Math 220, Linear Algebra II — Spring 2024
https://sites.aub.edu.lb/kmakdisi/
Problem set 4, due Thursday, February 22 at the beginning of class

Exercises from Artin:

Chapter 4, exercises 4.6, 5.1, 5.3, 5.7, 6.1 (caution: A might not be diagonalizable), 6.2, M.9.

Additional Exercises (also required):

Exercise A4.1: Find a basis $\mathcal{B}$ with respect to which **both** of the following linear transformations on $\mathbf{R}^3$ become diagonalized (the matrices below are the matrices with respect to the standard basis):

$$S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{pmatrix}.$$

Cultural note: A necessary condition for two linear transformations to be simultaneously diagonalized is for them to commute, i.e., $ST = TS$. But this is not sufficient. (Bonus: prove these statements.)

Exercise A4.2: Consider the following matrix representing a linear transformation from $\mathbf{R}^4$ to $\mathbf{R}^4$ (with respect to the standard basis):

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & -2 \end{pmatrix}.$$

a) Show that the subspace $V = \operatorname{span}\{\vec{e}_3, \vec{e}_4\} \subset \mathbf{R}^4$ is invariant for A.

b) Find a one-dimensional invariant subspace $L \subset V$ for A.

c) Find and factor the full characteristic polynomial of A, and, for each eigenvalue, find a basis for the associated eigenspace.

d) Show that A is not diagonalizable.

Exercise A4.3: Let $T : \mathbf{R}^4 \to \mathbf{R}^4$ be given by the matrix $A_T = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$.

a) Find a basis for each of $\ker T$ and $\operatorname{Image} T$.

b) Deduce the geometric multiplicity $m_g(0)$ of the eigenvalue 0.

c) Deduce also (without complicated calculations!) the existence of an eigenvector $\vec{\mathbf{v}}$ with a **nonzero** eigenvalue λ. What is λ?

d) Use the above to find the characteristic polynomial $p_T(x) = \det(xI - T)$.

Exercise A4.4: (Circulant matrices) Recall that in $\mathbf{C}$, the set of **fifth roots of unity** is the set $\mu_5 = \{\zeta \in \mathbf{C} \mid \zeta^5 = 1\}$. By looking at the polar representation of complex numbers, we see that $\mu_5 = \{1, \exp(2\pi i/5), \exp(4\pi i/5), \exp(6\pi i/5), \exp(8\pi i/5)\}$. A similar statement holds for the set μ_n of nth roots of unity. All we really need for this problem is the fact that there are exactly n distinct nth roots of unity in $\mathbf{C}$.

a) Let A be the 5×5 **circulant matrix**

$$A = \begin{pmatrix} a & b & c & d & e \\ e & a & b & c & d \\ d & e & a & b & c \\ c & d & e & a & b \\ b & c & d & e & a \end{pmatrix}.$$

Show that the determinant $\det A$ is given by the following product:

$$\det A = \prod_{\zeta \in \mu_5} (a + \zeta b + \zeta^2 c + \zeta^3 d + \zeta^4 e).$$

Hint: find a basis of eigenvectors. Make sure to prove it actually is a basis.

b) Generalize to $n \times n$ circulant matrices.

Look at, but do not hand in, the following exercises:

Chapter 4, exercises 5.2, 5.4, 5.6, 6.3, 6.5, 6.7 (if A and B are not invertible, then AB and BA have the same characteristic polynomial — can you prove this? — but might not be similar; give an example), 6.10, M.6, M.10.