Math 261 — Fall 2022
Number Theory
`https://sites.aub.edu.lb/kmakdisi/`
Problem set 10, due Friday, November 18 at the beginning of class

Exercise 10.1: Let k be a nonzero integer and let $N \geq 2$ with N not a square. You will need to use a nontrivial unit $u = a + b\sqrt{N} \in \mathbf{Z}[\sqrt{N}]$. If you like, u can be a fundamental unit.

a) Show that if there exists one solution to the equation $x^2 - Ny^2 = k$, then there exist infinitely many. (Hint: $(x_0 + y_0\sqrt{N})u^\ell$.)

b) Use this to find four different solutions (not just via changing the signs of x and y) to the equation $x^2 - 3y^2 = 13$. (You can easily find a unit $u \in \mathbf{Z}[\sqrt{3}]$ by trial and error.)

c) Show that the equation $x^2 - 3y^2 = 5$ has no solutions. Suggestion: reduce mod 3.

Exercise 10.2: Find a fundamental unit in each of $\mathbf{Z}[\sqrt{7}]$ and $\mathbf{Z}[\sqrt{13}]$, and use to find three solutions to each of the equations $x^2 - 7y^2 = 1$, $x^2 - 13y^2 = 1$. (The solutions should be genuinely different, and not just obtained by changing signs.)

Exercise 10.3: Consider the equation $x^2 - Ny^2 = k$ for arbitrary $k \in \mathbf{Z}$, not just $k = \pm 1$. As always, assume that $N > 0$ is not a square. We view the problem in terms of finding an element $\alpha = x + y\sqrt{N} \in \mathbf{Z}[\sqrt{N}]$ whose norm is k.

a) Show that when $k = 0$, the only solution is the trivial one $(x, y) = (0, 0)$. (This is easy, but not as instant as you might think.)

b) From now on, suppose $k \neq 0, \pm 1$, and let $u \in \mathbf{Z}[\sqrt{N}]$ be a unit with $\operatorname{Norm} u = +1$ and $u > 1$. (Take u to be either a fundamental unit or its square.) Show that if a nontrivial solution α exists as above, then some suitable $\alpha' = u^\ell \alpha = x' + y'\sqrt{N}$ gives rise to a solution with $1 < \alpha' < u$. Here $\ell \in \mathbf{Z}$ can be positive or negative.

c) For the above α', consider its conjugate $\overline{\alpha'} = x' - y'\sqrt{N}$. Use the fact $\alpha'\overline{\alpha'} = k$ to obtain a bound on the size of $\overline{\alpha'}$. From the bounds on α' and its conjugate, obtain upper and lower bounds for the integers x', y'. Thus only finitely many pairs (x', y') need to be considered.

d) The above shows: to see if $x^2 - Ny^2 = k$ has any solutions, one has to try only a finite number of pairs (x', y') in a bounded range; then all other solutions are obtained from such solutions $x' + y'\sqrt{N}$ by multiplying by some power of the unit u. Illustrate this for the equations $x^2 - 10y^2 = 3$ and $x^2 - 10y^2 = 31$.

Exercise 10.4: Consider the equation $(*)$: $\qquad x^2 - 101y^2 = 1$.

a) In this part, we want to find all **rational** solutions to $(*)$, so $(x, y) \in \mathbf{Q}^2$. Starting from the easy solution $(x, y) = (-1, 0)$, parametrize all the other rational solutions using a variable $t \in \mathbf{Q}$ and a line of slope t passing through $(-1, 0)$ which intersects the hyperbola $(*)$ in a second rational point. Give formulas for the coordinates of this second rational point in terms of t.

b) Now we study **integral** solutions to $(*)$, so $(x, y) \in \mathbf{Z}^2$, and we are effectively solving Pell's equation. First, find by trial and error an integral solution $(s, t) \in \mathbf{Z}^2$ to the related equation $s^2 - 101t^2 = -1$. (It is no loss of generality to stick to $s, t > 0$; now find a solution with small values of s, t.) Use this and $\mathbf{Z}[\sqrt{101}]$ to produce a solution $(x_1, y_1) \in \mathbf{Z}^2$ of equation $(*)$.

(c) Use reduction mod 101 and some extra reasoning that there are no other integral solutions (x, y) to $(*)$ with $1 < x < x_1$. (This means that you have found the fundamental solution, analogous to our theorem about finding a solution with smallest y. But for this exercise, pretend you do not know this theorem.)