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On the Fourier coefficients of nonholomorphic Hilbert modular forms of half-integral weight

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Introduction.

Let $f(z) = \sum_{n\geq 1} a_n e^{2\pi inz}$ be a Hecke eigenform of half-integral weight $m+1/2$, and let $g(z) = \sum_{n\geq 1} b_n e^{2\pi inz}$ be the corresponding even-weight form, in the sense of [Sh 73]. In particular, g has weight $2m$, and belongs to the same eigenvalues of Hecke operators as f. If $n = q^2 r$ with squarefree r, then a_n is expressible in terms of a_r and the $\{b_j\}$. At the end of [Sh 77], Shimura suggested that a_r should be related to special values of Dirichlet series associated to g. This was borne out in [Wa 81], where Waldspurger proved the striking relation that for squarefree r, a_r^2 is essentially proportional to $\sum_{n\geq 1} \overline{\varphi_r}(n) b_n n^{-s} \big|_{s=m}$. Here we have twisted the standard Dirichlet series for g by a character φ_r obtained from the character of f and the quadratic character $\left(\frac{r}{\cdot}\right)$.

The purpose of this paper is to derive generalizations of Waldspurger's relation, with an explicit proportionality constant, in the case where f and g are nonholomorphic Hilbert modular forms over a totally real number field F. If $F = \mathbf{Q}$, such forms are also called Maass forms. The method of proof, which ought to generalize to arbitrary number fields, follows, with some simplifications, that in [Sh 93a], which treats the case of holomorphic Hilbert modular forms. Previous investigations into this topic have included work by Kohnen and Zagier ([Ko-Za 81] and [Ko 85]) in the holomorphic case, and Katok and Sarnak ([Ka-Sa 93]) in the nonholomorphic case. Both of these treatments deal only with forms on the upper half-plane (i.e. $F = \mathbf{Q}$), with some additional restrictions. Recent (not yet published) work of M. Furusawa suggests that the method in [Sh 93a] and in this paper should generalize to yield a similar formula, in the case of the correspondence between automorphic forms on $Sp(n)$ and on $O(2n+1)$.

Extending Shimura's work in [Sh 93a] to the nonholomorphic case involves two main difficulties. First, as the Fourier expansions of Maass forms involve Whittaker functions instead of exponentials, Mellin transforms and Rankin-Selberg convolutions produce more complicated "Gamma-factors" than usual; these factors must be explicitly evaluated, in order to yield precise versions of Waldspurger's relation. Second, whereas the Fourier expansions of holomorphic forms are indexed only by totally positive elements of the field F, the expansions of Maass forms are indexed by field elements of arbitrary signature; this makes the calculations rather more delicate. Section 3 of this paper explains in explicit detail how one overcomes both of these problems in constructing a Dirichlet series from Hilbert modular forms of arbitrary integral weight. The results in section 3 are in principle known from [Ma 53], [J-L], and [W], but are not found in this form in the literature (see the discussion at the beginning of section 3). This applies particularly to the appendix to section 3, where we explicitly compute the Mellin transforms of *all* possible Whittaker functions at the archimedean places.

Sections 1 through 4 contain nothing new, but rather set up precise definitions and normalizations of all automorphic forms, special functions, and parameters used in the calculation — this is essential for obtaining the explicit formulas in section 8. Section 5 presents the main results on the Shimura correspondence, which are well-known in outline, but which need to be stated quite precisely for our purposes. The central work is done in sections 6–8. Section 6 carries through a calculation originally due to Niwa [Ni 74] over $\mathbf{Q}$, following Shimura in [Sh 87], where he treats holomorphic Hilbert modular forms. Section 7 contains a careful analysis of certain Gauss sums, using which we are able to remove the technical restrictions (3.20b, c) of [Sh 93a]. This replaces Lemmas 5.4 and 5.5 of that paper. (The remaining technical restriction (3.20a) is quite minor, according to Lemma 1.2 of [Sh 93a].) Section 8 contains the main results of this paper, Theorem 8.1 and Theorem 8.4. These generalize respectively Theorems 3.4 and 3.2 of [Sh 93a], and follow closely Shimura's method there. We refer the reader to the introduction of Shimura's paper, pages 502–503, for an overview of his method.

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0. Notation.

Our notation follows closely that of [Sh 93a]. For any ring R with identity element, $R^\times$ is its group of units. Throughout this paper, we let F be a totally real number field of finite degree over $\mathbf{Q}$, with ring of integers $\mathfrak{o}$, absolute different $\mathfrak{d}$, and absolute discriminant D_F. We write $\mathbf{a}$ and $\mathbf{f}$ for the sets of archimedean and nonarchimedean primes of F, respectively, and write $F_\mathbf{A}$ for the adeles of F (and, consequently, $F_\mathbf{A}^\times$ for the ideles). Generally speaking, for any algebraic group $\mathcal{G}$ over F, $\mathcal{G}_v$ is the completion of $\mathcal{G}$ at the prime $v \in \mathbf{a} \cup \mathbf{f}$, which we view implicitly as a subgroup of the adelization $\mathcal{G}_\mathbf{A}$ of $\mathcal{G}$. $\mathcal{G}$ injects into $\mathcal{G}_\mathbf{A}$ diagonally, and we write $\mathcal{G}_\mathbf{a}$ and $\mathcal{G}_\mathbf{f}$ for the archimedean and nonarchimedean factors of $\mathcal{G}_\mathbf{A}$. For any set X, $X^\mathbf{a}$ is the set of tuples $(x_v)_{v\in\mathbf{a}}$, where each $x_v \in X$. In particular, we view F as being embedded into $\mathbf{R}^\mathbf{a}$ (which is of course identified with $F_\mathbf{a}$). We introduce the following notation, for arbitrary $x, y \in \mathbf{C}^\mathbf{a}$:

$$(0.1) \qquad x^y = \prod_{v\in\mathbf{a}} x_v^{y_v}, \qquad \|x\| = \sum_{v\in\mathbf{a}} x_v, \qquad \mathbf{e}_\mathbf{a}(x) = \exp(2\pi i\|x\|) = \prod_{v\in\mathbf{a}} e^{2\pi i x_v}.$$

(The powers are taken according to context.) Also let $u = (1, 1, \ldots, 1) \in \mathbf{Z}^\mathbf{a}$. Thus $\|u\| = [F : \mathbf{Q}]$ and $x^u = \mathrm{N}_{F/\mathbf{Q}}x$ for $x \in F$. We will also let x in the above equations be an element of $F_\mathbf{A}$ or of $F_\mathbf{A}^\times$, in which case one should interpret $\mathbf{e}_\mathbf{a}(x)$ and x^y by first projecting x to $F_\mathbf{a} = \mathbf{R}^\mathbf{a}$. We also write $\mathbf{e}_\mathbf{A}(x)$ for the fundamental additive character of $F_\mathbf{A}$:

$$(0.2) \qquad \mathbf{e}_\mathbf{A}(x) = \mathbf{e}_\mathbf{a}(x_\mathbf{a}) \prod_{v\in\mathbf{f}} \mathbf{e}_v(x_v), \qquad \mathbf{e}_\mathbf{A}(F) = 1,$$

where $\mathbf{e}_v : F_v \to \mathbf{T}$ is the usual local additive character that is trivial on $\mathfrak{d}_v^{-1}$ (see equations (3.1a,b) of [Sh 83]). $\mathbf{T}$ is the circle group $\{z \in \mathbf{C} \mid |z| = 1\}$.

For $x, y \in \mathbf{R}^{\mathbf{a}}$ (or $F_{\mathbf{A}}$), we write $x \geq y$ if $x_v \geq y_v$ for all $v \in \mathbf{a}$, and $x \gg 0$ if $x_v > 0$ for all $v \in \mathbf{a}$ (in other words, if x is totally positive). We shall write $\mathfrak{o}^\times$, $\mathfrak{o}_+^\times$, and $(\mathfrak{o}^\times)^2$ for the units, the totally positive units, and the squares of units in $\mathfrak{o}$. (Trivially, $\mathfrak{o}^\times \supset \mathfrak{o}_+^\times \supset (\mathfrak{o}^\times)^2$.) For $x \in F_{\mathbf{a}}^\times$ or in $F_{\mathbf{A}}^\times$, define $|x| \in \mathbf{R}^{\mathbf{a}}$ and the signature $\operatorname{sgn} x \in \{\pm 1\}^{\mathbf{a}}$ by

$$(0.3) \qquad |x|_v = |x_v|, \qquad (\operatorname{sgn} x)_v = x_v/|x_v|, \qquad \text{for } v \in \mathbf{a}.$$

We also write $|x|_{\mathbf{A}} = \prod_{v \in \mathbf{a} \cup \mathbf{f}} |x_v|_v$, where $|x_v|_v$ is the normalized valuation of x_v. For a finite idele $t \in F_{\mathbf{f}}^\times$ and a fractional ideal $\mathfrak{x}$ of F, let $\mathrm{N}(\mathfrak{x})$ be the norm of $\mathfrak{x}$, and let $t\mathfrak{x}$ be the fractional ideal such that $(t\mathfrak{x})_v = t_v \mathfrak{x}_v$ for all $v \in \mathbf{f}$. Then $|t|_{\mathbf{A}} = \mathrm{N}(t\mathfrak{o})^{-1}$.

For a matrix $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we write $a_x = a, \ldots, d_x = d$. We define the groups

$$(0.4) \qquad G = SL_2(F), \qquad \tilde{G} = GL_2(F), \qquad P = \{\alpha \in G \mid c_\alpha = 0\},$$

$$(0.5) \qquad \tilde{G}_{\mathbf{a}+} = GL_2^+(\mathbf{R})^{\mathbf{a}} = \{\alpha \in \tilde{G}_{\mathbf{a}} \mid \det \alpha \gg 0\}, \quad \tilde{G}_{\mathbf{A}+} = \tilde{G}_{\mathbf{a}+}\tilde{G}_{\mathbf{f}}, \quad \tilde{G}_+ = \tilde{G} \cap \tilde{G}_{\mathbf{A}+}.$$

If $\mathfrak{x}\mathfrak{y} \subset \mathfrak{o}$ for two fractional ideals $\mathfrak{x}$ and $\mathfrak{y}$ of F, we define the following congruence subgroups of $G, \tilde{G}, G_{\mathbf{A}}, \tilde{G}_{\mathbf{A}}$:

$$(0.6) \qquad \tilde{D}[\mathfrak{x}, \mathfrak{y}] = \tilde{G}_{\mathbf{a}+} \prod_{v \in \mathbf{f}} \tilde{D}_v[\mathfrak{x}, \mathfrak{y}], \qquad \tilde{D}_v[\mathfrak{x}, \mathfrak{y}] = \mathfrak{o}_v[\mathfrak{x}, \mathfrak{y}]^\times,$$

where $\mathfrak{o}_v[\mathfrak{x}, \mathfrak{y}]$ is the subring of $\mathrm{M}_2(F_v)$ given by

$$(0.7) \qquad \mathfrak{o}_v[\mathfrak{x}, \mathfrak{y}] = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(F_v) \mid a, d \in \mathfrak{o}_v, b \in \mathfrak{x}_v, c \in \mathfrak{y}_v\};$$

$$(0.8) \quad D[\mathfrak{x}, \mathfrak{y}] = G_{\mathbf{A}} \cap \tilde{D}[\mathfrak{x}, \mathfrak{y}], \qquad \tilde{\Gamma}[\mathfrak{x}, \mathfrak{y}] = \tilde{G} \cap \tilde{D}[\mathfrak{x}, \mathfrak{y}], \qquad \Gamma[\mathfrak{x}, \mathfrak{y}] = G \cap D[\mathfrak{x}, \mathfrak{y}],$$

$$(0.9) \qquad C' = D[2\mathfrak{d}^{-1}, 2\mathfrak{d}], \qquad C'' = C' \cup C'\varepsilon,$$

$$(0.10) \qquad \varepsilon \in G_{\mathbf{A}}, \qquad \varepsilon_{\mathbf{a}} = 1, \qquad \varepsilon_v = \begin{pmatrix} 0 & -\delta_v^{-1} \\ \delta_v & 0 \end{pmatrix} \quad \text{for } v \in \mathbf{f},$$

where we pick once and for all $\delta \in F_{\mathbf{f}}^\times$ with $\delta\mathfrak{o} = \mathfrak{d}$. In general, $\Gamma \subset G$ is a congruence subgroup if it contains as a subgroup of finite index the principal congruence subgroup $\Gamma(\mathfrak{z})$ of matrices congruent to the identity mod $\mathfrak{z}$, as in [Sh 85b], (1.7b). Let $\mathcal{H}$ be the complex upper half-plane. Then $\tilde{G}_{\mathbf{a}+}$ acts on $\mathcal{H}^{\mathbf{a}}$ in the usual way, by componentwise linear fractional transformations, and the stabilizer of $\mathbf{i} = (i, \ldots, i) \in \mathcal{H}^{\mathbf{a}}$ is $SO_2(\mathbf{R})^{\mathbf{a}}$. This action extends to $\tilde{G}_{\mathbf{A}+}$ and to the adelic metaplectic group $M_{\mathbf{A}}$, which projects to $G_{\mathbf{A}}$; see section 3 of [Sh 85a], section 1 of [Sh 87], and section 4 of this paper.

A Hecke character $\psi : F_{\mathbf{A}}^{\times} \to \mathbf{T}$ is any continuous unitary character which is trivial on $F^{\times}$, and which may be of infinite order. If $\mathfrak{c}$ is the conductor of ψ, we define an ideal character $\psi^*(\mathfrak{x})$, for $\mathfrak{x}$ a fractional ideal prime to $\mathfrak{c}$, by $\psi^*(\mathfrak{x}) = \psi(x)$, where $x \in F_{\mathbf{f}}^{\times}$ is such that $\mathfrak{x} = x\mathfrak{o}$ and $x_v = 1$ if $v \mid \mathfrak{c}$. If $\mathfrak{x}$ is not prime to $\mathfrak{c}$, we set $\psi^*(\mathfrak{x}) = 0$. We write ψ_v, $\psi_{\mathbf{a}}$, and $\psi_{\mathbf{f}}$ for the restrictions of ψ to $F_v^{\times}$, $F_{\mathbf{a}}^{\times}$, and $F_{\mathbf{f}}^{\times}$. Moreover, for $x \in F_{\mathbf{A}}^{\times}$ and an ideal $\mathfrak{z}$ divisible by the conductor $\mathfrak{c}$ of ψ, we write $\psi_{\mathfrak{z}}(x) = \prod_{v|\mathfrak{z}} \psi_v(x_v)$. Typically $x_v \in \mathfrak{o}_v^{\times}$ for $v \mid \mathfrak{z}$, in which case $\psi_{\mathfrak{z}}(x) = \psi_{\mathfrak{c}}(x)$.

So far, all our notation has been the same as that in [Sh 93a], except that we use $\mathcal{H}, \mathbf{f}, \mathfrak{o}$ here, as opposed to $H, \mathbf{h}, \mathfrak{g}$ there. Also note that the groups $D[\mathfrak{x}, \mathfrak{y}]$ and $\tilde{D}[\mathfrak{x}, \mathfrak{y}]$ are slightly different than in [Sh 87], where their archimedean factors are defined to be compact. We now introduce some more notation. Define the "floor" and "ceiling" of $x \in \mathbf{R}$ by

$$\lfloor x \rfloor = \sup\{n \in \mathbf{Z} \mid n \leq x\}, \qquad \lceil x \rceil = \inf\{n \in \mathbf{Z} \mid n \geq x\}, \tag{0.11}$$

and define the Pochhammer symbol $(z)_n$, for $z \in \mathbf{C}$ and $0 \leq n \in \mathbf{Z}$, by

$$(z)_n = z(z+1)\cdots(z+n-1), \tag{0.12}$$

where we understand that $(z)_0 = 1$. We also define, for all $z \in \mathbf{C}^{\mathbf{a}}$ and $0 \leq n \leq m \in \mathbf{Z}^{\mathbf{a}}$,

$$\binom{m}{n} = \prod_{v \in \mathbf{a}} \binom{m_v}{n_v}, \qquad \Gamma(z) = \prod_{v \in \mathbf{a}} \Gamma(z_v), \qquad (z)_n = \prod_{v \in \mathbf{a}} (z_v)_{n_v}, \tag{0.13}$$

where $\binom{m_v}{n_v}$ is the binomial coefficient and Γ is the Gamma-function.

