Math 220, Linear Algebra II — Spring 2024
https://sites.aub.edu.lb/kmakdisi/
Problem set 7, due Thursday, March 14 at the beginning of class

Exercises from Artin:

Chapter 8, exercises 3.3, 3.4, 4.15, 5.1 (do this problem more generally for a Hermitian inner product space), 6.12, 6.13.

Additional Exercises (also required):

Exercise A7.1: (Riesz representation theorem for finite-dimensional spaces) Let V be a finite-dimensional inner product space (over $F = \mathbf{R}$ or $\mathbf{C}$), and let $T : V \to F$ be a linear transformation. Show that there exists a unique vector $\vec{\mathbf{w}} \in V$ for which we have

$$\forall \vec{\mathbf{x}} \in V, \quad T(\vec{\mathbf{x}}) = \langle \vec{\mathbf{w}}, \vec{\mathbf{x}} \rangle.$$

Hint: First do this in scratch for the cases $V = \mathbf{R}^n$ and $V = \mathbf{C}^n$, where it is easy. For the general proof, choose an orthonormal basis $\mathcal{U} = \{\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_n\}$ for V, and construct the vector $\vec{\mathbf{w}}$ out of the $\vec{\mathbf{u}}_i$ and the values $T(\vec{\mathbf{u}}_i)$.

Exercise A7.2: (Least squares) The system of equations

$$\begin{cases} x_1 + 3x_2 + x_3 = -1 \\ -2x_1 + 6x_2 + 4x_3 = 2 \\ -x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 - x_2 + 8x_3 = 1 \end{cases}$$

has no solutions.

a) Explain why the lack of solutions is the same thing as saying that

$$\vec{\mathbf{b}} = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \text{ does NOT belong to the image of } A = \begin{pmatrix} 1 & 3 & 1 \\ -2 & 6 & 4 \\ -1 & 3 & 2 \\ 3 & -1 & 8 \end{pmatrix}.$$

b) Find a vector $\vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ \vdots \\ x_4 \end{pmatrix} \in \mathbf{R}^4$ for which $\|A\vec{\mathbf{x}} - \vec{\mathbf{b}}\|$ is as small as possible. (This is called the **least squares** solution, even though it not actually a solution, and the answer to this part involves an orthogonal projection somewhere.)

c) In general, if A is an $m \times n$ matrix that describes an **injective** linear transformation $T_A : \mathbf{R}^n \to \mathbf{R}^m$, and if $\vec{\mathbf{b}} \in \mathbf{R}^m$, then show that there exists a unique $\vec{\mathbf{x}} \in \mathbf{R}^n$ which minimizes the quantity $\|A\vec{\mathbf{x}} - \vec{\mathbf{b}}\|$.

d) Show that this $\vec{\mathbf{x}}$ is the unique solution of the system of equations (written in matrix form):

$$A^t A\vec{\mathbf{x}} = A^t \vec{\mathbf{b}}.$$

Hint: For what space W do we have $\ker A^t = W^\perp$?

Exercise A7.3: Let M be an invertible $n \times n$ matrix. Show that one can write $M = U\Delta$ for an upper triangular matrix Δ and an isometry U; i.e., U is an orthogonal or unitary matrix, depending on whether the field of scalars is $\mathbf{R}$ or $\mathbf{C}$. (Hint: Gram-Schmidt on the columns of M.) What happens if M is not invertible?

Exercise A7.4: (Least-squares linear regression) In $\mathbf{R}^7$, define the vectors

$$\vec{\mathbf{u}} = (1, 1, 1, 1, 1, 1, 1),$$
$$\vec{\mathbf{x}} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1, 4, 5, 6, 7, 9, 10),$$
$$\vec{\mathbf{y}} = (y_1, y_2, y_3, y_4, y_5, y_6, y_7) = (0.9, 1.6, 2.2, 1.9, 2.8, 2.9, 3.8).$$

a) Let $W = \operatorname{span}\{\vec{\mathbf{u}}, \vec{\mathbf{x}}\}$. Compute an orthogonal basis for W and use it to find $\vec{\mathbf{z}} = \operatorname{Proj}_W \vec{\mathbf{y}}$.

b) Find a, b such that $\vec{\mathbf{z}} = a\vec{\mathbf{x}} + b\vec{\mathbf{u}}$.

c) In $\mathbf{R}^2$, draw the points (x_1, y_1), (x_2, y_2), and so on (i.e., these are the points with coordinates $(1, 0.9), (4, 1.6), \ldots, (10, 3.8)$.) Also draw the line $y = ax + b$ corresponding to the values of a and b from part (b) above. Explain why the line passes very close to the points. (Hint: why is the "vector of errors" $\vec{\mathbf{e}} = \vec{\mathbf{y}} - a\vec{\mathbf{x}} - b\vec{\mathbf{u}}$ so short?)