Math 220, Linear Algebra II — Spring 2024

`https://sites.aub.edu.lb/kmakdisi/`

Problem set 1, due Thursday, February 1 at the beginning of class

Exercises from Artin:

Notes on notation: (i) $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ is the finite field with p elements, where p is a prime. (ii) $GL_n(F)$ is the multiplicative group of invertible $n \times n$ matrices with entries in F; you can think of it as the group of invertible linear transformations from F^n to itself. The subgroup $SL_n(F)$ consists of those matrices with determinant 1. (The symbols GL and SL refer to the *general linear group* and the *special linear group*.)

Chapter 2, exercises 5.5, 12.5.

Chapter 3, exercises 2.4, 2.5, 2.6, 4.5, 5.4 (hint: the formula for $|GL_2(\mathbf{F}_p)|$ reads better as $(p^2 - 1)(p^2 - p)$).

Additional Exercises (also required):

Exercise A1.1: Define $T : \mathbf{R}^4 \to \mathbf{R}^3$ by the matrix $A_T = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 0 \\ 4 & 8 & 5 & 4 \end{pmatrix}$.

a) Find a basis for each of $\ker T$ and $\operatorname{Image} T$.

b) Find new bases $\mathcal{B} = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4\}$ for $\mathbf{R}^4$ and $\mathcal{C} = \{\vec{\mathbf{w}}_1, \vec{\mathbf{w}}_2, \vec{\mathbf{w}}_3\}$ for $\mathbf{R}^3$ such that

$$T(\vec{\mathbf{v}}_1) = \vec{\mathbf{w}}_1, \qquad T(\vec{\mathbf{v}}_2) = \vec{\mathbf{w}}_2, \qquad T(\vec{\mathbf{v}}_3) = T(\vec{\mathbf{v}}_4) = \vec{0}.$$

Exercise A1.2: Let $T : V \to W$ be a linear transformation, and do **not** assume that V is finite-dimensional. Suppose we know that $\ker T$ and $\operatorname{Image} T$ are finite dimensional. **Prove** that in this case V is finite-dimensional and that $\dim V = \dim \ker T + \dim \operatorname{Image} T$.

Exercise A1.3: Let $\mathcal{P}_n = \mathcal{P}_n(\mathbf{R})$ be the space of polynomials of degree $\leq n$ with coefficients in $\mathbf{R}$.

Consider the linear transformation $T : \mathcal{P}_n \to \mathcal{P}_n$ defined by $T(f) = f + f'$. Show that T is bijective without necessarily finding the inverse T^{-1}.

Look at, but do not hand in, the following exercises:

Chapter 2, exercises 5.3, 12.2.

Chapter 3, exercises 4.2, 4.3, 5.5, M.4.

"Look At" Exercise L1.1: Let V be a vector space, and take an ordered list L of vectors $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k \in V$ (the vectors are necessarily distinct): so $L = (\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$. Let $c \in F$, let $i \neq j$, and consider the new list $L' = (\vec{\mathbf{v}}'_1, \dots, \vec{\mathbf{v}}'_k)$ that is given by $\vec{\mathbf{v}}'_\ell = \vec{\mathbf{v}}_\ell$ when $\ell \neq i$, but $\vec{\mathbf{v}}'_i = \vec{\mathbf{v}}_i + c\vec{\mathbf{v}}_j$. For example: L could be $(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3)$ and L' could be $(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2 - 7\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_3)$. In this example, $i = 2$, $j = 1$, and $c = -7$.

Show in general that (i) $\operatorname{span} L = \operatorname{span} L'$, and (ii) L is linearly independent if and only if L' is linearly independent.