1. Hilbert Modular Forms on $\mathcal{H}^{\mathbf{a}}$.

We summarize some basic facts about (nonholomorphic) Hilbert modular forms $f : \mathcal{H}^{\mathbf{a}} \to \mathbf{C}$, as presented in the first two sections of [Sh 85b].

Definition. A weight n will be either an element of $\mathbf{Z}^{\mathbf{a}}$, in which case it is called *integral*, or it will be an element of $\mathbf{Z}^{\mathbf{a}} + u/2$, in which case it will be called *half-integral*.

The weight n factor of automorphy $J_n(\tau, z)$ is defined for integral n by

$$J_n(\tau, z) = j(\tau, z)^n, \qquad \tau \in \tilde{G}_{\mathbf{a}+}, z \in \mathcal{H}^{\mathbf{a}}. \tag{1.1}$$

Here $j(\tau, z) = (j(\tau_v, z_v))_{v \in \mathbf{a}}$ and $j(\alpha, z) = (\det \alpha)^{-1/2}(cz + d)$ for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbf{R})$ and $z \in \mathbf{C}$.

If n is half-integral, we restrict ourselves to $\tau \in G \cap P_{\mathbf{A}} C''$. In Proposition 3.2 of [Sh 85a] and Proposition 2.3 of [Sh 87], Shimura defines for such τ a quasi-factor of automorphy $h(\tau, z)$ of weight $u/2$; that is,

$$h(\tau, z)^2 = \zeta \cdot j(\mathrm{pr}(\tau), z)^u, \quad \text{some } \zeta \in \mathbf{T}. \tag{1.2}$$

The set of such τ includes the congruence subgroup $\Gamma_{1/2} = G \cap C'' \supset \Gamma[2\mathfrak{d}^{-1}, 2\mathfrak{d}\,]$. Then define for half-integral n

$$J_n(\tau, z) = h(\tau, z)j(\tau, z)^{n-u/2}, \quad \tau \in G \cap P_{\mathbf{A}}C'', z \in \mathcal{H}^{\mathbf{a}}. \tag{1.3}$$

For n integral or half-integral, and for $f : \mathcal{H}^{\mathbf{a}} \to \mathbf{C}$, define

$$(f\|_n\tau)(z) = J_n(\tau, z)^{-1}f(\tau z), \tag{1.4}$$

where $\tau \in \tilde{G}_{\mathbf{a}+}$ or $G \cap P_{\mathbf{A}}C''$, so that $J_n(\tau, z)$ is defined. For an archimedean place $v \in \mathbf{a}$, and for a given weight n, we define differential operators ϵ_v, δ_v^n, and L_v^n, which act on C^∞ functions $f : \mathcal{H}^{\mathbf{a}} \to \mathbf{C}$ by equations (2.3a-c) of [Sh 85b]:

$$\epsilon_v f = -y_v^2\, \partial f/\partial \bar{z}_v, \qquad \delta_v^n f = y_v^{-n_v}\, \partial(y_v^{n_v} f)/\partial z_v, \tag{1.5}$$

$$L_v^n f = 4\delta_v^{n-(2)_v}\epsilon_v f = -n_v + 4\epsilon_v\delta_v^n f. \tag{1.6}$$

Here $n-(2)_v$ means a weight whose w-component at a place $w \neq v$ is n_w, and whose v-component is $n_v - 2$. (Alternatively, $(2)_v$ is the injection of 2 into the additive group $\mathbf{R}^{\mathbf{a}}$ at the v-place.) We quote relations (2.5a-c) of [Sh 85b]:

$$\delta_v^n(f\|_n\tau) = (\delta_v^n f)\|_{n+(2)_v}\tau, \qquad \epsilon_v(f\|_n\tau) = (\epsilon_v f)\|_{n-(2)_v}\tau, \tag{1.7}$$

$$L_v^n(f\|_n\tau) = (L_v^n f)\|_n\tau. \tag{1.8}$$

Now according to our definitions, we should have $\tau \in G \cap P_{\mathbf{A}}C''$ if the weight is half-integral. We note, however, that these relations continue to hold for any $\tau \in G_{\mathbf{a}}$, in the sense that we can arbitrarily (but noncanonically) pick some holomorphic function $h(\tau, z)$ satisfying (1.2), and then use the same choice of h in interpreting $\|_n$ and $\|_{n\pm(2)_v}$ above.

Let n be a weight, and let $\lambda \in \mathbf{C}^{\mathbf{a}}$. Let $\Gamma \subset G$ be a congruence subgroup (contained in $\Gamma_{1/2}$ if n is half-integral).

Definition. The space of automorphic eigenforms of weight n and eigenvalue λ with respect to Γ, written $\mathcal{M}_{n,\lambda}(\Gamma)$, is the set of C^∞ functions $f : \mathcal{H}^{\mathbf{a}} \to \mathbf{C}$ satisfying conditions (2.7a-c) of [Sh 85b]:

$$\forall \tau \in \Gamma,\ f\|_n\tau = f, \qquad \forall v \in \mathbf{a},\ L_v f = \lambda_v f, \tag{1.9}$$

and the third requirement that f grow slowly at all cusps. This means that given $\tau \in G$, there exist positive numbers A, B, and C, depending on f and τ, such that

$$y^{n/2}|(f\|_n\tau)(x + iy)| \leq Ay^{Cu}, \quad \text{if } y^u \geq B. \tag{1.10}$$

If n is half-integral, we may need to interpret $f\|_n\tau$ in (1.10) in terms of a noncanonically chosen $h(\tau, z)$, as in the remark following (1.8). We call $f\|_n\tau$ the *expansion of f at the cusp corresponding to τ*. We define $\mathcal{M}_{n,\lambda}$ to be the union of the $\mathcal{M}_{n,\lambda}(\Gamma)$ over all congruence groups Γ. If the Fourier expansion of f at all cusps has no constant term (see Proposition 1.1 below), then f is called a *cusp form*. The set of all cusp forms in $\mathcal{M}_{n,\lambda}$ (respectively in $\mathcal{M}_{n,\lambda}(\Gamma)$) is denoted $\mathcal{S}_{n,\lambda}$ (respectively $\mathcal{S}_{n,\lambda}(\Gamma)$). Cusp forms decay rapidly at the cusps (Proposition 2.1 of [Sh 85b]).

Proposition 1.1. *Any $f \in \mathcal{M}_{n,\lambda}$ has the Fourier expansion*

$$f(x+iy) = c_0(y;f) + \sum_{\xi \in F^\times} c(\xi;f) W_{\beta,\gamma}(\xi y)\mathbf{e_a}(\xi x), \tag{1.11}$$

for $z = x+iy \in \mathcal{H}^{\mathbf{a}}$, with complex numbers $c(\xi;f)$ for $\xi \in F^\times$, and a constant term *$c_0(y;f)$. $c(\xi;f)$ is nonzero only for ξ in a fractional ideal depending on f, but ξ is not constrained to be totally positive, as in the case of holomorphic forms.*

This is well-known; see for instance section 2 of [Sh 85b]. Here $\beta, \gamma \in \mathbf{C}^{\mathbf{a}}$ are parameters chosen so that

$$\forall v \in \mathbf{a}, \quad \beta_v \gamma_v = \lambda_v, \quad \beta_v + \gamma_v = 1 - n_v. \tag{1.12}$$

$W_{\beta,\gamma}$ is a certain Whittaker function, defined as a product of local Whittaker functions at the archimedean places: $W_{\beta,\gamma}(y) = \prod_{v\in\mathbf{a}} W_{\beta_v,\gamma_v}(y_v)$. To define these local Whittaker functions, let $\beta, \gamma \in \mathbf{C}$ and $y > 0$. Define the auxiliary function

$$V(y;\beta,\gamma) = e^{-y/2} y^{\beta}\Gamma(\beta)^{-1} \int_0^\infty e^{-yt}(1+t)^{-\gamma} t^{\beta-1}\, dt. \tag{1.13}$$

(The integral converges only if β has a positive real part, but it can then be analytically continued to all β.) Then, for $0 \neq y \in \mathbf{R}$, the local Whittaker function is

$$W_{\beta,\gamma}(y) = \begin{cases} V(4\pi y;\beta,\gamma), & \text{if } y > 0, \\ (-4\pi y)^{\beta+\gamma-1} V(-4\pi y; 1-\beta, 1-\gamma), & \text{if } y < 0. \end{cases} \tag{1.14}$$

These are the same definitions as in [Sh 85b], equations (2.16-2.19), except that the parameter α there is replaced by the parameter $1-\gamma$ here. With this notation, W and V become symmetric in β and γ. (This is not immediate from (1.13); see section 10 of [Sh 85b].) We remark that the constant term $c_0(y;f)$ is a linear combination of terms of the form y^p, where $p_v = \beta_v$ or γ_v at each archimedean place v; if $\beta = \gamma$, then a term of the form $y^\beta \log y^u$ may also occur. (Refer to section 3 of [Sh 85b] for more details.)

By Proposition 2.2 of [Sh 85b], the operators δ_v^n and ϵ_v send automorphic eigenforms to forms with a related weight and eigenvalue. Namely,

$$\epsilon_v \mathcal{M}_{n,\lambda}(\Gamma) \subset \mathcal{M}_{n-(2)_v, \lambda-(n_v-2)_v}(\Gamma), \tag{1.15}$$

$$\delta_v^n \mathcal{M}_{n,\lambda} \subset \mathcal{M}_{n+(2)_v, \lambda+(n_v)_v}(\Gamma). \tag{1.16}$$

The same statements hold for cusp forms. If f has parameters β and γ for its Whittaker functions, then $\epsilon_v f$ has parameters $\beta + (1)_v$ and $\gamma + (1)_v$, and $\delta_v^n f$ has parameters $\beta - (1)_v$ and $\gamma - (1)_v$. We quote equations (10.13a-b) of [Sh 85b]. For $\xi \in F^\times$ and $x + iy \in \mathcal{H}^{\mathbf{a}}$,

$$\begin{aligned} &\epsilon_v(\mathbf{e_a}(\xi x) W_{\beta,\gamma}(\xi y)) \\ &\quad = \begin{cases} \beta_v\gamma_v(8\pi i \xi_v)^{-1} \mathbf{e_a}(\xi x) W_{\beta+(1)_v,\gamma+(1)_v}(\xi y), & \text{if } \xi_v > 0, \\ (8\pi i \xi_v)^{-1} \mathbf{e_a}(\xi x) W_{\beta+(1)_v,\gamma+(1)_v}(\xi y), & \text{if } \xi_v < 0; \end{cases} \end{aligned} \tag{1.17}$$

$$(1.18)\ \delta_v^n(\mathbf{e_a}(\xi x)W_{\beta,\gamma}(\xi y)) = \begin{cases} (2\pi i\xi_v)\mathbf{e_a}(\xi x)W_{\beta-(1)_v,\gamma-(1)_v}(\xi y), & \text{if } \xi_v > 0, \\ (\beta_v - 1)(\gamma_v - 1)(2\pi i\xi_v)\mathbf{e_a}(\xi x)W_{\beta-(1)_v,\gamma-(1)_v}(\xi y), & \text{if } \xi_v < 0. \end{cases}$$

This gives the effect of ϵ_v on the Fourier coefficients of $f \in \mathcal{M}_{n,\lambda}$ (similarly for the effect of δ_v^n):

$$c(\xi; \epsilon_v f) = \begin{cases} \beta_v\gamma_v(8\pi i\xi_v)^{-1}c(\xi; f), & \text{if } \xi_v > 0, \\ (8\pi i\xi_v)^{-1}c(\xi; f), & \text{if } \xi_v < 0. \end{cases} \tag{1.19}$$

Given two C^∞ functions f and g on $\mathcal{H}^{\mathbf{a}}$ which are invariant under the weight n action of a congruence subgroup Γ, we define their inner product

$$\langle f, g\rangle = \mathrm{vol}(\Gamma\backslash\mathcal{H}^{\mathbf{a}})^{-1} \int_{\Gamma\backslash\mathcal{H}^{\mathbf{a}}} \overline{f(z)}g(z)y^n\, d_Hz, \tag{1.20}$$

whenever the integral converges. Here the volume is computed with respect to the invariant measure $d_Hz = y^{-2u}\prod_{v\in\mathbf{a}} dx_v dy_v$. $\langle f, g\rangle$ is independent of the choice of Γ, and converges, for example, if f grows slowly at the cusps and g is a cusp form. For any $\tau \in \tilde{G}_+$,

$$\langle f, g\rangle = \langle f\|_n\tau, g\|_n\tau\rangle. \tag{1.21}$$

(If n is half-integral, we may have to pick a noncanonical $h(\tau, z)$, as before.) If $v \in \mathbf{a}$, then the operators δ_v^n and ϵ_v are adjoints with respect to this inner product: if f transforms by weight $n + (2)_v$, and g transforms by weight n, then, as shown in Proposition 2.4 of [Sh 85b],

$$\langle \epsilon_v f, g\rangle = \langle f, \delta_v^n g\rangle. \tag{1.22}$$

In particular, if $f \in \mathcal{S}_{n,\lambda}$ and $\lambda_v = 0$, then f is holomorphic in z_v, since $\langle \epsilon_v f, \epsilon_v f\rangle = (1/4)\langle L_v^n f, f\rangle = 0$; hence $\epsilon_v f = 0$. We see that L_v^n is positive and self-adjoint with respect to this inner product, on the space of cusp forms; its eigenvalues are therefore real and positive, and its different eigenspaces are mutually orthogonal.

In addition to the δ_v^n and ϵ_v operators, which shift the weights of automorphic eigenforms, there are certain "flipping" operators K_v^n, which change the signs of the weights. As these operators are most useful on integral weight forms, we will assume in the rest of this section that n is an integral weight. These operators were originally introduced by Maass in [Ma 53], in the case $F = \mathbf{Q}$. Our treatment in the case of Hilbert modular forms follows a suggestion of Shimura.

Definition. For any $v \in \mathbf{a}$ and C^∞ function $f : \mathcal{H}^{\mathbf{a}} \to \mathbf{C}$, define $K_v^n f$ by

$$(K_v^n f)(z) = y_v^{n_v} f(z'), \tag{1.23}$$

where $z'_w = z_w$ for $w \neq v$, and $z'_v = -\overline{z_v}$. K_v^n commutes with δ_w^n and ϵ_w, for $w \neq v$, but it exchanges the roles of ϵ_v and δ_v^n:

$$K_v^{n-(2)_v}\epsilon_v = \delta_v^{(-1)_v n}K_v^n, \qquad \epsilon_v K_v^n = K_v^{n+(2)_v}\delta_v^n. \tag{1.24}$$

(These identities are equivalent, since $K_v^{(-1)_v n} K_v^n$ is the identity.) Here $(-1)_v n$ is just notation for the weight whose w-component at a place $w \neq v$ is n_w, and whose v-component is $-n_v$. (Alternatively, $(-1)_v$ is the injection of -1 into the multiplicative group $(\mathbf{R}^\times)^{\mathbf{a}}$ at the v-place.) If $L_v^n f = \lambda_v f$, then (1.6) and (1.24) imply that

$$L_v^{(-1)_v n} K_v^n f = (\lambda_v + n_v) K_v^n f. \tag{1.25}$$

As it turns out, K does not take automorphic eigenforms to eigenforms, because it fails to preserve invariance by a congruence subgroup. Indeed, for any $\tau \in \tilde{G}_{\mathbf{a}+}$,

$$(K_v^n f)\|_{(-1)_v n} \tau = K_v^n (f\|_n e_v \tau e_v), \tag{1.26}$$

where $e_v = \begin{pmatrix} (-1)_v & 0 \\ 0 & 1 \end{pmatrix} \in \tilde{G}_{\mathbf{a}}$. Thus if f transforms under a congruence group Γ, $K_v^n f$ will transform under $e_v \Gamma e_v$, which is not even a subgroup of G, unless $F = \mathbf{Q}$. To circumvent this difficulty, take any $\beta \in \tilde{G}$ such that $\operatorname{sgn}\det\beta = (-1)_v$, and define for $f \in \mathcal{M}_{n,\lambda}(\Gamma)$

$$K_\beta^n f = K_v^n (f\|_n \beta e_v) \in \mathcal{M}_{(-1)_v n, \lambda + (n_v)_v}(\beta^{-1}\Gamma\beta). \tag{1.27}$$

By (1.25) and (1.26), $K_\beta^n f$ satisfies (1.9); one also easily checks that $K_\beta^n f$ grows slowly at the cusps. Note that K_β^n respects cusp forms. Typically, we take $\beta = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}$, where $b \in F^\times$ has signature $(-1)_v$. Then we can compare the Fourier coefficients of f and $K_\beta^n f$:

$$c(\xi; K_\beta^n f) = |b|^{n/2} |4\pi\xi_v|^{-n_v} c(\xi b^{-1}; f). \tag{1.28}$$

This uses the following consequence of (1.14):

$$\forall y \in \mathbf{R}^{\mathbf{a}}, \quad W_{\beta,\gamma}((-1)_v y) = |4\pi y_v|^{-n_v} W_{\beta',\gamma'}(y). \tag{1.29}$$

Here β and γ are the parameters of f, as in (1.12), and β' and γ' are the parameters of $K_\beta^n f$. (One should be careful not to confuse the parameter $\beta \in \mathbf{C}^{\mathbf{a}}$ with the matrix $\beta = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}$.) In other words, for $w \neq v$, $\{\beta'_w, \gamma'_w\} = \{\beta_w, \gamma_w\}$, and $\{\beta'_v, \gamma'_v\} = \{1 - \beta_v, 1 - \gamma_v\} = \{\gamma_v + n_v, \beta_v + n_v\}$. (In the notation of (1.6) and (1.24), we could write $\beta' = (1)_v + (-1)_v\beta$ and $\gamma' = (1)_v + (-1)_v\gamma$, or alternatively $\beta' = \beta + (n_v)_v$ and $\gamma' = \gamma + (n_v)_v$, but this is cumbersome.)

