Math 220, Linear Algebra II — Spring 2024
`https://sites.aub.edu.lb/kmakdisi/`
Problem set 3, due Thursday, February 15 at the beginning of class

Exercises from Artin:
Chapter 1, exercises 5.1, 5.5, M.7.

Additional Exercises (also required):

Exercise A3.1: Let $T : V \to W$ be a linear transformation, with $\dim \ker T = \ell$. Let $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k \in V$ be **linearly independent** vectors. (Note: the $\vec{\mathbf{v}}_i$ are not necessarily a basis for V.) Prove that $\dim \operatorname{span} \{T(\vec{\mathbf{v}}_1), \ldots, T(\vec{\mathbf{v}}_k)\} \geq k - \ell$.

(Hint: use rank-nullity for the restriction of T to an appropriate subspace of V.)

Exercise A3.2: Let V, W be finite-dimensional vector spaces, and let $T : V \to W$ be a linear transformation.

a) Show that T is surjective **if and only if** there exists a linear transformation $S : W \to V$ that satisfies $TS = id_W$.

b) Show that T is injective **if and only if** there exists a linear transformation $S : W \to V$ that satisfies $ST = id_V$.

Suggestion: for both parts (a) and (b), one direction requires you to actually construct a specific S. To do this, choose bases of V and W in an appropriate way, and use the fact that linear transformations can be expressed in terms of matrices and vice-versa, once you have chosen bases. Alternatively, you can use the linear extension theorem if you are adept at using it.

Exercise A3.3: Which of the following matrices (over $\mathbf{R}$) are equivalent? Which are similar? Explain.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Exercise A3.4: Consider the matrices over $\mathbf{R}$

$$A = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 2 & 4 \\ 1 & 2 & 0 & 3 \\ 3 & 4 & 3 & 10 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}.$$

a) Find the determinants $\det A$ and $\det B$ by using column operations.
b) Repeat the calculation of part (a), using row operations.
c) What are the ranks $\operatorname{rank} B$ and $\operatorname{rank} A$ (in that order)? Justify.

Exercise A3.5: (Cramer's rule) You may have seen the following method to solve linear systems of equations: the solution to $\begin{pmatrix} 1 & -1 & 4 \\ -8 & 3 & 1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 6 \end{pmatrix}$ is given by

$$x_1 = \frac{\det \begin{pmatrix} -2 & -1 & 4 \\ 0 & 3 & 1 \\ 6 & -1 & 1 \end{pmatrix}}{\det \begin{pmatrix} 1 & -1 & 4 \\ -8 & 3 & 1 \\ 2 & -1 & 1 \end{pmatrix}}, \qquad x_2 = \frac{\det \begin{pmatrix} 1 & -2 & 4 \\ -8 & 0 & 1 \\ 2 & 6 & 1 \end{pmatrix}}{\det \begin{pmatrix} 1 & -1 & 4 \\ -8 & 3 & 1 \\ 2 & -1 & 1 \end{pmatrix}}, \qquad x_3 = \frac{\det \begin{pmatrix} 1 & -1 & -2 \\ -8 & 3 & 0 \\ 2 & -1 & 6 \end{pmatrix}}{\det \begin{pmatrix} 1 & -1 & 4 \\ -8 & 3 & 1 \\ 2 & -1 & 1 \end{pmatrix}}.$$

a) Prove Cramer's rule in the general case: let A be an $n \times n$ matrix with entries in F, and $\vec{x}, \vec{b} \in F^n$ with $A\vec{x} = \vec{b}$, then the ith coordinate x_i of $\vec{x}$ satisfies $x_i \det A = \det A_i$, where A_i is gotten from A by replacing the ith column with $\vec{b}$.

Hint: let the columns of A be $C_1, C_2, \dots C_n$. Then $A\vec{x} = \vec{b}$ means that $x_1 C_1 + \dots + x_n C_n = \vec{b}$; in the above example, for instance, we have $x_1 \begin{pmatrix} 1 \\ -8 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 6 \end{pmatrix}$. Now the columns of A_1 are $\vec{b}, C_2, \dots, C_n$, i.e., $\left(\sum_j x_j C_j\right), C_2, \dots, C_n$. Evaluate $\det A_1$ using properties of the determinant to conclude that $\det A_1 = x_1 \det A$. Generalize to all x_i. (Side note: the proof does not really need A to be invertible; however, if A is not invertible, then this forces each A_i to have determinant 0. It is interesting to think about what this means.)

b) Deduce from the above that if A is invertible, then $A^{-1} = (\det A)^{-1}\tilde{A}$, where the entries of the matrix $\tilde{A}$ are polynomials in the entries of A. Cultural note: when $F = \mathbf{R}$ or $\mathbf{C}$, this shows that A^{-1} depends **continuously** on A (provided, of course, that $\det A \neq 0$, so that the inverse exists in the first place). The matrix $\tilde{A}$ is called the **cofactor matrix** of A, or sometimes the **classical adjoint** or **adjugate** of A. See Section 1.6 of Artin.

Hint: the ith column of A^{-1} is $A^{-1}\vec{e}_i$, which can be found by solving the system $A\vec{x} = \vec{e}_i$.

Exercise A3.6: (Elementary matrices) Let

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 77 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 88 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

The E_i are certain types of 3×3 elementary matrices, and analogs exist for all $n \times n$ matrices.

a) For $i = 1, 2, 3$ find $E_i A$ and $A E_i$, and interpret in terms of row or column operations on A.

b) Which operations preserve $\ker A$? What about $\operatorname{Image} A$? Why?

c) What do these operations do to $\det A$? (Answer this question in two ways, once by finding $\det E_i$, and once from the determinant being multilinear and alternating in the rows or columns of A.)

d) Let $\vec{b} \in \mathbf{R}^3$. We know how to solve the linear system $A\vec{x} = \vec{b}$ for unknown $\vec{x} \in \mathbf{R}^3$ by Gaussian elimination. Show that each step in Gaussian elimination amounts to replacing A and $\vec{b}$ by EA and $E\vec{b}$ for a suitable elementary matrix E (in practice, we often do several steps at once).

Exercise A3.7: Prove directly from the definition of the determinant as a sum over $\sigma \in S_n$ that $\det A^t = \det A$.

(Hint: at one key moment, you will need to use the fact that $\operatorname{sign} \sigma^{-1} = \operatorname{sign} \sigma$. Make sure to also justify why this fact holds.)

Look at, but do not hand in, the following exercises:

Chapter 1, exercises 5.2, 5.3, M.8, M.9 (and, purely for entertainment, M.10).

Treil's book, "Linear Algebra Done Wrong": look through the chapter on determinants and do a selection of problems (some computational, some conceptual, some that look interesting for their own sake).