Math 261 — Fall 2022
Number Theory
`https://sites.aub.edu.lb/kmakdisi/`
Problem set 12, NOT DUE, for practice before the final

Exercise 12.1: a) Show that the quadratic forms $79x^2 + 64xy + 13y^2$ and $x^2 + 3y^2$ are equivalent.

b) Find all reduced quadratic forms of discriminant $\Delta = -12$, and show directly that they are mutually inequivalent.

c) Show that every prime p with $p \equiv 1 \pmod 3$ is represented by the form $x^2 + 3y^2$.

Exercise 12.2: a) Find all the reduced forms of discriminant -23, and show directly that they are mutually inequivalent. (There are three equivalence classes of positive definite forms of discriminant -23; they and their negatives give a total of six equivalence classes.)

b) Which primes are represented by at least one of these reduced forms? (Challenge: what condition on a prime determines which actual reduced form represents it?)

Exercise 12.3: Find all the reduced forms of discriminant 40, and show that some of them are actually equivalent. Show that there are exactly two equivalence classes of forms of discriminant 40. (Hint: in this example, it turns out that every form is equivalent to its negative, but this requires some proof. You will probably find it easiest to argue separately for each case. This phenomenon is related to the presence of a fundamental unit $3 + \sqrt{10}$ in $\mathbf{Z}[\sqrt{10}]$ whose norm is $3^2 - 10 = -1$.)

Exercise 12.4: Consider the quadratic forms

$$f(\begin{pmatrix} x \\ y \end{pmatrix}) = f(x, y) = 19x^2 + 21xy + 6y^2, \qquad g(\begin{pmatrix} x \\ y \end{pmatrix}) = g(x, y) = 4x^2 + 7xy + 4y^2.$$

a) Verify that f and g have the same discriminant.

b) Find a reduced form F equivalent to f, and keep track of the matrix $M \in SL(2, \mathbf{Z})$ for which $F = f \star M$.

c) Do the same for g. You should get the same form F equivalent to g, with $F = g \star N$.

d) Use your result in part c) to deduce that f and g are equivalent, and find $M' \in SL(2, \mathbf{Z})$ such that $f = g \star M'$. Deduce from this $x, y \in \mathbf{Z}$ such that $g(x, y) = 19$.

e) Find another positive definite form H with the same discriminant that you can prove is not equivalent to F.

Exercise 12.5: a) If p is a prime with $p \neq 2, 7$, show that

$$\left(\frac{-28}{p}\right) = \left(\frac{-7}{p}\right) = \left(\frac{p}{7}\right).$$

b) Show that if $p \equiv 1, 2, \text{ or } 4 \bmod 7$ then there exists a quadratic form $f(x, y) = px^2 + hxy + \ell y^2$ with discriminant -28.

c) Find all the reduced positive definite quadratic forms with discriminant -28.

d) Perform a reduction on the quadratic form $f = 79x^2 + 50xy + 8y^2$ (which has discriminant -28) to show that f is equivalent to the quadratic form $g = u^2 + 7w^2$. Do this while keeping track of the matrix M such that $f * M = g$, and use this to deduce a solution to the equation $79 = u_0^2 + 7w_0^2$.

e) Show that if $p \equiv 1, 2, \text{ or } 4 \bmod 7$ then it is possible to write $p = u_0^2 + 7w_0^2$ for suitable $u_0, w_0 \in \mathbf{Z}$. (Include a proof that the reduction of the form f from part b is indeed $u^2 + 7w^2$, and not another one of the solutions to part c.)