Given a form f, we can thus apply an operator of the form K_β^n at all places v where $n_v < 0$, and thus obtain a form of positive weight, with essentially the same Fourier coefficients. After applying ϵ_v to f enough times at each $v \in \mathbf{a}$, we see that any cusp form of integral weight has essentially the same Fourier coefficients as a cusp form which, at each $v \in \mathbf{a}$, is either of weight 0, of weight 1, or holomorphic.

A similar statement holds for forms of half-integral weight, where a given cusp form is essentially the same as a form which is either of weight $1/2$, of weight $3/2$, or holomorphic, at any $v \in \mathbf{a}$. If $n_v = 3/2$ but f is not holomorphic in z_v, we may be tempted to apply ϵ_v to f, thereby replacing n_v with $-1/2$; we would then apply

K^n_β, to get a form of weight $1/2$ at the v-place. This would be slightly misleading, because it would limit the Fourier coefficients that we could study. Essentially, the point of view taken in this paper only allows us to study directly the Fourier coefficients $c(\xi; f)$, where f is of half-integral weight, and ξ is totally positive. Thus, once we have brought f to the form where it is either holomorphic or of weight $1/2$ at each place v, let us say that we want to study the Fourier coefficient $c(\xi; f)$. At the places $v \in \mathbf{a}$ where f is holomorphic, $\xi_v > 0$ (otherwise (1.19) would imply that $c(\xi; f) = 0$). There may be some places, however, where the weight is $1/2$ and $\xi_v < 0$. At these places, we have to apply ϵ_v and a flipping operator K^n_β, in order to express $c(\xi; f)$ as a Fourier coefficient indexed by a totally positive ξ. Indeed, (1.28) tells us that flipping at the place v replaces the ξth Fourier coefficient with the $b\xi$th, where $\operatorname{sgn} b = (-1)_v$. This will bring back weight $3/2$ at these places.

2. Integral Weight Forms on $\tilde{G}_{\mathbf{A}}$.

We now consider forms $\mathbf{f}$ which are functions on $\tilde{G}_{\mathbf{A}}$, and relate them to the integral weight forms of the previous section. The discussion here follows section 6 of [Sh 87] and section 3 of [Sh 93a], with some references to Shimura's earlier paper [Sh 78]. Fix an integral weight $n \in \mathbf{Z}^{\mathbf{a}}$, and let $\lambda \in \mathbf{C}^{\mathbf{a}}$ be a set of eigenvalues. Let $\mathfrak{y}$ and $\mathfrak{z}$ be integral ideals, and let Ψ be a Hecke character with conductor dividing $\mathfrak{z}$, such that

$$\forall x \in F^\times_{\mathbf{a}},\ \Psi_{\mathbf{a}}(x) = \operatorname{sgn}(x)^n |x|^{2i\mu} \tag{2.1}$$

for some $\mu \in \mathbf{R}^{\mathbf{a}}$ such that $\|\mu\| = 0$. (For our purposes, the ideal $\mathfrak{y}$ will usually just be $\mathfrak{o}$, but it will be more convenient in section 5 to allow any integral $\mathfrak{y}$.)

Definition. $\mathcal{S}_{n,\lambda}(\mathfrak{y}, \mathfrak{z}, \Psi)$ (respectively $\mathcal{M}_{n,\lambda}(\mathfrak{y}, \mathfrak{z}, \Psi)$) is the set of functions $\mathbf{f} : \tilde{G}_{\mathbf{A}} \to \mathbf{C}$ such that for all $p \in \tilde{G}_{\mathbf{A}}$,

$$\forall s \in F^\times_{\mathbf{A}},\ \mathbf{f}(sp) = \Psi(s)\mathbf{f}(p); \tag{2.2}$$

$$\forall \alpha \in \tilde{G} \text{ and } w \in \tilde{G}_{\mathbf{f}} \cap \tilde{D}[\mathfrak{y}\mathfrak{d}^{-1}, \mathfrak{z}\mathfrak{d}\,],\ \mathbf{f}(\alpha p w) = \Psi_{\mathfrak{z}}(d_w)\mathbf{f}(p); \tag{2.3}$$

and for each $q \in \tilde{G}_{\mathbf{f}}$, there exists a form $f_q \in \mathcal{S}_{n,\lambda}$ (respectively $\mathcal{M}_{n,\lambda}$) such that

$$\forall p \in \tilde{G}_{\mathbf{a}+},\ \mathbf{f}((\det q)^{-1} q p) = (\det p)^{i\mu} (f_q \|_n p)(\mathbf{i}). \tag{2.4}$$

We will occasionally relax requirement (2.3) to require merely that $\mathbf{f}(\alpha p w) = \mathbf{f}(p)$ for $\alpha \in \tilde{G}$ and w in a sufficiently small open subgroup of $\tilde{G}_{\mathbf{f}}$.

Now $\tilde{G}_{\mathbf{A}}$ splits into a disjoint union of double cosets indexed by the ideal classes λ mod $\mathbf{a}$:

$$\tilde{G}_{\mathbf{A}} = \bigsqcup_{\lambda} \tilde{G} t_\lambda^{-1} x_\lambda \tilde{D}[\mathfrak{y}\mathfrak{d}^{-1}, \mathfrak{z}\mathfrak{d}\,], \tag{2.5}$$

where $x_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & t_\lambda \end{pmatrix}$ and the $t_\lambda \in F^\times_{\mathbf{f}}$ are a set of representatives for the idele classes $F^\times_{\mathbf{A}} / F^\times F^\times_{\mathbf{a}+} \prod_{v \in \mathbf{f}} \mathfrak{o}_v^\times$. (Thus the ideals $\mathfrak{t}_\lambda = t_\lambda \mathfrak{o}$ are a set of representatives for the ideal classes of F mod $\mathbf{a}$.) This means that $\mathbf{f}$ is completely determined by the collection of forms f_λ on $\mathcal{H}^{\mathbf{a}}$ which are given by

$$\forall p \in \tilde{G}_{\mathbf{a}+}, \quad \mathbf{f}(t_\lambda^{-1} x_\lambda p) = (\det p)^{i\mu} (f_\lambda \|_n p)(\mathbf{i}). \tag{2.6}$$

(Strictly speaking, to adhere to the notation of (2.4), f_λ ought to be called f_{x_λ}.) We will write $\mathbf{f} = (f_\lambda)_\lambda$. The transformation properties of $\mathbf{f}$ imply

$$\forall \gamma \in \tilde{\Gamma}[\mathfrak{y}\mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1}, \mathfrak{z}\mathfrak{d}\mathfrak{t}_\lambda],\ f_\lambda \|_n \gamma = \Psi_{\mathfrak{z}}(a_\gamma)(\det \gamma)^{i\mu} f_\lambda. \tag{2.7}$$

Conversely, given $\{f_\lambda\}_\lambda$ satisfying (2.7), they fit together to give a function $\mathbf{f}$ on $\tilde{G}_{\mathbf{A}}$ satisfying (2.3) and (2.4). (2.2) then corresponds to the following condition: for any $s \in F_{\mathbf{A}}^\times$ and any ideal class λ, write $st_\lambda^{-1} x_\lambda = \alpha t_\mu^{-1} x_\mu w$, for some $\alpha \in \tilde{G}_+$, $w \in \tilde{D}[\mathfrak{y}\mathfrak{d}^{-1}, \mathfrak{z}\mathfrak{d}]$, and ideal class μ. We then require that

$$\Psi(s) f_\lambda = \Psi_{\mathfrak{z}}(d_w)(\det w_{\mathbf{a}})^{-i\mu} f_\mu \|_n w_{\mathbf{a}}. \tag{2.8}$$

(Note that the action of $w_{\mathbf{a}}$ on forms is the same as that of $\alpha^{-1} \in \tilde{G}_+$.)

Definition. Given an ideal $\mathfrak{m}$ and a signature $\sigma \in \{\pm 1\}^{\mathbf{a}}$, define the $(\mathfrak{m}, \sigma)$th Fourier coefficient of $\mathbf{f} \in \mathcal{M}_{n,\lambda}(\mathfrak{y}, \mathfrak{z}, \Psi)$ by

$$\mathrm{c}(\mathfrak{m}, \sigma; \mathbf{f}) = |\xi|^{-n/2 - i\mu} c(\xi; f_\lambda), \quad \text{with } \xi \in F^\times,\ \operatorname{sgn} \xi = \sigma,\ \xi \mathfrak{t}_\lambda^{-1} = \mathfrak{m}. \tag{2.9}$$

(The requirement on ξ and λ determines ξ only up to a totally positive unit $u \in \mathfrak{o}_+^\times$, but the automorphy of f_λ by $\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$ implies that $\mathrm{c}(\mathfrak{m}, \sigma; \mathbf{f})$ is well-defined.) Note that $c(\xi; f_\lambda)$ is nonzero only if $\xi \in \mathfrak{y}^{-1}\mathfrak{t}_\lambda$; hence $\mathrm{c}(\mathfrak{m}, \sigma; \mathbf{f})$ is nonzero only if $\mathfrak{m} \subset \mathfrak{y}^{-1}$. Similarly, we define constant terms for $\mathbf{f}$ by

$$\mathrm{c}_0(\mathfrak{m}, t; \mathbf{f}) = (\xi^{-1} t)^{n/2 + i\mu} c_0(\xi^{-1} t; f_\lambda), \tag{2.10}$$

for $t \in \mathbf{R}^{\mathbf{a}}$ and an ideal $\mathfrak{m}$. Here ξ and λ are chosen so that $\operatorname{sgn} \xi = \operatorname{sgn} t$ and $\xi \mathfrak{t}_\lambda^{-1} = \mathfrak{m}$. This definition of the Fourier coefficients is independent of the choice of $\mathfrak{t}_\lambda$, since we can obtain the Fourier coefficients directly from the expansion

$$\begin{aligned} &\mathbf{f}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) \\ &\quad = \mathrm{c}_0(y\mathfrak{o}, y_{\mathbf{a}}; \mathbf{f}) + \sum_{\xi \in F^\times} |\xi y_{\mathbf{a}}|^{n/2 + i\mu}\, \mathrm{c}(\xi y \mathfrak{o}, \operatorname{sgn} \xi y_{\mathbf{a}}; \mathbf{f}) W_{\beta,\gamma}(\xi y_{\mathbf{a}}) \mathbf{e}_{\mathbf{A}}(\xi x). \end{aligned} \tag{2.11}$$

Here $x \in F_{\mathbf{A}}$ and $y \in F_{\mathbf{A}}^\times$; the archimedean component $y_{\mathbf{a}}$ of y may have any signature. β and γ are the usual parameters for the Whittaker functions.

We will later have use for the following

Lemma 2.1. *If* $\mathbf{f} \in \mathcal{M}_{n,\lambda}(\mathfrak{y}', \mathfrak{z}, \Psi)$*, and if* $\mathrm{c}(\mathfrak{m}, \sigma; \mathbf{f}) = 0$ *unless* $\mathfrak{m} \subset \mathfrak{y}^{-1}$ *for some* $\mathfrak{y} \supset \mathfrak{y}'$*, then* $\mathbf{f} \in \mathcal{M}_{n,\lambda}(\mathfrak{y}, \mathfrak{z}, \Psi)$.

Proof: The assumption above implies that each f_λ is invariant under $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, for $b \in \mathfrak{y}\mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1}$. By Lemma 3.4 of [Sh 87], the set of such matrices and the elements of the group $\tilde{\Gamma}[\mathfrak{y}'\mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1}, \mathfrak{z}\mathfrak{d}\mathfrak{t}_\lambda]$ generate $\tilde{\Gamma}[\mathfrak{y}\mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1}, \mathfrak{z}\mathfrak{d}\mathfrak{t}_\lambda]$. This implies the automorphy of $\mathbf{f}$ with respect to the larger group. ∎

We extend the differential operators δ_v and ϵ_v to act on forms on $\tilde{G}_{\mathbf{A}}$. For $v \in \mathbf{a}$ and $\mathbf{f} = (f_\lambda)_\lambda \in \mathcal{M}_{n,\lambda}(\mathfrak{y}, \mathfrak{z}, \Psi)$, define

$$\epsilon_v \mathbf{f} = (\epsilon_v f_\lambda)_\lambda \in \mathcal{M}_{n-(2)_v, \lambda-(n_v-2)_v}(\mathfrak{y}, \mathfrak{z}, \Psi). \tag{2.12}$$

The $\epsilon_v f_\lambda$ inherit relations (2.7) and (2.8), so they do indeed fit together to yield a form $\epsilon_v \mathbf{f}$ as claimed. If $\mathbf{f}$ is in fact a cusp form, then so is $\epsilon_v \mathbf{f}$. We can similarly define $\delta_v \mathbf{f} \in \mathcal{M}_{n+(2)_v, \lambda+(n_v)_v}(\mathfrak{y}, \mathfrak{z}, \Psi)$ to be $(\delta_v^n f_\lambda)_\lambda$, where we omit the superscript n from δ_v because the weight n is known from the transformation of $\mathbf{f}$ under $SO(2)^{\mathbf{a}}$, as seen from (2.4).

Equations (1.19) and (2.9) yield

$$\mathrm{c}(\mathfrak{m}, \sigma; \epsilon_v \mathbf{f}) = \begin{cases} (8\pi i)^{-1} \beta_v \gamma_v \, \mathrm{c}(\mathfrak{m}, \sigma; \mathbf{f}), & \text{if } \sigma_v > 0, \\ -(8\pi i)^{-1} \, \mathrm{c}(\mathfrak{m}, \sigma; \mathbf{f}), & \text{if } \sigma_v < 0. \end{cases} \tag{2.13}$$

The corresponding formulas for the Fourier coefficients of $\delta_v \mathbf{f}$ are easy to write down, since $4\delta_v \epsilon_v \mathbf{f} = \lambda_v \mathbf{f} = \beta_v \gamma_v \mathbf{f}$. This shows that the definition of ϵ_v and δ_v is independent of the choice of t_λ. It follows by induction that for any $r \geq 1$,

$$\mathrm{c}(\mathfrak{m}, \sigma; (\epsilon_v)^r \mathbf{f}) = \begin{cases} (-1)^r (\beta_v)_r (\gamma_v)_r (-8\pi i)^{-r} \, \mathrm{c}(\mathfrak{m}, \sigma; \mathbf{f}), & \text{if } \sigma_v > 0, \\ (-8\pi i)^{-r} \, \mathrm{c}(\mathfrak{m}, \sigma; \mathbf{f}), & \text{if } \sigma_v < 0. \end{cases} \tag{2.14}$$

(We have used the Pochhammer symbol of (0.12).)

