Math 220, Linear Algebra II — Spring 2024
`https://sites.aub.edu.lb/kmakdisi/`
Problem set 6, due Thursday, March 7 at the beginning of class

Exercises from Artin:

Chapter 8, exercises 2.2, 4.1, 4.3, 4.10, 4.11ac (see the additional exercises before doing 4.11).

Additional Exercises (also required):

Exercise A6.1: For V an inner product space (over $\mathbf{R}$, but not necessarily finite-dimensional), and for a subset $S \subset V$, recall that the **orthogonal space** to S is the set $S^\perp$ defined by:

$$S^\perp = \{\vec{\mathbf{v}} \in V \mid \forall \vec{\mathbf{w}} \in S, \ \langle \vec{\mathbf{w}}, \vec{\mathbf{v}} \rangle = 0\}.$$

a) Show that $S^\perp$ is a subspace of V.

b) Show that $S^\perp = (\operatorname{span} S)^\perp$. Thus we will generally look at subspaces $W \subset V$ and consider the subspace $W^\perp$. Conclude that if A is an $m \times n$ matrix, representing a linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$, then $\ker A = (\operatorname{Image} A^t)^\perp$, where the orthogonal space is taken using the standard inner product on $\mathbf{R}^n$.

c) Show that one always has $W \subset (W^\perp)^\perp$, and $W \cap W^\perp = \{\vec{0}\}$. Thus W and $W^\perp$ are independent subspaces. (This uses the fact that the inner product is positive definite.)

d) Show that if V is finite-dimensional, then $\dim W + \dim W^\perp = \dim V$. Deduce that $V = W \oplus W^\perp$.

Hint: One way is to take a basis for W and extend it to a basis for V, then apply Gram-Schmidt to get (why?) an orthonormal basis for V that contains an orthonormal basis for W. This allows you to express everything using this orthonormal basis. Another way is to show directly the existence of the orthogonal projection, but this may take more work. A third way, which works only in $\mathbf{R}^n$, is to relate this result to the fact that row rank = column rank, and to use the second part of (b) above.

Exercise A6.2: We work in $\mathbf{R}^5$ with the standard inner product. To save space, we shall write vectors as rows (Artin writes rows and puts transposes, as you may have noticed).

a) Show that the set $S = \{(1, 2, 1, 1, 0), (2, 1, -3, -1, 0)\}$ is an orthogonal set in $\mathbf{R}^5$. Write $W = \operatorname{span} S$.

b) Find a basis for $W^\perp$ by directly computing $S^\perp$ from the definition. You will find that $W^\perp$ is 3-dimensional, and you can find a basis $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ for $W^\perp$ where $\vec{\mathbf{v}}_1 = \vec{\mathbf{e}}_5 = (0, 0, 0, 0, 1)$ is the standard basis vector, and $\vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ are orthogonal to $\vec{\mathbf{e}}_5$.

c) With respect to the basis $\{\vec{\mathbf{e}}_5, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ for $W^\perp$ that you found above, find a number $\mu \in \mathbf{R}$ so that $\{\vec{\mathbf{e}}_5, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3 - \mu\vec{\mathbf{v}}_2\}$ are an orthogonal set. Show that this orthogonal set is still a basis for $W^\perp$. What is the relation with Gram-Schmidt?

d) Write $\vec{\mathbf{e}}_2 = (0, 1, 0, 0, 0)$ as a sum of a vector in W and another vector in $W^\perp$.

Exercise A6.3: Let V be an inner product space, and let W be a subspace of V. As before, write $W^\perp$ for the orthogonal complement of W. Let $\vec{\mathbf{v}} \in V$ be written as $\vec{\mathbf{v}} = \vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2$, with $\vec{\mathbf{v}}_1 \in W$ and $\vec{\mathbf{v}}_2 \in W^\perp$ (so $\vec{\mathbf{v}}_1$ is the orthogonal projection of $\vec{\mathbf{v}}$ onto W). Show that $\vec{\mathbf{v}}_1$ is the closest vector in W to $\vec{\mathbf{v}}$. Here "closest" is measured using the usual distance $d(\vec{\mathbf{w}}, \vec{\mathbf{w}}') = |\vec{\mathbf{w}} - \vec{\mathbf{w}}'| = \sqrt{\langle \vec{\mathbf{w}} - \vec{\mathbf{w}}', \vec{\mathbf{w}} - \vec{\mathbf{w}}' \rangle}$. Include at least one picture as part of your explanation!

Exercise A6.4: Let V be an inner product space, and let $W_1, W_2, \ldots, W_r$ be subspaces of V that are pairwise orthogonal: this means that if $i \neq j$, then for all $\vec{\mathbf{x}} \in W_i$ and $\vec{\mathbf{y}} \in W_j$, we have $\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle = 0$. Prove that the subspace $W_1 + W_2 + \cdots + W_r \subset V$ is in fact a direct sum:

$$W_1 + W_2 + \cdots + W_r = W_1 \oplus W_2 \oplus \cdots \oplus W_r.$$

Use this to show that a collection of nonzero pairwise orthogonal vectors is linearly independent.

Look at, but do not hand in, the following exercises:

Chapter 5, exercises 1.4, M.3, M.4, M.5, M.6, M.7. Also read Theorem 5.2.3 (the point is that the set of diagonalizable matrices is a dense subset of $M_n(\mathbf{C})$).

Chapter 8, exercises 1.1, 2.1, 4.2.