**Math 242, Topics in Algebra – Spring 2023**
`https://sites.aub.edu.lb/kmakdisi/`
**Problem set 6, due Tuesday, April 4 at the beginning of class**

**Exercises from Fraleigh:**

Section 33, exercises 4, 5, 10, 12, 13.

Section 48, exercises 5, 6, 7, 8, 9, 10, 11. (These are fairly quick).

**Additional Exercises (also required):**

**Exercise A6.1:** Show that all the roots of $x^3 - 5x + 2$ are constructible real numbers, while none of the roots of $x^3 - 6x + 2$ is constructible.

**Exercise A6.2:** Let $\mathbf{F}_{25}$ be the finite field with 25 elements. Show that there exists an irreducible polynomial $f \in \mathbf{F}_{25}[x]$ with $\deg f = 242$.

Generalize to irreducible polynomials of any degree over any finite field.

**Exercise A6.3:** Consider the polynomial $f = x^4 + x^2 + x + 3 \in \mathbf{Z}_5[x]$.

a) Factor this polynomial into irreducibles in $\mathbf{Z}_5[x]$, and prove that the resulting factors are irreducible. (Nothing tricky is involved.)

b) How many roots does this polynomial have over each of (i) $\mathbf{F}_5 = \mathbf{Z}_5$, (ii) $\mathbf{F}_{125}$, and (iii) $\mathbf{F}_{25}$? (You do not need to find the actual roots; just count how many there are.)

c) Deduce the GCDs $\gcd(f, x^5 - x)$, $\gcd(f, x^{125} - x)$, and $\gcd(f, x^{25} - x)$ in $\mathbf{F}_5[x]$.

**Exercise A6.4:** Let $p$ be a prime number, and let $\mathbf{F}_p = \mathbf{Z}_p$ as usual. Take $\alpha \in \overline{\mathbf{F}}_p$ with $[\mathbf{F}_p(\alpha) : \mathbf{F}_p] = 4$. Write $E = \mathbf{F}_p(\alpha)$, so $E = \mathbf{F}_{p^4}$.

a) Show that $\alpha, \alpha^p, \alpha^{p^2}, \alpha^{p^3}$ are all distinct and that

$$\operatorname{irr}(\alpha, \mathbf{F}_p) = (x - \alpha)(x - \alpha^p)(x - \alpha^{p^2})(x - \alpha^{p^3}).$$

b) Show that the number of monic irreducible polynomials of degree 4 in $\mathbf{F}_p[x]$ is exactly $(p^4 - p^2)/4$.

c) By considering $\mathbf{F}_{p^6}$, find the number of monic irreducible polynomials of degree 6 in $\mathbf{F}_p[x]$. (Be careful with the counting.)

**Look at, but do not hand in, the following exercises:**

Section 33, exercises 1, 6, 7, 8, 9, 11, 14.

Section 48, exercises 23, 24, 25, 26, 28.