Given two forms $\mathbf{f} = (f_\lambda)_\lambda$ and $\mathbf{g} = (g_\lambda)_\lambda$ of the same weight, with one of them a cusp form, we define their inner product to be

$$\langle \mathbf{f}, \mathbf{g} \rangle = \sum_\lambda \langle f_\lambda, g_\lambda \rangle, \tag{2.15}$$

This is independent of the choice of representatives t_λ, and the operators δ_v and ϵ_v are, as before, adjoints with respect to this inner product. Moreover, for any $w \in \tilde{G}_{\mathbf{f}}$,

$$\langle \mathbf{f}(pw), \mathbf{g}(pw) \rangle = \langle \mathbf{f}(p), \mathbf{g}(p) \rangle. \tag{2.16}$$

Indeed, the above equality and the independence of $\langle \mathbf{f}, \mathbf{g} \rangle$ of the choice of t_λ both follow from (1.21) and from the following observation: if $q \in \tilde{G}_{\mathbf{f}}$, and $(\det q)^{-1} q \in \tilde{G} t_\lambda^{-1} x_\lambda \tilde{D}[\mathfrak{y}\mathfrak{d}^{-1}, \mathfrak{z}\mathfrak{d}]$, then there exist $\alpha \in \tilde{G}$ and $\zeta \in \mathbf{T}$ such that for any $\mathbf{f}$, $f_q = \zeta f_\lambda \|_n \alpha$. (Here f_q is as in (2.4).)

We shall assume in the rest of this section that $\mathfrak{y} = \mathfrak{o}$. We define Hecke operators as in section 2 of [Sh 78]. This yields, for any integral ideal $\mathfrak{n}$, an operator $\mathfrak{T}(\mathfrak{n})$, which maps $\mathcal{M}_{n,\lambda}(\mathfrak{o}, \mathfrak{z}, \Psi)$ to itself and preserves the subspace of cusp forms. These Hecke operators commute with each other and with the operators ϵ_v and δ_v, for any $v \in \mathbf{a}$. Their action on the Fourier coefficients of forms is given by

$$\mathrm{c}(\mathfrak{m}, \sigma; \mathfrak{T}(\mathfrak{n})\mathbf{f}) = \sum_{\mathfrak{a} \supset \mathfrak{m}+\mathfrak{n}, \mathfrak{a}+\mathfrak{z}=\mathfrak{o}} \Psi^*(\mathfrak{a}) \mathrm{N}(\mathfrak{a}^{-1}\mathfrak{n}) \, \mathrm{c}(\mathfrak{a}^{-2}\mathfrak{m}\mathfrak{n}, \sigma; \mathbf{f}), \tag{2.17}$$

for all ideals $\mathfrak{m}$ and signatures σ. If $\mathbf{f}$ is an eigenform of all the $\mathfrak{T}(\mathfrak{n})$, with $\mathfrak{T}(\mathfrak{n})\mathbf{f} = \chi(\mathfrak{n})\mathbf{f}$, we call χ the *system of eigenvalues* for $\mathbf{f}$. The Fourier coefficients of such an

$\mathbf{f}$ then satisfy $\mathrm{c}(\mathfrak{m},\sigma;\mathbf{f}) = \mathrm{N}(\mathfrak{m})^{-1}\chi(\mathfrak{m})\,\mathrm{c}(\mathfrak{o},\sigma;\mathbf{f})$. If $\mathrm{c}(\mathfrak{o},u;\mathbf{f}) = 1$, then we call our eigenform *normalized*. (Recall that $u = (1,1\ldots,1) \in \mathbf{Z}^{\mathbf{a}}$.) As usual, we have the identity of formal Dirichlet series

$$(2.18)\qquad \sum_{\mathfrak{m}} \chi(\mathfrak{m})\mathrm{M}(\mathfrak{m}) \\ = \prod_{\mathfrak{p}|\mathfrak{z}}(1-\chi(\mathfrak{p})\mathrm{M}(\mathfrak{p}))^{-1}\prod_{\mathfrak{p}\nmid\mathfrak{z}}(1-\chi(\mathfrak{p})\mathrm{M}(\mathfrak{p})+\Psi^*(\mathfrak{p})\mathrm{N}(\mathfrak{p})\mathrm{M}(\mathfrak{p}^2))^{-1},$$

where the $\mathrm{M}(\mathfrak{m})$ are a formal system of multiplicative symbols in the ideals of F; that is to say, $\mathrm{M}(\mathfrak{o}) = 1$, $\mathrm{M}(\mathfrak{mn}) = \mathrm{M}(\mathfrak{m})\mathrm{M}(\mathfrak{n})$ for any two ideals $\mathfrak{m}$ and $\mathfrak{n}$, and the $\mathrm{M}(\mathfrak{p})$ are independent indeterminates for different prime ideals $\mathfrak{p}$.

We conclude this section by defining the *inverter* $J_{\mathfrak{z}}$, which sends $\mathcal{S}_{n,\lambda}(\mathfrak{o},\mathfrak{z},\Psi)$ to $\mathcal{S}_{n,\lambda}(\mathfrak{o},\mathfrak{z},\Psi^{-1})$. Namely, define the matrix $\pi = \begin{pmatrix} 0 & -1 \\ \delta^2 s & 0 \end{pmatrix} \in \tilde{G}_{\mathbf{f}}$, with an idele $s \in F_{\mathbf{f}}^{\times}$ such that $s\mathfrak{o} = \mathfrak{z}$. Then define

$$(2.19)\qquad (J_{\mathfrak{z}}\mathbf{f})(p) = \Psi(\det p)^{-1}\mathbf{f}(p\pi), \quad \forall p \in \tilde{G}_{\mathbf{A}}.$$

As observed in [Sh 93a], equations (3.16) and (3.17), this definition is independent of the choices of δ and of s. Moreover, the inverter interacts nicely with the Hecke operators: if $\mathfrak{n}$ is prime to $\mathfrak{z}$, then

$$(2.20)\qquad J_{\mathfrak{z}}\mathfrak{T}(\mathfrak{n})\mathbf{f} = \Psi^*(\mathfrak{n})\mathfrak{T}(\mathfrak{n})J_{\mathfrak{z}}\mathbf{f}.$$

3. Dirichlet Series from Integral Weight Forms.

Let $\mathbf{f} = (f_\lambda)_\lambda \in \mathcal{M}_{n,\lambda}(\mathfrak{y},\mathfrak{z},\Psi)$ be a form on $\tilde{G}_{\mathbf{A}}$, as in the previous section. In this section, we will use the "flipping operators" of section 1 to impose conditions on $\mathbf{f}$, which will cause all of the Fourier coefficients of $\mathbf{f}$ to be expressible in terms of the Fourier coefficients $\mathrm{c}(\mathfrak{m},u;\mathbf{f})$ with a totally positive signature. For example, if $\mathbf{f}$ is holomorphic, then $\mathrm{c}(\mathfrak{m},\sigma;\mathbf{f}) = 0$ unless $\sigma = u$. We will then express the Dirichlet series

$$(3.1)\qquad D(s,\mathbf{f},\chi) = \sum_{\mathfrak{m}} \mathrm{c}(\mathfrak{m},u;\mathbf{f})\chi^*(\mathfrak{m})\mathrm{N}(\mathfrak{m})^{-s}$$

in terms of the Mellin transforms of the various f_λ, with a completely explicit expression for the Gamma-factors at the archimedean places. Here χ is a certain unramified Hecke character, which we use in order to be able to be able to recover the $\mathrm{c}(\mathfrak{m},u;\mathbf{f})$ from a knowledge of $D(s,\mathbf{f},\chi)$ for all such χ.

Although the material in this section is in principle known, it does not seem to be explicitly stated in the literature, except to some extent in [Ma 53], for $F = \mathbf{Q}$. Maass does not quite calculate the Gamma-factor $Y(s,\beta,\gamma,\sigma)$ of (3.15) and (3.16) in the general case, but instead reduces first to the case of forms that are either of weight 0, of weight 1, or holomorphic (as explained at the end of section 1, this is not a loss of generality). Jacquet and Langlands [J-L] and Weil [W] deal with this topic over arbitrary number fields, but avoid explicitly calculating $Y(s,\beta,\gamma,\sigma)$.

We first generalize the "flipping operator" K_v to $\tilde{G}_{\mathbf{A}}$. For $v \in \mathbf{a}$ and $\mathbf{f} \in \mathcal{M}_{n,\lambda}(\mathfrak{y}, \mathfrak{z}, \Psi)$, define $K_v\mathbf{f} \in \mathcal{M}_{(-1)_v n, \lambda+(n_v)_v}(\mathfrak{y}, \mathfrak{z}, \Psi)$ by

$$(K_v\mathbf{f})(p) = \mathbf{f}(pe_v), \quad p \in \tilde{G}_{\mathbf{A}}. \tag{3.2}$$

(Recall that $e_v = \begin{pmatrix} (-1)_v & 0 \\ 0 & 1 \end{pmatrix} \in \tilde{G}_{\mathbf{a}}$ or $\tilde{G}_{\mathbf{A}}$, according to context, as was defined in section 1.) It is immediate that $K_v\mathbf{f}$ satisfies (2.2) and (2.3). As for (2.4), for each ideal class λ define an ideal class $\mu = \mu(\lambda)$, such that $b\mathfrak{t}_\lambda^{-1} = \mathfrak{t}_\mu^{-1}$ with $b \in F^\times$ such that $\operatorname{sgn} b = (-1)_v$. Let K_β^n be the flipping operator of (1.27), with respect to $\beta = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$K_v\mathbf{f} = (g_\lambda)_\lambda, \qquad g_\lambda = |b|^{i\mu} K_\beta^n f_\mu \in \mathcal{M}_{(-1)_v n, \lambda+(n_v)_v}. \tag{3.3}$$

(The exponent μ is determined by $\Psi_{\mathbf{a}}$, as in (2.1).) As in the case of forms on $\mathcal{H}^{\mathbf{a}}$, the operator K_v commutes with δ_w, ϵ_w and K_w, for $w \neq v$, and $K_v\delta_v = \epsilon_v K_v$. K_v also respects cusp forms, and treats Fourier coefficients well:

$$\mathrm{c}(\mathfrak{m}, \sigma; K_v\mathbf{f}) = (4\pi)^{-n_v}\, \mathrm{c}(\mathfrak{m}, (-1)_v\sigma; \mathbf{f}). \tag{3.4}$$

This follows either from (1.29) and (2.11), or from (3.3) and (1.28). From now on, we will assume, without loss of generality, that $n \geq 0$. (Otherwise, apply K_v to $\mathbf{f}$ at each place v where $n_v < 0$.) We will also assume that $\mathbf{f}$ is a cusp form, even though this is not always necessary; it will sometimes be enough to assume that the constant terms $\mathrm{c}_0(\mathfrak{m}, t; \mathbf{f})$ are zero.

Definition. We define another flipping operator Z_v, acting on $\mathbf{f} \in \mathcal{S}_{n,\lambda}(\mathfrak{y}, \mathfrak{z}, \Psi)$, by

$$Z_v\mathbf{f} = K_v(\epsilon_v)^{n_v}\mathbf{f} \in \mathcal{S}_{n,\lambda}(\mathfrak{y}, \mathfrak{z}, \Psi). \tag{3.5}$$

As before, the weight n is determined by $\mathbf{f}$, so there is no need to include it as a superscript on Z_v. We will be more interested in Z_v than in K_v. (To see why $Z_v\mathbf{f}$ has the same weight n and eigenvalue λ as $\mathbf{f}$, compare their β and γ parameters.)

Now $Z_vZ_v = K_v(\epsilon_v)^{n_v}K_v(\epsilon_v)^{n_v} = (\delta_v)^{n_v}(\epsilon_v)^{n_v}$, since $K_v\epsilon_v = \delta_v K_v$ and K_vK_v is the identity. Thus, by induction, or by comparing Fourier coefficients,

$$Z_vZ_v\mathbf{f} = 4^{-n_v}(\beta_v)_{n_v}(\gamma_v)_{n_v}\mathbf{f} = E_v^2\mathbf{f}, \tag{3.6}$$

where we use our choice of β_v and γ_v to pick a square root E_v of $4^{-n_v}(\beta_v)_{n_v}(\gamma_v)_{n_v}$:

$$E_v = (i/2)^{n_v}(\beta_v)_{n_v} = (-i/2)^{n_v}(\gamma_v)_{n_v}. \tag{3.7}$$

If we exchange β_v and γ_v, E_v gets multiplied by $(-1)^{n_v}$. Thus, if n_v is even, E_v is symmetric in β_v and γ_v, and is thus expressible in terms of n_v and λ_v. (Indeed for any r, $(\beta_v)_r(\gamma_v)_r = \prod_{k=0}^{r-1}(\lambda_v + (1-n_v)k + k^2)$; also, if n_v is even, then $E_v = 2^{-n_v}(\beta_v)_{n_v/2}(\gamma_v)_{n_v/2}$.)

Since the $\{Z_v\}_{v\in\mathbf{a}}$ and the Hecke operators all commute, it is sensible to break up $\mathcal{S}_{n,\lambda}(\mathfrak{y}, \mathfrak{z}, \Psi)$ into simultaneous eigenspaces for the Z_v, with eigenvalues $\pm E_v$.

This can be done, as Z_v is certainly semisimple if $E_v \neq 0$. On the other hand, if $E_v = 0$, then $(\delta_v)^{n_v}(\epsilon_v)^{n_v}\mathbf{f} = 0$, so $\langle(\epsilon_v)^{n_v}\mathbf{f}, (\epsilon_v)^{n_v}\mathbf{f}\rangle = 0$ for $\mathbf{f}$ is a cusp form. Hence $Z_v\mathbf{f} = K_v(\epsilon_v)^{n_v}\mathbf{f} = 0$, and $\mathbf{f}$ is already trivially an eigenform of Z_v. It is straightforward to show from (2.14) and (3.4) that $Z_v\mathbf{f} = \pm E_v\mathbf{f}$ if and only if for all ideals $\mathfrak{m}$ and all signatures σ with $\sigma_v > 0$,

$$\begin{aligned}
\text{(3.8)} \qquad \mathrm{c}(\mathfrak{m}, (-1)_v\sigma; \mathbf{f}) &= \pm(\beta_v)_{n_v}\, \mathrm{c}(\mathfrak{m}, \sigma; \mathbf{f}) \\
&= \pm(-1)^{\lfloor n_v/2\rfloor}(\beta_v)_{\lceil n_v/2\rceil}(\gamma_v)_{\lfloor n_v/2\rfloor}\, \mathrm{c}(\mathfrak{m}, \sigma; \mathbf{f}).
\end{aligned}$$

Thus if $\mathbf{f}$ is an eigenfunction of the Z_v, we can retrieve all the Fourier coefficients of $\mathbf{f}$ from a knowledge of only the $\mathrm{c}(\mathfrak{m}, u; \mathbf{f})$. Our definition of $D(s, \mathbf{f}, \chi)$ in (3.1), which makes reference only to the Fourier coefficients with signature u, gives a meaningful Dirichlet series only in this case.

