Math 220, Linear Algebra II — Spring 2024
https://sites.aub.edu.lb/kmakdisi/
Problem set 5, due Thursday, February 29 at the beginning of class

Exercises from Artin:
Chapter 5, exercises 1.1, 1.2, 1.3.
Chapter 8, exercises 4.6, 4.7, 4.8, 4.9.

Additional Exercises (also required):

Exercise A5.1: a) Let $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2 \in \mathbf{R}^2$, and let $a, b, c \in \mathbf{R}$. What is the relation between $\det(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$ and $\det(a\vec{\mathbf{v}}_1, b\vec{\mathbf{v}}_1 + c\vec{\mathbf{v}}_2)$?

b) If $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$ is an orthonormal basis for $\mathbf{R}^2$, show that $\det(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$ is either 1 or -1.

c) Using (a) and (b), explain geometrically why $|\det(a\vec{\mathbf{v}}_1, b\vec{\mathbf{v}}_1 + c\vec{\mathbf{v}}_2)|$ computes the area of the parallelogram with sides $a\vec{\mathbf{v}}_1$ and $b\vec{\mathbf{v}}_1 + c\vec{\mathbf{v}}_2$, when $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$ is orthonormal. (Just draw a picture and say a few sentences. You can pretend that $a, c > 0$ if you wish, but it would be nice to also think about the cases when one or both of a, c might be negative.)

d) Show that every basis of $\mathbf{R}^2$ of the form $\{a\vec{\mathbf{v}}_1, b\vec{\mathbf{v}}_1 + c\vec{\mathbf{v}}_2\}$ for some choice of orthonormal basis $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$ and some choice of $a, b, c \in \mathbf{R}$ with $a, c \neq 0$. (Hint: Gram-Schmidt. You can even restrict to showing this result for $a, c > 0$, which is better for the picture in part (c).)

Culture: Thus the (nonzero) determinant of a basis describes the area of a parallelogram, with a sign that corresponds to the orientation of the basis.

e) Generalize the above to $\mathbf{R}^3$. Small bonus: do this for any $\mathbf{R}^n$.

Exercise A5.2: Let V be a finite-dimensional inner product space over $\mathbf{R}$. Let $\vec{\mathbf{z}} \neq \vec{0}$, and consider the linear transformation $S_{\vec{\mathbf{z}}} : V \to V$, given by

$$S_{\vec{\mathbf{z}}}(\vec{\mathbf{x}}) = \vec{\mathbf{x}} - 2\frac{\langle \vec{\mathbf{x}}, \vec{\mathbf{z}} \rangle}{\langle \vec{\mathbf{z}}, \vec{\mathbf{z}} \rangle}\vec{\mathbf{z}}.$$

a) Show that $S_{\vec{\mathbf{z}}}(\vec{\mathbf{x}})$ is the reflection of $\vec{\mathbf{x}}$ with respect to the hyperplane $\{\vec{\mathbf{z}}\}^\perp$. Draw a simple picture as part of your explanation.

b) Show that $S_{\vec{\mathbf{z}}}$ is an orthogonal transformation of V.

c) Let $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in V$ be vectors with $\|\vec{\mathbf{v}}\| = \|\vec{\mathbf{w}}\| \neq 0$. Suppose $\dim V \geq 2$. Show that there exists a choice of $\vec{\mathbf{z}}$ for which $S_{\vec{\mathbf{z}}}(\vec{\mathbf{v}}) = \vec{\mathbf{w}}$. (One choice of $\vec{\mathbf{z}}$ works "almost all the time"; for the exceptional case, you will need the assumption that $\dim V \geq 2$. I strongly encourage you to draw pictures as part of your reasoning.)

Exercise A5.3: Let V be a vector space over any field, and let $W \subset V$ be a subspace. We consider the quotient vector space $\overline{V} = V/W$.

In solving this problem, try not to assume that either V or W is finite-dimensional, and try not to choose any bases. For the purposes of this problem, the word "natural" below means something that you can define without any choice of coordinates. Some authors would use the word "canonical" instead.

a) Show that there is a "natural" bijection between subspaces $\overline{Z} \subset \overline{V}$ and "intermediate" subspaces Z with $W \subset Z \subset V$. Be sure to explain how to go from Z to $\overline{Z}$ and how to go from $\overline{Z}$ to Z.

b) Show that $Z_1 \subset Z_2$ (where in fact $W \subset Z_1 \subset Z_2 \subset V$) if and only if $\overline{Z}_1 \subset \overline{Z}_2$.

c) In the situation of part (b), describe a "natural" isomorphism between the quotients Z_2/Z_1 and $\overline{Z_2}/\overline{Z_1}$.