Math 242, Topics in Algebra – Spring 2023
`https://sites.aub.edu.lb/kmakdisi/`
Problem set 10, due Tuesday, May 2 at the beginning of class

Exercises from Fraleigh:
Section 53, exercises 25, 26.
Section 54, exercises 4, 5, 9, 10, 11, 12.

Additional Exercises (also required):

Exercise A10.1: (Adapted from Artin, *Algebra*) Let E be a Galois extension of $\mathbf{Q}$ such that $G(E/\mathbf{Q})$ is isomorphic to the Klein 4-group. Show that $E = \mathbf{Q}(\sqrt{d}, \sqrt{e})$ for some $d, e \in \mathbf{Q}$.

Exercise A10.2: (Also adapted from Artin) Let F be a field of characteristic p, let $a \in F$, and assume that the polynomial $f(x) = x^p - x - a \in F[x]$ is irreducible. (Note: one can show that this particular $f(x)$ is irreducible if and only if it has no roots in F.) Let $\alpha \in \overline{F}$ be a root of f.
a) Show that $\alpha + 1$ is also a root of f.
b) Show that $F(\alpha)$ is a Galois extension of F.
c) Show that $G(F(\alpha)/F)$ is a cyclic group of order p.

Exercise A10.3: Let $\alpha = \sqrt{2 + \sqrt{3}}$.
a) Find $f(x) = \mathrm{irr}(\alpha, \mathbf{Q})$. As always, justify. Show that the roots of $f(x)$ are $\alpha, -\alpha, 1/\alpha, -1/\alpha$. Deduce that the splitting field of $f(x)$ over $\mathbf{Q}$ is $K = \mathbf{Q}(\alpha)$.
b) Find the Galois group $G = \mathrm{Gal}(K/\mathbf{Q})$. This means to "do the complete job", as Fraleigh says in Exercise 54.8. In other words: (i) describe all the elements of G in terms of their effect on α and its conjugates, (ii) and identify all of the subgroups $H_1, H_2, \ldots < G$, as well as the associated intermediate fields $F_1, F_2, \ldots$. This means in particular that you should find generators of the intermediate fields, i.e., express each such field F_i as $\mathbf{Q}(\beta_i)$ for some suitable β_i.
c) Identify the splitting field K as a familiar extension of $\mathbf{Q}$ from the course. (The easiest way is to replace each β_i by an "easier" generator of F_i.)

Exercise A10.4: Let $p \neq 2$ be a prime and let E be the splitting field of $x^p - 2$ over $\mathbf{Q}$. Note that $E = \mathbf{Q}(\sqrt[p]{2}, \zeta)$, where ζ is a primitive pth root of unity.
a) Show that $[E : \mathbf{Q}] = p(p-1)$. (Caution: it is neither obvious that $x^p - 2$ is irreducible over $\mathbf{Q}(\zeta)$, nor that $\Phi_p(x) = (x^p - 1)/(x - 1)$ is irreducible over $\mathbf{Q}(\sqrt[p]{2})$. In fact, for some nonprime p, these statements are false!)
b) Show that $G(E/\mathbf{Q})$ is isomorphic to the (multiplicative) group of matrices

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbf{Z}_p^*, b \in \mathbf{Z}_p \right\}.$$

(This group is sometimes called the $ax + b$ group; can you see why?)

Look at, but do not hand in, the following exercises:
Section 53, exercises 17, 18, 19.
Section 54, exercise 8.