In particular, if n_v is even for all v, then a form satisfying $Z_v\mathbf{f} = E_v\mathbf{f}$ for all v will be called *even*. We will be paying special attention to this case when we discuss the Shimura correspondence between forms of half-integral weight and "even" forms. Note that a holomorphic form of weight $n \geq 2u$ is automatically even, since then $Z_v\mathbf{f} = 0$. The following result is easy:

Lemma 3.1. $\mathbf{f}$ *is even if and only if for any* $v \in \mathbf{a}$, *and for any signature* σ *with* $\sigma_v > 0$, *one of the following three equivalent conditions holds,*

$$\text{(3.9)} \qquad \mathrm{c}(\mathfrak{m}, (-1)_v\sigma; \mathbf{f}) = (-1)^{n_v/2}(\beta_v)_{n_v/2}(\gamma_v)_{n_v/2}\, \mathrm{c}(\mathfrak{m}, \sigma; \mathbf{f});$$

$$\text{(3.10)} \qquad \mathrm{c}(\mathfrak{m}, (-1)_v\sigma; (\epsilon_v)^{n_v/2}\mathbf{f}) = \mathrm{c}(\mathfrak{m}, \sigma; (\epsilon_v)^{n_v/2}\mathbf{f});$$

$$\text{(3.11)} \qquad K_v(\epsilon_v)^{n_v/2}\mathbf{f} = (\epsilon_v)^{n_v/2}\mathbf{f}.$$

Moreover, if $\mathbf{f}$ *is even, then so is its "inversion"* $J_{\mathfrak{z}}\mathbf{f}$, *as defined in (2.19).*

Proof: The equivalence of the above three conditions with being "even" is immediate. The last statement is therefore true, since $J_{\mathfrak{z}}$ commutes with K_v and ϵ_v. ∎

We are now ready to construct Dirichlet series attached to automorphic forms which are simultaneous eigenforms of all the Z_v. We thus fix a parity $P \in \{0,1\}^{\mathbf{a}}$, and assume that

$$\text{(3.12)} \qquad \forall v \in \mathbf{a},\ Z_v\mathbf{f} = (-1)^{P_v}E_v\mathbf{f}.$$

Let μ be determined by $\Psi_{\mathbf{a}}$ as in (2.1), and let χ be an unramified Hecke character of F such that

$$\text{(3.13)} \qquad \forall x \in F_{\mathbf{a}}^{\times},\ \chi_{\mathbf{a}}(x) = |x|^{i\rho},$$

for some $\rho \in \mathbf{R}^{\mathbf{a}}$ such that $\|\rho\| = 0$. Let $K = \{y \in \mathbf{R}^{\mathbf{a}} \mid y \gg 0\}/\mathfrak{o}_+^{\times}$, and let the real part of $s \in \mathbf{C}$ be sufficiently large. We first evaluate the Mellin transform

$$\text{(3.14)} \qquad I_\lambda(s) = \int_K f_\lambda(iy)y^{n/2+su-u+i\rho+i\mu}\,dy, \quad dy = \prod_{v\in\mathbf{a}} dy_v.$$

Proposition 3.2. *For $\sigma \in \{\pm 1\}$ and for $s, \beta, \gamma \in \mathbf{C}$ such that $\beta + \gamma = 1 - n$ with $0 \le n \in \mathbf{Z}$, define $Y(s, \beta, \gamma, \sigma)$ by*

$$Y(s, \beta, \gamma, 1) = \int_0^\infty V(y; \beta, \gamma) y^{s+n/2-1}\, dy, \tag{3.15}$$

$$Y(s, \beta, \gamma, -1) = (-1)^{\lfloor n/2 \rfloor} (\beta)_{\lceil n/2 \rceil} (\gamma)_{\lfloor n/2 \rfloor} \int_0^\infty V(y; 1-\beta, 1-\gamma) y^{s-n/2-1}\, dy. \tag{3.16}$$

Then for $\mathbf{f}$ satisfying (3.12),

$$I_\lambda(s) = \sum_{\xi \in F^\times/\mathfrak{o}_+^\times} \mathrm{c}(\xi \mathfrak{t}_\lambda^{-1}, u; \mathbf{f}) \chi^*(\xi \mathfrak{o}) \mathrm{N}(\xi \mathfrak{o})^{-s} (4\pi)^{\|-n/2-su\|} \times \prod_{v \in \mathbf{a}} (\operatorname{sgn} \xi_v)^{P_v} Y(s + i\rho_v + i\mu_v, \beta_v, \gamma_v, \operatorname{sgn} \xi_v). \tag{3.17}$$

An explicit expression for Y is given in Proposition 3a.1 below.

Proof: (1.11) and (2.9) immediately yield

$$I_\lambda(s) = \sum_{\xi \in F^\times/\mathfrak{o}_+^\times} \mathrm{c}(\xi \mathfrak{t}_\lambda^{-1}, \operatorname{sgn} \xi; \mathbf{f}) |\xi|^{-su-i\rho} \times \int_{y \gg 0} W_{\beta,\gamma}(y \operatorname{sgn} \xi) y^{n/2+su-u+i\rho+i\mu}\, dy. \tag{3.18}$$

(Note that $|\xi|^{-su-i\rho}$ can be written as $\chi^*(\xi \mathfrak{o}) \mathrm{N}(\xi \mathfrak{o})^{-s}$.) By (3.8) and (3.12),

$$\mathrm{c}(\xi \mathfrak{t}_\lambda^{-1}, \operatorname{sgn} \xi; \mathbf{f}) = \mathrm{c}(\xi \mathfrak{t}_\lambda^{-1}, u; \mathbf{f}) \prod_{v \in \mathbf{a}, \xi_v < 0} (-1)^{P_v} (-1)^{\lfloor n_v/2 \rfloor} (\beta_v)_{\lceil n_v/2 \rceil} (\gamma_v)_{\lfloor n_v/2 \rfloor}. \tag{3.19}$$

The proposition follows upon writing $W_{\beta,\gamma}$ in terms of the V function. ∎

Now let us assemble the I_λ into the Dirichlet series $D(s, \mathbf{f}, \chi)$ of (3.1).

Proposition 3.3. *Under the assumptions and notation of Proposition 3.2,*

$$\begin{aligned} \sum_\lambda \chi^*(\mathfrak{t}_\lambda^{-1}) \mathrm{N}(\mathfrak{t}_\lambda^{-1})^{-s} I_\lambda(s) &= (4\pi)^{\|-n/2-su\|} D(s, \mathbf{f}, \chi) \\ &\quad \times \prod_{v \in \mathbf{a}} \Big(Y(s+i\rho_v+i\mu_v, \beta_v, \gamma_v, 1) + (-1)^{P_v} Y(s+i\rho_v+i\mu_v, \beta_v, \gamma_v, -1) \Big). \end{aligned} \tag{3.20}$$

Moreover, the last factor above is a nonzero meromorphic function of s, unless $n_v = 0$ and $P_v = 1$ for some $v \in \mathbf{a}$.

Proof: Apply Proposition 3.2, and observe that for any ideal $\mathfrak{m}$ and signature σ, there is exactly one $\mathfrak{t}_\lambda$ and $\xi \in F^\times/\mathfrak{o}_+^\times$ such that $\operatorname{sgn} \xi = \sigma$ and $\xi \mathfrak{t}_\lambda^{-1} = \mathfrak{m}$. It then follows from the explicit evaluation of Y in Proposition 3a.1 that $Y(s, \beta_v, \gamma_v, 1) + (-1)^{P_v} Y(s, \beta_v, \gamma_v, -1)$ is always a nonvanishing meromorphic function of s, unless $n_v = 0$ and $P_v = 1$. We remark that if this is the case for some "bad" places $v \in \mathbf{a}$, we can apply $(2\pi i)^{-1} \delta_v$ to $\mathbf{f}$ at all such places. This replaces $\mathbf{f}$ with another form $\mathbf{f}'$ with $\mathrm{c}(\mathfrak{m}, u; \mathbf{f}') = \mathrm{c}(\mathfrak{m}, u; \mathbf{f})$ and $Z_v \mathbf{f}' = (-1)^{P_v} \mathbf{f}'$. Then $D(s, \mathbf{f}', \chi) = D(s, \mathbf{f}, \chi)$, and $\mathbf{f}'$ now has weight 2 at the previously "bad" archimedean places. ∎

Appendix: Calculation of Y in Terms of Gamma-functions.

We wish to obtain an explicit formula for the function $Y(s,\beta,\gamma,\sigma)$ defined in (3.15) and (3.16). Recall that $\beta+\gamma = 1-n$ with $0 \le n \in \mathbf{Z}$. We shall prove:

Proposition 3a.1. *If n is even, then*

$$\text{(3a.1)} \qquad Y(s,\beta,\gamma,\sigma) = 2^{2s-1}\pi^{-1/2}\sum_{q=0}^{n/2}\binom{n/2}{q}2^q\sigma^q(s+1-n/2+q)_{n/2-q} \times \Gamma((s+n/2+q+\beta)/2)\Gamma((s+n/2+q+\gamma)/2).$$

If n is odd, then

$$\text{(3a.2)} \quad Y(s,\beta,\gamma,\sigma) = 2^{2s-1}\pi^{-1/2}\sum_{q=0}^{\lfloor n/2\rfloor}\binom{\lfloor n/2\rfloor}{q}2^q\sigma^q(s+1-n/2+q)_{\lfloor n/2\rfloor-q} \times \Big(\Gamma((s+1+n/2+q+\beta)/2)\Gamma((s+n/2+q+\gamma)/2) + \sigma\Gamma((s+n/2+q+\beta)/2)\Gamma((s+1+n/2+q+\gamma)/2)\Big).$$

Moreover, $Y(s,\beta,\gamma,1)\pm Y(s,\beta,\gamma,-1)$ is a nonzero meromorphic function of s, with the one exception that if $n=0$, then $Y(s,\beta,\gamma,1)-Y(s,\beta,\gamma,-1)=0$.

Proof: We first reduce the above formulas to the case where $n=0$ or $n=1$. (1.13) implies the following equations (which are equivalent to (1.17) and (1.18)):

$$\text{(3a.3)} \qquad V(y;\beta,\gamma) = (1/2)V(y;\beta+1,\gamma+1) + (\beta+\gamma+1)y^{-1}V(y;\beta+1,\gamma+1) - \frac{\partial}{\partial y}V(y;\beta+1,\gamma+1),$$

$$\text{(3a.4)} \qquad -\beta\gamma y^{-2}V(y;1-\beta,1-\gamma) = (-1/2)V(y;-\beta,-\gamma) - \frac{\partial}{\partial y}V(y;-\beta,-\gamma).$$

Thus Y satisfies the recurrence

$$\text{(3a.5)} \quad Y(s,\beta,\gamma,\sigma) = (s+1-n/2)Y(s,\beta+1,\gamma+1,\sigma)+(\sigma/2)Y(s+1,\beta+1,\gamma+1,\sigma),$$

where we note that $\beta+1$ and $\gamma+1$ correspond to $n-2$. It then follows by induction that for any integer $r\ge 0$,

$$\text{(3a.6)} \quad Y(s,\beta,\gamma,\sigma) = \sum_{q=0}^{r}\binom{r}{q}2^{-q}\sigma^q(s+1-n/2+q)_{r-q}Y(s+q,\beta+r,\gamma+r,\sigma).$$

In particular, picking $r=\lfloor n/2\rfloor$, we can reduce the problem to evaluating Y for $n=0$ or $n=1$. In the former case V is essentially a K-Bessel function, and its Mellin transform Y is well-known; we nonetheless include a derivation of the result, as it shall be useful in the latter case.

Let us thus deal with the case $n = 0$. Then $\beta + \gamma = 1$, so $Y(s, \beta, \gamma, 1) = Y(s, \beta, \gamma, -1)$ from the definition. We may assume that the real parts of β and s are sufficiently large, express V as in (1.13), and invoke analytic continuation. We then obtain (remembering that $-\gamma = \beta - 1$)

$$Y(s, \beta, \gamma, \pm 1) = \Gamma(\beta)^{-1}\Gamma(s + \beta) \int_0^\infty (t + 1/2)^{-s-\beta}(1 + t)^{\beta-1}t^{\beta-1}\, dt. \tag{3a.7}$$

Substituting $u = 4t(1 + t) = 4(t + 1/2)^2 - 1$ now yields

$$\Gamma(\beta)^{-1}\Gamma(s + \beta)2^{s-\beta} \int_0^\infty u^{\beta-1}(1 + u)^{(-s-\beta-1)/2}\, du. \tag{3a.8}$$

This last integral is known, being a Beta-integral (see [H], (8.7-4)). Simplify the resulting expression using Legendre's duplication formula for the Gamma-function ([H], (8.4-18)). This yields

$$Y(s, \beta, \gamma, \pm 1) = 2^{2s-1}\pi^{-1/2}\Gamma((s + \beta)/2)\Gamma((s + \gamma)/2), \quad \text{if } n = 0. \tag{3a.9}$$

It remains to evaluate Y in the case $n = 1$; this time, we will have to proceed differently according to the sign of σ. If $\sigma = 1$, we obtain as before

$$\begin{aligned} &Y(s, \beta, \gamma, 1) \\ &\quad = \Gamma(\beta)^{-1}\Gamma(s + 1/2 + \beta) \int_0^\infty (t + 1/2)^{-s-1/2-\beta}(1 + t)^{-\gamma}t^{\beta-1}\, dt. \end{aligned} \tag{3a.10}$$

Since $-\gamma = \beta$, $(1 + t)^{-\gamma}t^{\beta-1} = (t + 1/2)(1 + t)^{\beta-1}t^{\beta-1} + 1/2(1 + t)^{\beta-1}t^{\beta-1}$. This reduces us to a sum of integrals of the same kind as the integral in (3a.7), which we know how to evaluate. Working similarly for $\sigma = -1$, we obtain the final result that if $n = 1$,

$$\begin{aligned} &Y(s, \beta, \gamma, \pm 1) = 2^{2s-1}\pi^{-1/2} \\ &\times \Big(\Gamma((s{+}3/2{+}\beta)/2)\Gamma((s{+}1/2{+}\gamma)/2) \pm \Gamma((s{+}1/2{+}\beta)/2)\Gamma((s{+}3/2{+}\gamma)/2)\Big). \end{aligned} \tag{3a.11}$$

This concludes our evaluation of Y in the cases $n = 0$ and $n = 1$, and, together with the inductive relation (3a.6), completes our proof of equations (3a.1) and (3a.2).

It now follows from these explicit expressions for Y that if $P \in \{0, 1\}$, then $Y(s, \beta, \gamma, 1) + (-1)^P Y(s, \beta, \gamma, -1)$ is a nonzero meromorphic function of s, unless $n = 0$ and $P = 1$. For instance, if n is even, then this function is the product of $2^{2s}\Gamma((s + n/2 + P + \beta)/2)\Gamma((s + n/2 + P + \gamma)/2)$ with a polynomial in s whose leading coefficient is positive. ∎

4. Half-integral Weight Forms on the Adele Group.

We now turn our attention to forms of half-integral weight on the adelic metaplectic group $M_{\mathbf{A}}$, following sections 1 and 2 of [Sh 93a] and sections 3 and 5 of [Sh 87]. Throughout this section, let $0 \leq m \in \mathbf{Z}^{\mathbf{a}}$, and let $k = m + u/2$ be a half-integral weight. Let $\lambda \in \mathbf{C}^{\mathbf{a}}$ be a collection of eigenvalues, and let $\beta, \gamma \in \mathbf{C}^{\mathbf{a}}$ be the parameters for the Whittaker functions corresponding to (k, λ).

Definition. Given two integral ideals $\mathfrak{b}$ and $\mathfrak{b}'$, and a Hecke character ψ whose conductor divides $\mathfrak{c} = 4\mathfrak{b}\mathfrak{b}'$, define $\mathcal{S}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi)$ (respectively $\mathcal{M}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi)$) to be the set of $f \in \mathcal{S}_{k,\lambda}$ (respectively in $\mathcal{M}_{k,\lambda}$) satisfying

$$f\|_k\gamma = \psi_{\mathfrak{c}}(a_\gamma)f, \ \forall\gamma \in \Gamma[2\mathfrak{b}\mathfrak{d}^{-1}, 2\mathfrak{b}'\mathfrak{d}\,]. \tag{4.1}$$

It should be noted that for a given f, ψ is only determined up to multiplication by a Hecke character that is unramified at all $v \in \mathbf{f}$. As observed in Lemma 1.2 of [Sh 93a], and in the paragraph preceding it, it is no loss of generality for us to assume that ψ is *normalized* for f; namely, we shall assume from now on that

$$\forall x \in F_{\mathbf{a}}^\times, \psi_{\mathbf{a}}(x) = \operatorname{sgn}(x)^m|x|^{i\mu}, \tag{4.2}$$

for some $\mu \in \mathbf{R}^{\mathbf{a}}$ such that $\|\mu\| = 0$. Note that if $f \in \mathcal{M}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi)$, then $\epsilon_v f \in \mathcal{M}_{k-(2)_v,\lambda-(k_v-2)_v}(\mathfrak{b}, \mathfrak{b}', \psi)$, for any $v \in \mathbf{a}$, and similarly for δ_v^k; also, ψ is still normalized for $\epsilon_v f$ and $\delta_v^k f$.

