Math 220, Linear Algebra II — Spring 2024
`https://sites.aub.edu.lb/kmakdisi/`
Problem set 8, NOT DUE — for practice before the midterm

Exercises from Artin:

Chapter 8, exercises 6.9, 6.15.

Additional Exercises:

Exercise A8.1: Show that the complex matrix $A = \begin{pmatrix} 2 & i \\ i & 0 \end{pmatrix}$ is **not** diagonalizable, even though $A^t = A$. Why does this not contradict the statement of the spectral theorem?

Exercise A8.2: Let V be an inner product space (over $\mathbf{R}$, but the proof is the same for $\mathbf{C}$), and let $P : V \to V$ be a self-adjoint linear transformation such that $P^2 = P$. (This means that $P \circ P = P$.) Show that P is the orthogonal projection operator onto the subspace $W = \operatorname{Image} P$. Find the characteristic polynomial of P.

Exercise A8.3: Let V be a finite-dimensional inner product space, and let $T : V \to V$ be a linear transformation.

a) Use the spectral theorem to show that if T is self-adjoint, then $id_V + T^2 : V \to V$ is an invertible linear transformation.

b) Give an example of V and T such that $id_V + T^2$ is not invertible. (Of course, such a T cannot be self-adjoint.) Preferably find such an example over $\mathbf{R}$; if you have trouble finding the example, then just settle for an example over $\mathbf{C}$, which is easier.

Exercise A8.4: Let V be a finite-dimensional inner product space, and assume given a **self-adjoint** linear transformation $T : V \to V$ such that $T^3 = T$.

(i) Show that the only possible eigenvalues of T are $\lambda = 0$, 1, or -1. Call the corresponding eigenspaces $V^{(0)}, V^{(1)}, V^{(-1)}$. (It is possible that some of these eigenspaces are just $\{\vec{0}\}$: for example, if $T = id_V$, then $V^{(0)} = V^{(-1)} = \{\vec{0}\}$.)

(ii) Show that every vector $\vec{\mathbf{v}} \in V$ has a decomposition $\vec{\mathbf{v}} = \vec{\mathbf{v}}_0 + \vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_{-1}$, with $\vec{\mathbf{v}}_0 \in V^{(0)}$, $\vec{\mathbf{v}}_1 \in V^{(1)}$, and $\vec{\mathbf{v}}_{-1} \in V^{(-1)}$.

(iii) Define a linear transformation $P : V \to V$ by $P = (1/2)(T^2 + T)$. Show that P is the orthogonal projection onto $V^{(1)}$.

(iv) Find another "polynomial" $Q = aT^2 + bT + cI$ for suitable $a, b, c \in \mathbf{R}$, such that Q is the orthogonal projection onto $V^{(0)}$.

Exercise A8.5: Consider the complex matrix $M = \begin{pmatrix} 3 & 3+i \\ 3-i & 6 \end{pmatrix}$.

a) Why do we know without any calculation that M is diagonalizable?

b) Find an **orthonormal** basis $\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2\}$ of $\mathbf{C}^2$ consisting of eigenvectors of M, and find the corresponding eigenvalues. (Note: do this by the "usual" way.)

c) For $z_1, z_2 \in \mathbf{C}$, let

$$f(z_1, z_2) = 3z_1\overline{z_1} + (3+i)z_2\overline{z_1} + (3-i)z_1\overline{z_2} + 6z_2\overline{z_2}.$$

Find explicit constants $C_1, C_2 > 0$ for which you can show that for all $z_1, z_2 \in \mathbf{C}$, we have $C_1(|z_1|^2 + |z_2|^2) \le f(z_1, z_2) \le C_2(|z_1|^2 + |z_2|^2)$.