Math 261 — Fall 2022
Number Theory
`https://sites.aub.edu.lb/kmakdisi/`
Problem set 11, due Friday, November 25 at the beginning of class

Exercise 11.1: a) Show that if two quadratic forms are (possibly improperly) equivalent, then they represent the same integers.

b) Show that if two quadratic forms are equivalent, then they properly (sometimes called "primitively") represent the same integers.

Reminder: we say that $f(x,y) = ax^2+bxy+cy^2$ and $F(u,w) = Au^2+Buw+Cw^2$ are equivalent iff there exists a change of variables $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}$ transforming f into F, where $p,q,r,s \in \mathbf{Z}$ and $ps-qr = \pm 1$. We have introduced the notation $F = f * \begin{pmatrix} p & q \\ r & s \end{pmatrix}$.

Note that this problem is really about changes of variables, and the solution has almost nothing to do with quadratic forms.

Exercise 11.2: Identify which of the following quadratic forms are definite or indefinite, and use Exercise 11.1 to show that no two of them are equivalent. (Exceptionally in this exercise, do not use the fact that equivalent forms would have the same discriminant.)

$$x^2+3y^2, \qquad x^2-3y^2, \qquad -x^2-3y^2, \qquad 3x^2+2xy+5y^2, \qquad 2x^2+2xy+2y^2.$$

Exercise 11.3: a) Let $f(x,y) = 3x^2+2xy+5y^2$. Show that the three smallest integers represented by f are 3, 5, and 6; find all pairs $(x,y) \in \mathbf{Z}^2$ such that $f(x,y) \in \{3,5,6\}$. Do the same for the form $g(x,y) = 3x^2-2xy+5y^2$. (This is easy, since f and g are "improperly" equivalent, i.e., by a transformation of determinant -1.)

b) Show that f and g are not "properly" equivalent. One way to do this is to try which matrices M transform the corresponding pairs (x,y) that you found in part a to each other, and to show that any such M must have determinant -1.

Exercise 11.4: Show that the quadratic form $5x^2-13y^2$ represents the numbers ± 5 but does not represent the numbers $0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 6$. Suggestion: first use congruences modulo any of the numbers 5, 13, and 4 to eliminate the values $0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 6$. Then find explicit solutions that represent ± 5 by trial and error. (You may want to narrow down the search a bit using congruences.)

Exercise 11.5: a) Find the eigenvalues λ_1, λ_2 and eigenvectors $\vec{u}_1, \vec{u}_2$ of $M = \begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix}$. Make sure to normalize the vectors $\vec{u}_1$ and $\vec{u}_2$ to have length 1. Observe that $\vec{u}_1 \perp \vec{u}_2$. (This is a consequence of the **spectral theorem** in finite dimension: any real matrix M that is symmetric, i.e., $M^t = M$, can be diagonalized using an orthonormal basis of eigenvectors.)

b) What are the minimum and maximum values of $f(x,y) = 3x^2+2xy+5y^2$ on the circle $x^2+y^2=1$? What about on $x^2+y^2=k$? What is the relation to part (a) above?

c) Find all $(x,y) \in \mathbf{Z}^2$ for which $f(x,y) \le 15$. Suggestion: use the minimum on $x^2+y^2=k$ in part (b) to show that $f(x,y) \le 15 \Rightarrow x^2+y^2 \le k$ for a suitable choice of k.

If you could not do parts (a) and (b), try instead to solve (c) by completing the square in f and arguing by hand.