As described in equations (3.5) through (3.12) of [Sh 87], and equations (1.7) through (1.10) of [Sh 93a], we can associate to $f \in \mathcal{M}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi)$ a function $f_{\mathbf{A}}$ on $M_{\mathbf{A}}$, defined by

$$f_{\mathbf{A}}(\alpha x) = \psi_{\mathfrak{c}}(a_x)^{-1}(f\|_k x)(\mathbf{i}), \ \forall\alpha \in G \text{ and } x \in \operatorname{pr}^{-1}(D[2\mathfrak{b}\mathfrak{d}^{-1}, 2\mathfrak{b}'\mathfrak{d}\,]). \tag{4.3}$$

Here $M_{\mathbf{A}}$ is the adelic metaplectic group, and $\operatorname{pr} : M_{\mathbf{A}} \to G_{\mathbf{A}}$ is the projection. This projection map splits on certain subgroups of $G_{\mathbf{A}}$, and there are lifts $r : G \to M_{\mathbf{A}}$ and $r_P : P_{\mathbf{A}} \to M_{\mathbf{A}}$, which agree on P. We use r to view G as a subgroup of $M_{\mathbf{A}}$, and write $a_x = a_{\operatorname{pr}(x)}$; similarly for the other matrix entries of x. We note that the factor of automorphy $h(\tau, z)$ mentioned in (1.2) is in fact defined on all of $\operatorname{pr}^{-1}(P_{\mathbf{A}}C'')$. In this context, the action of x on $\mathcal{H}^{\mathbf{a}}$ and the factor of automorphy $j(x, z)$ are both defined using the projection of x to $G_{\mathbf{A}}$ and from thence to $G_{\mathbf{a}}$. Thus $f\|_k x$ makes sense for $x \in \operatorname{pr}^{-1}(P_{\mathbf{A}}C'')$. $f_{\mathbf{A}}$ then satisfies

$$f_{\mathbf{A}}(\alpha x w) = \psi_{\mathfrak{c}}(a_w)^{-1}J_k(w, \mathbf{i})^{-1}f_{\mathbf{A}}(x), \tag{4.4}$$

whenever $x \in M_{\mathbf{A}}$, $\alpha \in G$, and $w \in \operatorname{pr}^{-1}(D[2\mathfrak{b}\mathfrak{d}^{-1}, 2\mathfrak{b}'\mathfrak{d}\,])$ such that $w\mathbf{i} = \mathbf{i}$. Conversely, given $f_{\mathbf{A}}$ satisfying (4.4), we can retrieve f as a function on $\mathcal{H}^{\mathbf{a}}$ from (4.3); this f will then satisfy (4.1). We remark that the correspondence between f and $f_{\mathbf{A}}$ extends to any $f \in \mathcal{M}_{k,\lambda}$, as in [Sh 87].

We now turn to the Fourier expansion of a form $f \in \mathcal{M}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi)$.

Proposition 4.1. *Let $f \in \mathcal{M}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi)$. For $\xi \in F^\times$ and a fractional ideal $\mathfrak{m}$, one can define Fourier coefficients $\mu(\xi, \mathfrak{m}; f, \psi) \in \mathbf{C}$ — which we will sometimes write $\mu_f(\xi, \mathfrak{m})$, even though they depend on the choice of ψ — and constant terms $\mu_0(\mathfrak{m}, y; f, \psi)$ for $y \in \mathbf{R}^{\mathbf{a}}$, $y \gg 0$. These coefficients and constant terms satisfy*

$$\begin{aligned} f_{\mathbf{A}}\left(r_P\begin{pmatrix} t & s \\ 0 & t^{-1}\end{pmatrix}\right) &= t_{\mathbf{a}}^m|t|_{\mathbf{A}}^{1/2}\psi_{\mathbf{f}}(t)^{-1}\mu_0(t\mathfrak{o}, t_{\mathbf{a}}^2; f, \psi) \\ &+ t_{\mathbf{a}}^m|t|_{\mathbf{A}}^{1/2}\psi_{\mathbf{f}}(t)^{-1}\sum_{\xi\in F^\times}\mu(\xi, t\mathfrak{o}; f, \psi)W_{\beta,\gamma}(\xi t_{\mathbf{a}}^2/2)\mathbf{e}_{\mathbf{A}}(\xi ts/2), \end{aligned} \tag{4.5}$$

for $t \in F_{\mathbf{A}}^{\times}$ and $s \in F_{\mathbf{A}}$. Furthermore, $\mu_f(\xi, \mathfrak{m}) = 0$ unless $\xi \in \mathfrak{m}^{-2}\mathfrak{b}^{-1}$ (but ξ may have any signature), and

$$\mu_f(\xi b^2, \mathfrak{m}) = b^m \psi_{\mathbf{a}}(b)\mu_f(\xi, b\mathfrak{m}) = |b|^{m+i\mu}\mu_f(\xi, b\mathfrak{m}),\ \forall b \in F^{\times}. \tag{4.6}$$

Also, $\mu_0(\mathfrak{m}, y; f, \psi) = b^m \psi_{\mathbf{a}}(b)\mu_0(b\mathfrak{m}, b^2 y; f, \psi)$. We can obtain the μ and the μ_0 from the expansion

$$\begin{aligned}\psi_{\mathfrak{c}}(d_\beta)^{-1} J_k(\beta, \beta^{-1}z) f(\beta^{-1}z) &= |r|_{\mathbf{A}}^{1/2} \frac{\psi_{\mathbf{f}}}{\psi_{\mathfrak{c}}}(r)^{-1}\mu_{0,f}(r\mathfrak{o}, y) \\ &+ \sum_{\xi \in F^{\times}} |r|_{\mathbf{A}}^{1/2} \frac{\psi_{\mathbf{f}}}{\psi_{\mathfrak{c}}}(r)^{-1}\mu_f(\xi, r\mathfrak{o}) W_{\beta,\gamma}(\xi y/2)\mathbf{e}_{\mathbf{a}}(\xi x/2),\end{aligned} \tag{4.7}$$

for $r \in F_{\mathbf{f}}^{\times}$, $\beta \in G \cap \begin{pmatrix} r & 0 \\ 0 & r^{-1}\end{pmatrix} D[2\mathfrak{b}\mathfrak{d}^{-1}, 2\mathfrak{b}'\mathfrak{d}]$, and $z = x + iy \in \mathcal{H}^{\mathbf{a}}$.

Proof: This is a generalization of Proposition 3.1 of [Sh 87] and Proposition 1.1 of [Sh 93a] to the nonholomorphic case. The proof proceeds as in [Sh 87], with virtually no modification. Observe that for $v \in \mathbf{a}$, (1.17) and (4.7) imply

$$\mu(\xi, \mathfrak{m}; \epsilon_v f, \psi) = \begin{cases} \beta_v \gamma_v (8\pi i \xi_v)^{-1}\mu(\xi, \mathfrak{m}; f, \psi), & \text{if } \xi_v > 0, \\ (8\pi i \xi_v)^{-1}\mu(\xi, \mathfrak{m}; f, \psi), & \text{if } \xi_v < 0. \end{cases} \tag{4.8}$$

We note that [Sh 87] and [Sh 93a] use the notation $\lambda(\xi, \mathfrak{m}; f, \psi)$ for the Fourier coefficients, but we are already using the symbol λ to denote simultaneously an ideal class (as in $\mathfrak{t}_\lambda$) and the eigenvalue of the operators L_v^k. ∎

Propositions 1.4 and 1.5 of [Sh 93a] also generalize straightforwardly to nonholomorphic forms:

Proposition 4.2. *(1) Let $f \in \mathcal{M}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi)$, and take a fractional ideal $\mathfrak{y}$ and a totally positive $\tau \in F$. Let ϵ_τ be the Hecke character associated to the quadratic extension $F(\sqrt{\tau})/F$. Then there exists a form $h \in \mathcal{M}_{k,\lambda}(\tau\mathfrak{y}^{-2}\mathfrak{b} \cap \mathfrak{o}, \tau^{-1}\mathfrak{y}^2\mathfrak{b}' \cap \mathfrak{o}, \psi\epsilon_\tau)$ which corresponds to f, in the sense that*

$$\mu(\xi, \mathfrak{m}; h, \psi\epsilon_\tau) = \mu(\tau\xi, \mathfrak{y}^{-1}\mathfrak{m}; f, \psi), \tag{4.9}$$

$$\mu_0(\mathfrak{m}, y; h, \psi\epsilon_\tau) = \mu_0(\mathfrak{y}^{-1}\mathfrak{m}, y; f, \psi), \tag{4.10}$$

for every $\xi \in F^{\times}$, fractional ideal $\mathfrak{m}$, and totally positive $y \in \mathbf{R}^{\mathbf{a}}$. Also,

$$\mu_h(\xi, \mathfrak{m}) = 0 \quad \text{unless } \tau\xi \in \mathfrak{b}^{-1}\mathfrak{y}^2\mathfrak{m}^{-2}. \tag{4.11}$$

(2) Now suppose $\tau^{-1}\mathfrak{y}^2\mathfrak{b}' \subset \mathfrak{o}$, and take $h \in \mathcal{M}_{k,\lambda}(\tau\mathfrak{y}^{-2}\mathfrak{b} \cap \mathfrak{o}, \tau^{-1}\mathfrak{y}^2\mathfrak{b}', \psi\epsilon_\tau)$ satisfying (4.11), or even the weaker condition $\mu_h(\xi, \mathfrak{y}) = 0$ unless $\tau\xi \in \mathfrak{b}^{-1}$. Then h comes from a unique form f via the above correspondence.

(3) If f is a cusp form, then so is the corresponding form h, and

$$\langle f, f\rangle = \tau^{-k}\mathrm{N}(\mathfrak{y})\langle h, h\rangle. \tag{4.12}$$

Proof: Note first that $\psi\epsilon_\tau$ is again normalized for h. The proof of (1) proceeds as in [Sh 93a], by writing $p(z) = f(z/\tau)$ and defining

$$h_{\mathbf{A}}(x) = (\psi\epsilon_\tau)(r)\mathrm{N}(\mathfrak{y})^{-1/2}p_{\mathbf{A}}(xr_P\begin{pmatrix} r & 0 \\ 0 & r^{-1}\end{pmatrix}), \tag{4.13}$$

for some $r \in F_{\mathbf{f}}^\times$ with $r\mathfrak{o} = \mathfrak{y}^{-1}$. Alternatively, if $\beta \in G \cap \begin{pmatrix} r & 0 \\ 0 & r^{-1}\end{pmatrix}U$, for a sufficiently small open subgroup U of C'', then for some $\zeta \in \mathbf{T}$,

$$h(z) = \zeta\mathrm{N}(\mathfrak{y})^{-1/2}J_k(\beta, \beta^{-1}z)p(\beta^{-1}z). \tag{4.14}$$

The only additional difficulty lies in showing that h is an eigenfunction of the L_v^k operators. This follows from (1.8), since we can view $\sigma = \begin{pmatrix} \tau^{-1/2} & 0 \\ 0 & \tau^{1/2}\end{pmatrix}$ as a matrix in $G_{\mathbf{a}}$ (in other words, $\sigma_v = \begin{pmatrix} \tau_v^{-1/2} & 0 \\ 0 & \tau_v^{1/2}\end{pmatrix}$, where we take the positive square root at each $v \in \mathbf{a}$). Then $p = \tau^{k/2}f\|_k r_P(\sigma)$. Thus $L_v^k p = \lambda_v p$, and similarly (4.14) implies that $L_v^k h = \lambda_v h$, for $v \in \mathbf{a}$. Incidentally, if h corresponds to f, then $\tau_v^{-1}\epsilon_v h$ corresponds to $\epsilon_v f$, and similarly for δ_v^k. (4.11) then follows from the fact that $\mu_f(\xi, \mathfrak{m}) = 0$ unless $\xi \in \mathfrak{b}^{-1}\mathfrak{m}^{-2}$. Statements (2) and (3) hold by the same reasoning as in [Sh 93a]. ∎

We now introduce the Hecke operators $\{T_v\}_{v\in\mathbf{f}}$, which depend on $\mathfrak{b}$, $\mathfrak{b}'$, and ψ. The operators T_v act on $\mathcal{M}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi)$, preserving the space of cusp forms; they are defined by the same expansion in terms of double cosets as in equation (2.4) of [Sh 93a], and equations (5.5) and (5.6) of [Sh 87]. (In [Sh 87], the operators were called $T_{\mathfrak{p}}$, where $\mathfrak{p}$ is the prime corresponding to v.) For $f \in \mathcal{M}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi)$, the same proof as in Proposition 5.4 of [Sh 87] gives us that for $\xi \in \mathfrak{m}^{-2}\mathfrak{b}^{-1}$,

$$\mu(\xi, \mathfrak{m}; T_v f, \psi) = \mu(\xi, \mathfrak{m}\mathfrak{p}; f, \psi), \quad \text{if } v \mid \mathfrak{c}; \tag{4.15}$$

$$\begin{aligned}\mu(\xi, \mathfrak{m}; T_v f, \psi) = \mu(\xi, \mathfrak{m}\mathfrak{p}; f, \psi) + \psi^*(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{-1}\left(\frac{\xi c^2}{\mathfrak{p}}\right)\mu(\xi, \mathfrak{m}; f, \psi) \\ + \psi^*(\mathfrak{p})^2\mathrm{N}(\mathfrak{p})^{-1}\mu(\xi, \mathfrak{m}\mathfrak{p}^{-1}; f, \psi), \quad \text{if } v \dagger \mathfrak{c}.\end{aligned} \tag{4.16}$$

Here $\mathfrak{p}$ is the prime corresponding to v; $c \in F_v$ generates $\mathfrak{m}_v$; and $\left(\frac{d}{\mathfrak{p}}\right)$, for $d \in \mathfrak{o}_v$, is the quadratic residue symbol (taken to be 0 when $d \in \mathfrak{p}_v$). Note that if $\xi \notin \mathfrak{m}^{-2}\mathfrak{b}^{-1}$, then the left-hand sides of (4.15) and (4.16) are 0, even though the right-hand sides need not be. As usual, the Hecke operators commute with each other, and with ϵ_v and δ_v^k, for $v \in \mathbf{a}$. Another standard fact, shown in Lemma 2.8 of [Sh 93a] and Proposition 5.3 of [Sh 87], is that

$$\langle f, T_v g\rangle = \psi^*(\mathfrak{p})^2\langle T_v f, g\rangle, \quad \text{if } v \dagger \mathfrak{c}, \tag{4.17}$$

if v corresponds to $\mathfrak{p}$, and f and g are C^∞ functions on $\mathcal{H}^{\mathbf{a}}$, which transform under (4.1), under a suitable criterion of convergence. Proposition 2.2 of [Sh 93a] also carries over: let $f \in \mathcal{M}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi)$ be an eigenform of all the T_v, with $T_v f = \omega_v f$. Take a totally positive $\tau \in F^\times$, and define ϵ_τ as in Proposition 4.2. Write $\tau\mathfrak{b} = \mathfrak{q}^2\mathfrak{r}$, with a fractional ideal $\mathfrak{q}$ and a squarefree integral ideal $\mathfrak{r}$. Then we have the identity of formal Dirichlet series (ω_v is the Hecke eigenvalue corresponding to the prime $\mathfrak{p}$):

$$(4.18) \qquad \sum_{\mathfrak{m} \subset \mathfrak{o}} \mu_f(\tau, \mathfrak{q}^{-1}\mathfrak{m})\mathrm{M}(\mathfrak{m}) = \mu_f(\tau, \mathfrak{q}^{-1}) \prod_{\mathfrak{p} \nmid \mathfrak{r}\mathfrak{c}} (1 - (\psi\epsilon_\tau)^*(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{-1}\mathrm{M}(\mathfrak{p}))$$
$$\times \prod_{\mathfrak{p} | \mathfrak{c}} (1 - \omega_v \mathrm{M}(\mathfrak{p}))^{-1} \prod_{\mathfrak{p} \nmid \mathfrak{c}} (1 - \omega_v \mathrm{M}(\mathfrak{p}) + \psi^*(\mathfrak{p})^2 \mathrm{N}(\mathfrak{p})^{-1}\mathrm{M}(\mathfrak{p}^2))^{-1}.$$

As in the case of integral weight forms, one can define an inverter for forms of half-integral weight, following Lemma 2.3 and equation (2.5) of [Sh 93a]. Take $\omega \in G \cap U\varepsilon^{-1}$, for U a sufficiently small open subgroup of C'', and $\varepsilon \in G_{\mathbf{f}}$ as in (0.10). Then define the *inversion* f^* of f by

$$(4.19) \qquad f^* = \psi(\delta) f\|_k \omega \in \mathcal{M}_{k,\lambda}(\mathfrak{b}', \mathfrak{b}, \overline{\psi}).$$

(This is independent of the choice of either ω or δ.) Alternatively, take the unique $\tilde{\varepsilon} \in M_{\mathbf{A}}$ such that $\mathrm{pr}(\tilde{\varepsilon}) = \varepsilon$, and $h(\tilde{\varepsilon}, z) = 1$. Then

$$(4.20) \qquad f^*_{\mathbf{A}}(x) = \psi(\delta) f_{\mathbf{A}}(x\tilde{\varepsilon}).$$

