Math 242, Topics in Algebra – Spring 2023
`https://sites.aub.edu.lb/kmakdisi/`
Problem set 5, due Tuesday, March 21 at the beginning of class

Reminder: Our midterm is on Friday, March 24 from 3-4:30pm in Bliss 205. Students with a letter of accommodations, please come at 2pm to get extra time.

Exercises from Fraleigh:
Section 31, exercises 14, 17, 23, 28, 29, 30, 37.
Section 32, exercises 1, 3, 4, 5, 6, 7, 8.

Additional Exercises (also required):

Exercise A5.1: Let $f(x) = x^4 + x + 1 \in \mathbf{Z}_2[x]$, and let α be a root of f in the algebraic closure $\overline{\mathbf{Z}_2}$ of $\mathbf{Z}_2$.

a) Show that $f(x)$ is irreducible in $\mathbf{Z}_2[x]$, so that $\mathbf{F}_{16} = \mathbf{Z}_2(\alpha)$ is a finite field of order 16. (In fact, it is "the" finite field of order 16; we will see later that two finite fields of the same order are isomorphic.)

b) Show that all the other roots of $f(x)$ belong to $\mathbf{F}_{16}$, find these roots, and deduce the factorization of $f(x)$ in $\mathbf{F}_{16}[x]$ into linear factors.

Exercise A5.2: Let $\alpha = \sqrt{1+\sqrt{3}}$, and let $\beta = \sqrt{1-\sqrt{3}}$.

a) Find $\mathrm{irr}(\alpha, \mathbf{Q})$ and show that its roots are $\{\alpha, -\alpha, \beta, -\beta\}$.

b) Show that $\mathrm{irr}(\beta, \mathbf{Q}(\alpha)) = x^2 + (\sqrt{3} - 1)$. This involves showing not only that β is a root of this polynomial, but also that this polynomial actually has coefficients in $\mathbf{Q}(\alpha)$ and that it is irreducible in $\mathbf{Q}(\alpha)[x]$.

c) Deduce a basis for $\mathbf{Q}(\alpha, \beta)$ as a vector space over $\mathbf{Q}$.

d) Deduce a basis for $\mathbf{Q}(\alpha, \beta)$ as a vector space over $\mathbf{Q}(\sqrt{3})$.

e) What is different if we work instead with the elements $\sqrt{2 \pm \sqrt{3}}$?

Look at, but do not hand in, the following exercises:
Section 31, exercises 19, 26, 27, 31, 32, 33, 34, 35, 36.

"Look At" exercise L5.1 — not to be handed in: An element $\alpha \in \mathbf{C}$ is said to be an **algebraic integer** if it is the root of a **monic** polynomial with **integral** coefficients:

$$\alpha^n + c_{n-1}\alpha^{n-1} + \cdots + c_0 = 0, \qquad c_i \in \mathbf{Z}.$$

(If we only imposed the weaker requirement that $c_i \in \mathbf{Q}$, then we would be requiring α to be algebraic over $\mathbf{Q}$ — in that case, α would be called an algebraic number.)

a) Show that if β is an algebraic number, then one can find an integer n such that $n\beta$ is an algebraic integer. (Thus we can write $\beta = \alpha/n$, where α is an algebraic integer.)

b) Show that if $\alpha \in \mathbf{Q}$ is an algebraic integer, then in fact $\alpha \in \mathbf{Z}$.

c) (Quite hard to answer on your own — you should look up some references) If α, β are algebraic integers, show that $\alpha + \beta$ and $\alpha\beta$ are algebraic integers.