Math 220, Linear Algebra II — Spring 2024
`https://sites.aub.edu.lb/kmakdisi/`
Problem set 2, due Thursday, February 8 at the beginning of class

Exercises from Artin:

Chapter 3, exercises 6.1, 6.3. (Recall that in some soft copies of the book, these would be 5.1. 5.3). In 6.1, what happens if $\mathbf{R}$ is replaced by another field?

Chapter 4, exercises 1.3, 1.4, 3.3, 4.1, 4.2, 6.4, 6.8. (Notes: (i) In exercise 4.2, assume that $2 \neq 0$ in your field; the exercise does not work over $\mathbf{F}_2$. (ii) For 4.1 and 6.8, first do exercise A2.1. Actually, 4.1 only needs you to know the definition of an invariant subspace as given in exercise A2.1).

Additional Exercises (also required):

Exercise A2.1: a) Let V be finite-dimensional, and let $T : V \to V$ be a linear transformation. Assume that T has an invariant subspace W with $W \subset V$, and $W \neq \{\vec{0}\}, W \neq V$. The phrase "invariant subspace" means that $T(W) \subset W$.

Show that there exists a basis $\mathcal{B}$ for V for which the matrix ${}_{\mathcal{B}}[T]_{\mathcal{B}}$ is block upper triangular, i.e., ${}_{\mathcal{B}}[T]_{\mathcal{B}} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ with A, B, D matrices.

b) Viewing $T\big|_W : W \to W$, explain why A is the matrix of $T\big|_W$ with respect to a suitable basis.

c) Carefully explain how T gives rise to a linear transformation $\overline{T} : V/W \to V/W$, and show that D is the matrix of $\overline{T}$ with respect to a suitable basis.

Exercise A2.2: Define a linear transformation $T : \mathcal{P}_3 \to \mathcal{P}_3$ by

$$T(f) = (x+1)^2 f'' - 4xf' + 6f.$$

Here $\mathcal{P}_3$ is the space of polynomials of degree at most 3, with coefficients in $\mathbf{R}$.

a) Find the matrix $A = {}_{\mathcal{B}}[T]_{\mathcal{B}}$, where $\mathcal{B}$ is the basis $\mathcal{B} = \{1, x, x^2, x^3\}$ for $\mathcal{P}_3$.

b) Find a basis for each of $\operatorname{Image} T$ and $\ker T$. Justify your reasoning. Make sure you give elements of $\mathcal{P}_3$. (Suggestion: work with the matrix A viewed as a linear transformation from $\mathbf{R}^4$ to $\mathbf{R}^4$. Then translate your results to the setting of $\mathcal{P}_3$.)

c) Find the eigenvalues of T, and, for each eigenvalue, find one eigenvector. Again, these should be elements of $\mathcal{P}_3$. (Suggestion: work again with A and its characteristic polynomial, then translate the results back to $\mathcal{P}_3$.)

d) Show that ${}_{\mathcal{B}}[T]_{\mathcal{B}}$ is not similar to a diagonal matrix.

Look at, but do not hand in, the following exercises:

Chapter 3, exercise 7.1, M.1, M.2.

Chapter 4, exercises 2.1, 2.2, 2.5, 3.2, 4.3, 4.4, 4.8, 6.6, 6.7, M.5.

(See the back for an additional "Look at" exercise.)

"Look At" Exercise L2.1, not to be handed in: This exercise sketches a somewhat artificial proof that the row rank and the column rank of a matrix are equal. We will prove this result "officially" later in the course. In this exercise, A is a $p \times n$ matrix with rows $R_1, \ldots, R_p$ and columns $C_1, \ldots, C_n$. Here is an example with a 3×5 matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ 102 & 202 & 302 & 402 & 502 \end{pmatrix},$$

$$R_1 = (1,2,3,4,5), \qquad R_2 = (1,1,1,1,1), \qquad R_3 = (102,202,302,402,502),$$

$$C_1 = \begin{pmatrix} 1 \\ 1 \\ 102 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 2 \\ 1 \\ 202 \end{pmatrix}, \qquad \ldots, \qquad C_5 = \begin{pmatrix} 5 \\ 1 \\ 502 \end{pmatrix}.$$

a) Viewing $A = A_T$ for a linear transformation $T : F^n \to F^p$, assume that some row R_i is a linear combination of the other rows. (For example, in the matrix above, take $i = 3$ and $R_3 = 100R_1 + 2R_2$.) Let B be the $(p-1) \times n$ matrix obtained by removing row R_i from A. (In our example, $B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$.) Viewing B as the matrix of a linear transformation $U : F^n \to F^{p-1}$, show that $\ker T = \ker U$.

Hint: why are the systems of linear equations for $\ker T$ and $\ker U$ equivalent?

From now on, we will just write $\ker A$ and $\ker B$ for matrices, instead of mentioning the linear transformations. So $\ker A$ is what we previously called $\ker T$.

b) Suppose the row rank of A is r. Show that $\dim \ker A \leq n - r$.

Hint: remove "redundant" rows from A "without changing its kernel", until you obtain an $r \times n$ matrix M with the same kernel. Show using Rank-Nullity that the kernel of (the linear transformation given by) M has dimension at least $n - r$.

(Remark: $r \leq n$, because the rows belong to F^n. Also note that once we finish the exercise, we will see that $\dim \ker A = n - r$.)

c) Deduce from part (b) and Rank-Nullity for (the transformation given by) A that $r \geq$ the column rank of A. Thus "row rank $\geq$ column rank".

d) By applying part (c) to the row and column ranks of the transposed matrix A^t, show that the row rank of the original matrix A is $\leq$ the column rank of the original matrix A. Thus, from (c) and (d), the row and column ranks of A are equal.