We shall call the operator $f \mapsto f^*$ the *inverter*. Unlike the inverter of (2.19), this one does not depend on the level. This inverter commutes with ϵ_v and δ_v^k, and takes cusp forms to cusp forms. Lemmas 2.6 and 2.7 of [Sh 93a] remain valid:

Proposition 4.3. *Let $f \in \mathcal{M}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi)$, and fix τ and $\mathfrak{y}$ as in Proposition 4.2. If h corresponds to f, as in (4.9), then $T_v h$ corresponds to $T_v f$, for all $v \in \mathbf{f}$ such that $\mathfrak{y}_v^2 = \tau\mathfrak{o}_v$. If furthermore $\mathfrak{y}^2 \subset \tau\mathfrak{b}$, then $T_v h$ corresponds to $T_v f$ for any v not dividing $\tau^{-1}\mathfrak{b}^{-1}\mathfrak{y}^2$. As for the action of the inverter, $(T_v f)^* = \psi^*(\mathfrak{p})^2 T_v(f^*)$, for any v corresponding to a prime $\mathfrak{p} \nmid \mathfrak{c}$. Moreover, h^* corresponds up to a constant to f^*, with respect to τ^{-1} and $\mathfrak{y}^{-1}$. Namely,*

$$(4.21) \qquad \mu(\xi, \mathfrak{m}; h^*, \overline{\psi\epsilon_\tau}) = \tau^k \mathrm{N}(\mathfrak{y})^{-1} \mu(\xi/\tau, \mathfrak{y}\mathfrak{m}; f^*, \overline{\psi}).$$

5. Description and First Properties of the Theta-function.

The purpose of the next few sections is to study the Shimura correspondence between forms of even weight and forms of half-integral weight. We will express the correspondence as a convolution with a certain theta-function of two variables. From now until the end of this paper, fix the following

Notation: Let $k = m + u/2$ be a half-integral weight, with $m \in \mathbf{Z}^{\mathbf{a}}$ and $m \geq 0$. Let λ be a set of eigenvalues, and let $\mathfrak{b}$ and $\mathfrak{b}'$ be integral ideals of $\mathfrak{o}$. Put $\mathfrak{c} = 4\mathfrak{b}\mathfrak{b}'$ and let ψ be a Hecke character with conductor dividing $\mathfrak{c}$, which is normalized as in (4.2). Define $\mu \in \mathbf{R}^{\mathbf{a}}$ by (4.2) (so $\|\mu\| = 0$). Pick any totally positive $\tau \in F^\times$, and write $\tau\mathfrak{b} = \mathfrak{q}^2\mathfrak{r}$ with $\mathfrak{r}$ a squarefree integral ideal. Let ϵ_τ be the Hecke character associated to the quadratic extension $F(\sqrt{\tau})/F$, and put $\varphi = \psi\epsilon_\tau$, which is also normalized, with the same μ. Let $\mathfrak{h}$ be the conductor of φ, and note that $\mathfrak{h}$ divides

$\mathfrak{rc}$. (This is because the conductor of ϵ_τ divides $4\mathfrak{r} \cap 4\mathfrak{b}$, so $\mathfrak{h}$ even divides $4\mathfrak{r} \cap \mathfrak{c}$.) Take $f \in \mathcal{S}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi)$, and let $h \in \mathcal{S}_{k,\lambda}(\mathfrak{o}, \mathfrak{rbb}', \varphi)$ be the form corresponding to f by Proposition 4.2, with $\mathfrak{y} = \mathfrak{qr}$. In other words,

$$\mu_h(\xi, \mathfrak{m}) = \mu_f(\tau\xi, \mathfrak{q}^{-1}\mathfrak{r}^{-1}\mathfrak{m}), \quad \text{for all } \xi, \mathfrak{m}. \tag{5.1}$$

Write β and γ for the parameters of the Whittaker functions corresponding to weight k and eigenvalue λ, as in (1.12). Then the parameters corresponding to weight $2m$ and eigenvalue 4λ are 2β and 2γ.

For $z, w \in \mathcal{H}^{\mathbf{a}}$ and $p \in \tilde{G}_{\mathbf{A}}$, we are going to define theta-functions $\Theta(z, p; \eta)$ and $\theta(z, w; \eta_\lambda)$, such that $\Theta(z, p; \eta)$ transforms in the z-variable like an element of $\mathcal{M}_{k,\lambda}(\mathfrak{o}, 4^{-1}\mathfrak{rc}, \varphi)$; $\Theta(z, p; \eta)$ transforms like an element of $\mathcal{M}_{2m,4\lambda}(\mathfrak{r}, 2^{-1}\mathfrak{c}, \psi^2)$ in the p-variable; and $\Theta(z, p; \eta) = (\theta(z, w; \eta_\lambda))_\lambda$, in the sense of (2.6). (The meaning of η will be explained after we state Theorem 5.1 and Theorem 5.2 below.)

Theorem 5.1. *Let $f \in \mathcal{S}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi)$, and let h correspond to it as in (5.1). Define $\Theta(z, p; \eta)$ by (5.8) below, where η is defined as in (5.22), and so depends on τ and φ. Define $\mathbf{g}_\tau$ by*

$$C\mathbf{g}_\tau(p) = \int_\Phi \Theta(z, p; \eta) h(z) y^k d_H z, \qquad p \in \tilde{G}_{\mathbf{A}}, \tag{5.2}$$

$$C = 2^{1+\|m+u\|} i^{\|m\|} \mathrm{N}(\mathfrak{rc}), \qquad \Phi = \Gamma[2\mathfrak{d}^{-1}, 2^{-1}\mathfrak{rcd}]\backslash\mathcal{H}^{\mathbf{a}}. \tag{5.3}$$

Then the following assertions hold:

(1) $\mathbf{g}_\tau \in \mathcal{M}_{2m,4\lambda}(\mathfrak{o}, 2^{-1}\mathfrak{c}, \psi^2)$, and $\mathbf{g}_\tau$ is "even" in the sense of Lemma 3.1. (Note that the conductor of ψ^2 divides $2^{-1}\mathfrak{c} = 2\mathfrak{bb}'$, since $a \equiv 1 \pmod{2\mathfrak{bb}'}$ implies that $a^2 \equiv 1 \pmod{4\mathfrak{bb}'}$.)

(2) We have the identity of formal Dirichlet series

$$\begin{aligned} &\sum_{\mathfrak{m}} \mathrm{c}(\mathfrak{m}, u; \mathbf{g}_\tau)\mathrm{M}(\mathfrak{m}) \\ &\quad = \sum_{\mathfrak{m}} \mu_f(\tau, \mathfrak{q}^{-1}\mathfrak{m})\mathrm{M}(\mathfrak{m}) \prod_{\text{prime } \mathfrak{p} \nmid \mathfrak{rc}} (1 - \varphi^*(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{-1}\mathrm{M}(\mathfrak{p}))^{-1}, \end{aligned} \tag{5.4}$$

where $\mathfrak{m}$ runs over the ideals of F, and the $\mathrm{M}(\mathfrak{m})$ are a system of formal multiplicative symbols in the ideals of F (see (2.18)).

(3) If $v \in \mathbf{f}$ corresponds to $\mathfrak{p}$, and $T_v f = \omega_v f$, then $\mathfrak{T}(\mathfrak{p})\mathbf{g}_\tau = \mathrm{N}(\mathfrak{p})\omega_v \mathbf{g}_\tau$.

(4) If f is orthogonal to all the weight k theta-series $\theta_m(bz)$ of [Sh 87], (4.5), with $b \gg 0$, then $\mathbf{g}_\tau$ is in fact a cusp form. If, in addition, $T_v f = \omega_v f$ for every $v \in \mathbf{f}$, and $\mu_f(\xi, \mathfrak{m}) \neq 0$ for some $\xi \gg 0$ and some $\mathfrak{m}$, then there exists in $\mathcal{S}_{2m,4\lambda}(\mathfrak{o}, 2^{-1}\mathfrak{c}, \psi^2)$ a system χ of Hecke eigenvalues, with $\chi(\mathfrak{p}) = \mathrm{N}(\mathfrak{p})\omega_v$. $\mathbf{g}_\tau$ is then $\mu_f(\tau, \mathfrak{q}^{-1})$ times the normalized "even" eigenform belonging to χ. Warning: χ need not be a primitive system of eigenvalues; it could come from a primitive form at lower level.

Theorem 5.2. *Let $\mathbf{g} = (g_\lambda)_\lambda \in \mathcal{S}_{2m,4\lambda}(\mathfrak{o}, 2^{-1}\mathfrak{c}, \psi^2)$, and set*

$$\tilde{h}(z) = \sum_\lambda \langle \theta(z, w; \eta_\lambda), g_\lambda(w) \rangle = \langle \Theta(z, p; \eta), \mathbf{g}(p) \rangle, \tag{5.5}$$

using η and η_λ of (5.22) and (5.23). Then $\tilde{h} \in \mathcal{S}_{k,\lambda}(\mathfrak{o}, 4^{-1}\mathfrak{r}\mathfrak{c}, \varphi)$, and $\mu_{\tilde{h}}(\xi, \mathfrak{m}) = 0$ unless $\xi \in \mathfrak{r}\mathfrak{m}^{-2}$. This shows that $\tilde{h}$ comes from a form $\tilde{f} \in \mathcal{S}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi)$, analogously to (5.1). Moreover, for almost all $v \in \mathbf{f}$ corresponding to a prime $\mathfrak{p}$, if $\mathfrak{T}(\mathfrak{p})\mathbf{g} = \chi(\mathfrak{p})\mathbf{g}$, then $T_v\tilde{f} = \chi(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{-1}\tilde{f}$.

We shall prove preliminary forms of Theorem 5.1 and Theorem 5.2 in this section, and fill in the remaining details in the next two sections. For any w, w', and $z \in \mathbf{C}^{\mathbf{a}}$, and $\xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(F_{\mathbf{A}})$, we define the symbols $[\xi, w, w']$, $[\xi, w]$, and $\mathrm{R}\,[\xi, z, w]$, all of which are elements of $\mathbf{C}^{\mathbf{a}}$, by:

$$(5.6)\quad [\xi, w, w'] = \left((-1 \quad w_v)\xi_v \begin{pmatrix} w'_v \\ 1 \end{pmatrix} \right)_{v\in\mathbf{a}} = (c_v w_v w'_v + d_v w_v - a_v w'_v - b_v)_{v\in\mathbf{a}},$$

$$(5.7)\qquad [\xi, w] = [\xi, w, w], \qquad \mathrm{R}\,[\xi, z, w] = (z_v \det \xi_v + \frac{iy_v}{2v_v^2}|[\xi, w]_v|^2)_{v\in\mathbf{a}}.$$

Here y and v are the imaginary parts of z and w, respectively.

Let $V = \{\xi \in \mathrm{M}_2(F) |\, \mathrm{trace}\,\xi = 0\}$, let $\mathcal{S}(V_{\mathbf{f}})$ denote the space of Schwartz class functions on $V_{\mathbf{f}}$, and let $\mathcal{S}(V_v)$ denote the space of Schwartz class functions on V_v, for any $v \in \mathbf{f}$. We will view any $\eta \in \mathcal{S}(V_{\mathbf{f}})$ as a function on $V_{\mathbf{A}}$ via the projection onto $V_{\mathbf{f}}$. Restricting η to $V \subset V_{\mathbf{A}}$, we get a "locally constant function" on V in the sense of [Sh 85a]. For $z, w \in \mathcal{H}^{\mathbf{a}}$, and $p \in \tilde{G}_{\mathbf{A}}$, we now define our theta-functions

$$(5.8)\qquad \Theta_m(z, p; \eta) = y^{u/2} \sum_{\xi\in V} \eta(p^{-1}\xi p)\varphi(\det p)\left[(p^{-1}\xi p)_{\mathbf{a}}, -\mathbf{i}\right]^m \mathbf{e}_{\mathbf{a}}(2^{-1}\mathrm{R}\left[(p^{-1}\xi p)_{\mathbf{a}}, z, \mathbf{i}\right]),$$

$$(5.9)\qquad \theta_m(z, w; \eta) = y^{u/2}v^{-2m} \sum_{\xi\in V} \eta(\xi)[\xi, \overline{w}]^m \mathbf{e}_{\mathbf{a}}(2^{-1}\mathrm{R}\,[\xi, z, w]).$$

(In [Sh 93a], the theta-function of (5.9) is also denoted by Θ.) Again, y and v are the imaginary parts of z and w. The parameters m and η may be dropped if they are known from the context. $\varphi = \psi\epsilon_\tau$ is the Hecke character from before. For $\eta \in \mathcal{S}(V_{\mathbf{f}})$, and for t_λ and x_λ as in (2.5), define

$$(5.10)\qquad \eta_\lambda(y) = \varphi(t_\lambda)^{-1}\eta(x_\lambda^{-1} y x_\lambda).$$

Then, analogously to (2.6),

$$(5.11)\qquad \forall p \in \tilde{G}_{\mathbf{a}+},\ \Theta(z, t_\lambda^{-1}x_\lambda p; \eta) = (\det p)^{i\mu} J_{2m}(p, \mathbf{i})^{-1}\theta(z, p\mathbf{i}; \eta_\lambda).$$

The transformation properties of the theta-functions are given by

$$(5.12)\qquad \forall s \in F_{\mathbf{A}}^\times,\ \Theta(z, sp; \eta) = \varphi^2(s)\Theta(z, p; \eta) = \psi^2(s)\Theta(z, p; \eta),$$

$$(5.13)\qquad \forall \alpha \in \tilde{G} \text{ and } w \in \tilde{G}_{\mathbf{f}},\ \Theta_m(z, \alpha p w; \eta) = \varphi(\det w)\Theta_m(z, p; \eta^{w^{-1}}),$$

$$(5.14) \qquad \forall \alpha \in \tilde{G}_+,\ J_{2m}(\alpha, w)^{-1}\theta_m(z, \alpha w; \eta) = \theta_m(z, w; \eta^\alpha),$$

$$(5.15) \qquad \forall \gamma \in G \cap P_{\mathbf{A}}C'',\ \overline{J_k(\gamma, z)^{-1}}\theta(\gamma z, w; {}^\gamma\eta) = \theta(z, w; \eta).$$

Here we define η^α by

$$(5.16) \qquad \eta^q(\xi) = \eta(q\xi q^{-1}), \qquad q, \xi \in \tilde{G}_{\mathbf{A}} \text{ or } \tilde{G}_{\mathbf{f}}.$$

Equations (5.12), (5.13), and (5.14) are then immediate. For the definition of ${}^\gamma\eta$ in (5.15), valid only for $\gamma \in G \cap P_{\mathbf{A}}C''$, see section 7 of [Sh 85a], section 11 of [Sh 87], section 5 of [Sh 93a], and [Sh 93b]. The action $(\gamma, \eta) \mapsto {}^\gamma\eta$ arises from the representation of $M_{\mathbf{A}}$ on $\mathcal{S}(V_{\mathbf{A}})$ associated to the quadratic form det on V. We follow [Sh 93a] (especially the discussion before Lemma 5.1 there) in defining ${}^\gamma\eta$ slightly differently from its meaning in [Sh 87]. (The actions agree on $G \cap P_{\mathbf{A}}C'$.) Moreover, $\eta^\alpha = \eta$ and ${}^\gamma\eta = \eta$ for α and γ in sufficiently small congruence subroups (depending on η). A stronger result holds:

Proposition 5.2.1. *Fix η and m, and write $\theta(z, w)$ for the function of (5.9). Assume $f \in \mathcal{S}_{k,\lambda}$, and $g \in \mathcal{S}_{2m,4\lambda}$, and define*

$$(5.17) \qquad f_\theta(w) = \left\langle \overline{\theta(z, w)}, f(z) \right\rangle, \qquad g_\theta(z) = \langle \theta(z, w), g(w) \rangle, \qquad z, w \in \mathcal{H}^{\mathbf{a}}.$$

Then $f_\theta \in \mathcal{M}_{2m,4\lambda}$ (not necessarily a cusp form!), and $g_\theta \in \mathcal{S}_{k,\lambda}$.

