Math 242, Topics in Algebra – Spring 2023
https://sites.aub.edu.lb/kmakdisi/
Problem set 4, due Tuesday, March 14 at the beginning of class

Exercises from Fraleigh:
Section 30, exercises 2, 3, 5, 6, 11, 12, 13, 24, 25.
Section 31, exercises 1, 3, 5, 10.

Additional Exercises (also required):

Exercise A4.1, copied from Jacobson Basic Algebra I: Let $w = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$ (in $\mathbf{C}$). Note that $w^{12} = 1$ but $w^r \neq 1$ if $1 \leq r < 12$ (so w is a generator of the cyclic group of 12th roots of 1). Show that $[\mathbf{Q}(w) : \mathbf{Q}] = 4$ and determine the minimum [meaning irreducible] polynomial of w over $\mathbf{Q}$.

[Note: it may be easier to find the minimum polynomial first, and then deduce the degree $[\mathbf{Q}(w) : \mathbf{Q}]$.]

Exercise A4.2: In this problem, we write $\zeta = \exp(2\pi i/7) \in \mathbf{C}$. As you know, $\zeta^7 = 1$, and in fact we have the following factorizations over $\mathbf{C}$:

$$x^7 - 1 = (x-1)(x-\zeta)(x-\zeta^2)(x-\zeta^3)(x-\zeta^4)(x-\zeta^5)(x-\zeta^6),$$
$$f(x) = (x-\zeta)(x-\zeta^2)(x-\zeta^3)(x-\zeta^4)(x-\zeta^5)(x-\zeta^6),$$

where

$$f(x) = \frac{x^7-1}{x-1} = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \in \mathbf{Z}[x].$$

a) Show that $f(x)$ is irreducible in $\mathbf{Z}[x]$. (Hint: we have seen this in the lecture. Write $x = y + 1$ and show that $f(y+1)$ satisfies the Eisenstein criterion.)

b) Let $\alpha = \zeta + \zeta^{-1}$. Show that α is a root of $g(x) = x^3 + x^2 - 2x - 1$ and find a degree 2 polynomial $h(x) \in (\mathbf{Q}(\alpha))[x]$ for which $h(\zeta) = 0$.

c) By looking at degrees of field extensions, deduce that $g(x)$ is irreducible in $\mathbf{Q}[x]$ and that $h(x)$ is irreducible in $(\mathbf{Q}(\alpha))[x]$. (Note: one can show irreducibility of $g(x)$ directly, and irreducibility of $h(x)$ by observing that $h(x)$ is irreducible in $\mathbf{R}[x]$ and that $\mathbf{Q}(\alpha) \subset \mathbf{R}$, but the point is to have some practice with degrees of algebraic extensions.)

Look at, but do not hand in, the following exercises:
Section 30, exercises 7, 8, 10, 16, 17, 19, 20, 21, 23, 26, 27.
Section 31, exercises 6, 7, 11, 12, 13.