)$ are all orthogonal to each other (the $T(\vec{\mathbf{v}}_i)$ need not have length 1, however, and some of them may even be zero). Hint: T^*T is self-adjoint.

d) Show that every $n \times n$ matrix M can be written $M = U_1 D U_2$, where D is diagonal and U_1 and U_2 are both orthogonal (or unitary).

e) Deduce lower and upper bounds on $\|T(\vec{\mathbf{v}})\|$ in terms of $\|\vec{\mathbf{v}}\|$ and the entries on the diagonal of D above, where M is the matrix of T with respect to some orthonormal basis.

Exercise A9.3: On $\mathcal{P}_n$, define the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$. (Prove for yourself, but do not hand in, the fact that this is positive definite.) Define $T : \mathcal{P}_n \to \mathcal{P}_n$ by

$$Tf = \frac{d}{dx}\left((1 - x^2)\frac{df}{dx}\right).$$

a) Show that T is self-adjoint with respect to this inner product.

b) What are its eigenvalues?

c) Show that for $0 \le k \le n$, there exists an eigenvector P_k, of degree k and leading coefficient 1; so $P_k(x) = x^k + a_{k,k-1}x^{k-1} + \ldots + a_{k,0}$. Show that the different P_k's are orthogonal. What is the eigenvalue of P_k? (For more fun: find an expression for its coefficients $a_{k,i}$. Challenge: find a nice expression for $\langle P_k, P_k \rangle$.)

d) What breaks down in each of parts 1–3, if instead of T we consider the linear transformation $f \mapsto d^2 f/dx^2$?

Remark: the P_k's are called the Legendre polynomials.

Look at, but do not hand in, the following exercises:

Chapter 8, exercises 4.13, 4.14, 6.4, 6.8, 6.19, 6.20, M.7.