Proof: The inner products in (5.17) make sense, thanks to (5.14) and (5.15), and converge nicely, since f and g are cusp forms. One obtains from the transformation properties of $\theta(z, w)$ that f_θ and g_θ are C^∞ functions on $\mathcal{H}^{\mathbf{a}}$ that transform like automorphic forms of weight $2m$ and weight k, respectively. By the estimates in [Sh 82], pages 614 and 615, f_θ and g_θ grow slowly at the cusps. Moreover, these estimates imply that g_θ decays rapidly at the cusps, by simply expanding θ as a sum over ξ and integrating termwise in (5.17). Our only possible worry is the contribution to the inner product from the term for $\xi = 0$, when $m = 0$. This contribution vanishes, being proportional to $\eta(0)y^{u/2}$ times the inner product of $g(w)$ with the constant function 1. (This inner product vanishes because $g(w)$ is a cusp form of weight 0.) The rest of the proposition will follow once we show that f_θ and g_θ are eigenfunctions of the L_v operators, with the stated eigenvalues. This follows from the self-adjointness of L_v and from the following lemma. ∎

Lemma 5.3. *Fix $v \in \mathbf{a}$, and write L_z^k and L_w^{2m} for $L_{z,v}^k$ and $L_{w,v}^{2m}$; similarly for δ_z^k, δ_w^{2m}, ϵ_z, and ϵ_w. Then*

$$(5.18) \qquad L_z^k \overline{\theta_m} = 1/4\, \overline{L_w^{2m}\theta_m};$$

$$(5.19) \qquad \pi i \epsilon_z \overline{\theta_m} = \overline{\delta_w^{2m-(2)_v}\delta_w^{2m-(4)_v}\theta_{m-(2)_v}}, \quad \text{if } m_v \geq 2;$$

$$(5.20) \qquad 4\delta_z^{k-(2)_v}\overline{\delta_w^{2m-(4)_v}\theta_{m-(2)_v}} = \pi i \overline{\epsilon_w\theta_m}, \quad \text{if } m_v \geq 2.$$

Proof: Note that (5.18) follows from (5.19) and (5.20) (at least if $m_v \geq 2$), since the differential operators in w commute with those in z. One proves these identities by applying the operators term-by-term to $\theta(z, w)$, and showing that the above identities hold for each term $y^{u/2}v^{-2m}\eta(\xi)[\xi, \overline{w}]^m \mathbf{e}_{\mathbf{a}}(2^{-1}\mathrm{R}\,[\xi, z, w])$. This is extremely messy, but presents no conceptual difficulties. We remark that (5.18) already appears in [S 75]. ∎

We also note the following consequence of (5.20):

$$\epsilon_w \epsilon_w f_\theta = (-4/i\pi)\epsilon_w \delta_w^{2m-(4)_v}(\epsilon_z f)_\theta = (\text{constant})(\epsilon_z f)_\theta. \tag{5.21}$$

We now define the function $\eta \in \mathcal{S}(V_{\mathbf{f}})$ that we shall use in our theta-function. η is the product of local functions $\eta_v \in \mathcal{S}(V_v)$, for $v \in \mathbf{f}$: $\eta(y) = \prod_{v\in\mathbf{f}} \eta_v(y_v)$. η_v is given by $\eta_v\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right) = 0$ unless $a \in \mathfrak{o}_v$, $b \in 2\mathfrak{c}_v^{-1}\mathfrak{d}_v^{-1}$, and $c \in 2^{-1}\mathfrak{c}_v\mathfrak{d}_v$, in which case

$$\eta_v\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right) = \begin{cases} 1, & \text{if } v \dagger \mathfrak{rc}; \\ \sum_{t\in 2^{-1}\mathfrak{r}_v^{-1}/2^{-1}\mathfrak{c}_v, 2t\mathfrak{r}_v=\mathfrak{o}_v} \varphi_v(t)^{-1}\mathbf{e}_v(bt), & \text{if } v \mid \mathfrak{rc}. \end{cases} \tag{5.22}$$

By (5.10) we similarly get $\eta_\lambda(y) = \prod_{v\in\mathbf{f}}(\eta_\lambda)_v(y_v)$, where $(\eta_\lambda)_v\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right) = 0$ unless $a \in \mathfrak{o}_v$, $b \in 2(\mathfrak{c}\mathfrak{d}\mathfrak{t}_\lambda)_v^{-1}$, and $c \in 2^{-1}(\mathfrak{c}\mathfrak{d}\mathfrak{t}_\lambda)_v$, whereupon

$$(\eta_\lambda)_v\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right) = \begin{cases} \varphi_v(t_\lambda)^{-1}, & \text{if } v \dagger \mathfrak{rc}; \\ \sum_{t\in 2^{-1}(\mathfrak{r}^{-1}\mathfrak{t}_\lambda)_v/2^{-1}(\mathfrak{c}\mathfrak{t}_\lambda)_v, 2t(\mathfrak{r}\mathfrak{t}_\lambda^{-1})_v=\mathfrak{o}_v} \varphi_v(t)^{-1}\mathbf{e}_v(bt), & \text{if } v \mid \mathfrak{rc}. \end{cases} \tag{5.23}$$

$\theta(z, w; \eta_\lambda)$ and $\Theta(z, p; \eta)$ then satisfy

$$\forall \gamma \in \Gamma[2\mathfrak{d}^{-1}, 2^{-1}\mathfrak{rcd}\,],\ \overline{J_k(\gamma, z)^{-1}}\theta(\gamma z, w; \eta_\lambda) = \varphi_{\mathfrak{rc}}(a_\gamma)^{-1}\theta(z, w; \eta_\lambda), \tag{5.24}$$

$$\forall \gamma \in \Gamma[2\mathfrak{d}^{-1}, 2^{-1}\mathfrak{rcd}\,],\ \overline{J_k(\gamma, z)^{-1}}\Theta(\gamma z, p; \eta) = \varphi_{\mathfrak{rc}}(a_\gamma)^{-1}\Theta(z, p; \eta), \tag{5.25}$$

$$\forall \alpha \in \tilde{\Gamma}[\mathfrak{r}\mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1}, 2^{-1}\mathfrak{c}\mathfrak{d}\mathfrak{t}_\lambda\,],\ J_{2m}(\alpha, w)^{-1}\theta(z, \alpha w; \eta_\lambda) = \varphi_{\mathfrak{rc}}(a_\alpha^2/\det\alpha)\theta(z, w; \eta_\lambda), \tag{5.26}$$

$$\forall \alpha \in \tilde{G} \text{ and } w \in \tilde{G}_{\mathbf{f}} \cap \tilde{D}[\mathfrak{r}\mathfrak{d}^{-1}, 2^{-1}\mathfrak{c}\mathfrak{d}\,],\ \Theta(z, \alpha p w; \eta) = \psi_{2^{-1}\mathfrak{c}}^2(d_w)\Theta(z, p; \eta). \tag{5.27}$$

(5.24) (which implies (5.25)) means that ${}^{\gamma}\eta_\lambda = \varphi_{\mathfrak{rc}}(a_\gamma)\eta_\lambda$ for $\gamma \in \Gamma[2\mathfrak{d}^{-1}, 2^{-1}\mathfrak{rcd}\,]$. This follows from Proposition 11.7 of [Sh 87], with M, $\mathfrak{x}$, $\mathfrak{y}$, and $\mathfrak{z}$ there replaced by $V \cap \mathfrak{o}[2\mathfrak{r}\mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1}, 2^{-1}\mathfrak{c}\mathfrak{d}\mathfrak{t}_\lambda\,]$, $\mathfrak{o}$, $4(\mathfrak{rc})^{-1}\mathfrak{d}^{-2}$, and $\mathfrak{rc}$. (See also Lemma 5.1 of [Sh 93a], and the discussion preceding it.) (5.26) and (5.27) mean that

$$\forall \alpha \in \tilde{D}[\mathfrak{r}\mathfrak{d}^{-1}, 2^{-1}\mathfrak{c}\mathfrak{d}\,],\ \eta^\alpha = \varphi_{\mathfrak{rc}}(a_\alpha^2/\det\alpha)\eta, \tag{5.28}$$

$$\forall \alpha \in \tilde{D}[\mathfrak{r}\mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1}, 2^{-1}\mathfrak{c}\mathfrak{d}\mathfrak{t}_\lambda\,],\ {\eta_\lambda}^\alpha = \varphi_{\mathfrak{rc}}(a_\alpha^2/\det\alpha)\eta_\lambda. \tag{5.29}$$

These identities can be proved locally, by direct calculation at each $v \in \mathbf{f}$.

We are now in a position to prove part of Theorem 5.1 and Theorem 5.2. The proofs will be completed in sections 6 and 7.

Theorem 5.1, preliminary version. *Given $f \in \mathcal{S}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi)$, define $\mathbf{g}_\tau$ as in Theorem 5.1. Then $\mathbf{g}_\tau \in \mathcal{M}_{2m,4\lambda}(\mathfrak{r}, 2^{-1}\mathfrak{c}, \psi^2)$ and $\mathbf{g}_\tau$ is "even." The fact that $\mathbf{g}_\tau$ actually belongs to $\mathcal{M}_{2m,4\lambda}(\mathfrak{o}, 2^{-1}\mathfrak{c}, \psi^2)$ and the rest of the statement of the theorem will follow once we prove (in section 6) the identity (5.4) and the fact that $\mathbf{g}_\tau$ is a cusp form if f is orthogonal to the relevant theta-series.*

Proof: Using the notation of Theorem 5.1, define $g_{\tau\lambda}(w)$ by

$$Cg_{\tau\lambda}(w) = \int_\Phi \theta(z, w; \eta_\lambda) h(z) y^k d_H z. \tag{5.30}$$

Then $g_{\tau\lambda}(w)$ is a constant times $\left\langle \overline{\theta(z, w; \eta_\lambda)}, h(z) \right\rangle$, and so by Proposition 5.2.1, $g_{\tau\lambda} \in \mathcal{M}_{2m,4\lambda}$. (5.12), (5.27), and (5.11) imply that $\mathbf{g}_\tau = (g_{\tau\lambda})_\lambda$ satisfies the conditions corresponding to (2.2), (2.3), and (2.6), and so belongs to $\mathcal{M}_{2m,4\lambda}(\mathfrak{r}, 2^{-1}\mathfrak{c}, \psi^2)$. By Lemma 3.1, $\mathbf{g}_\tau$ is "even"if for any $v \in \mathbf{a}$, $(\epsilon_v)^{m_v}\mathbf{g}_\tau(p)$ (which is of weight 0 at the v-place) is invariant under replacing p by pe_v. (Recall that $e_v = \begin{pmatrix} (-1)_v & 0 \\ 0 & 1 \end{pmatrix}$.) Trivially,

$$\Theta(z, pe_v) = \Theta(z, p), \quad \text{if } m_v = 0. \tag{5.31}$$

Also, if $m_v = 1$, $\epsilon_v \mathbf{g}_\tau$ is given by integrating h against $\epsilon_v \Theta(z, p)$, where ϵ_v acts on the p-variable. One can then verify that $\epsilon_v \Theta(z, p)$ equals

$$\begin{aligned}(5.32)\, y^{u/2} \sum_{\xi \in V} & \eta(p^{-1}\xi p)\varphi(\det p)\big[(p^{-1}\xi p)_\mathbf{a}, -\mathbf{i}\big]^{m-(1)_v} \mathbf{e}_\mathbf{a}(2^{-1}\mathrm{R}\left[(p^{-1}\xi p)_\mathbf{a}, z, \mathbf{i}\right]) \\ & \times i\big[p^{-1}\xi p, \mathbf{i}, -\mathbf{i}\big]_v (m_v - \frac{\pi y_v}{2} \big|\big[(p^{-1}\xi p)_\mathbf{a}, \mathbf{i}\big]_v\big|^2),\end{aligned}$$

which is also invariant under replacing p by pe_v, provided $m_v = 1$. The case where m_v is arbitrary can be reduced to these two cases by (5.21). Indeed, applying ϵ_v twice to $\mathbf{g}_\tau$ is the same as replacing h by a constant multiple of $\epsilon_v h$ and decreasing m_v by 2. (Note that ψ and φ remain normalized throughout this process.)

Proving (5.4) will imply the remaining statements in the theorem, except for the first part of assertion (4) (about cuspidality). Indeed, (5.4) implies that $\mathrm{c}(\mathfrak{m}, u; \mathbf{g}_\tau) = 0$ unless $\mathfrak{m} \subset \mathfrak{o}$, because on the right-hand side $\mu_f(\tau, \mathfrak{q}^{-1}\mathfrak{m}) = 0$ unless $\tau \in \mathfrak{b}^{-1}\mathfrak{q}^2\mathfrak{m}^{-2}$, which forces $\mathfrak{m} \subset \mathfrak{o}$, since $\tau\mathfrak{b} = \mathfrak{q}^2\mathfrak{r}$ with $\mathfrak{r}$ integral and squarefree. Since $\mathbf{g}_\tau$ is "even," $\mathrm{c}(\mathfrak{m}, \sigma; \mathbf{g}_\tau) = 0$ for all signatures σ, unless $\mathfrak{m} \subset \mathfrak{o}$. By Lemma 2.1, $\mathbf{g}_\tau \in \mathcal{M}_{2m,4\lambda}(\mathfrak{o}, 2^{-1}\mathfrak{c}, \psi^2)$. This will complete the proof of assertions (1) and (2). Assertion (3) is then a direct consequence of (5.4), by comparing Fourier coefficients (this is somewhat messy). Finally, the second part of assertion (4) merely means that we can find some τ so that $\mathbf{g}_\tau \neq 0$. By assumption, there exist $\xi \gg 0$ and $\mathfrak{m}$ such that $\mu_f(\xi, \mathfrak{m}) \neq 0$. For $\tau = \xi$, (5.4) implies that the formal Dirichlet series for $\mathbf{g}_\tau$ is nonzero. ∎

Theorem 5.2, preliminary version. *(1) Take any $g \in \mathcal{S}_{2m,4\lambda}$, and an ideal class λ. Put $\tilde{h}(z) = \langle \theta(z, w; \eta_\lambda), g(w) \rangle$. Then $\tilde{h} \in \mathcal{S}_{k,\lambda}(\mathfrak{o}, 4^{-1}\mathfrak{r}\mathfrak{c}, \varphi)$.*

(2) Now take $\mathbf{g} \in \mathcal{M}_{2m,4\lambda}(\mathfrak{o}, 2^{-1}\mathfrak{c}, \psi^2)$, and define $\tilde{h}$ by (5.5). Then for almost all $v \in \mathbf{f}$ corresponding to a prime $\mathfrak{p}$, if $\mathfrak{T}(\mathfrak{p})\mathbf{g} = \chi(\mathfrak{p})\mathbf{g}$, then $T_v\tilde{h} = \chi(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{-1}\tilde{h}$.

Thus, if $\tilde{h}$ corresponds to a form $\tilde{f} \in \mathcal{S}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi)$, then $\tilde{f}$ has almost all the same Hecke eigenvalues as $\tilde{h}$, and the rest of Theorem 5.2 follows.

Proof: By Proposition 5.2.1, $\tilde{h} \in \mathcal{S}_{k,\lambda}$ in both cases. (5.24) then implies that $\tilde{h} \in \mathcal{S}_{k,\lambda}(\mathfrak{o}, 4^{-1}\mathfrak{r}\mathfrak{c}, \varphi)$ in both cases. The relation between the Hecke eigenvalues of $\mathbf{g}$ and $\tilde{h}$ follows from Theorem 6.7 of [Sh 88]. (This type of relation goes back to Shintani's paper [S 75].) The statement on the Hecke eigenvalues of $\tilde{f}$ then follows from Proposition 4.3. Note that we can always define a form $\tilde{f}$ by $\mu_{\tilde{f}}(\xi, \mathfrak{m}) = \mu_{\tilde{h}}(\xi/\tau, \mathfrak{q}\mathfrak{r}\mathfrak{m})$, analogously to (5.1). *A priori*, however, we know only that $\tilde{f} \in \mathcal{S}_{k,\lambda}(\mathfrak{r}\mathfrak{b}, \mathfrak{b}', \psi)$. Removing $\mathfrak{r}$ from the level of $\tilde{f}$ will be the job of section 7, and this will complete the proof of Theorem 5.2. ∎

6. More on Theorem 5.1.

The purpose of this section is to derive the equality (5.4) of section 5. This will complete our proof of Theorem 5.1, except for some remarks that we shall make at the end of this section, concerning the conditions under which $\mathbf{g}_\tau$ is a cusp form. We will keep the same notation as in section 5. It will be enough to show (5.4) in the case where the multiplicative symbols are of the form

$$\mathrm{M}(\mathfrak{m}) = \chi^*(\mathfrak{m})\mathrm{N}(\mathfrak{m})^{-s}. \tag{6.1}$$

Here the real part of s is sufficiently large, and χ is any Hecke character with conductor $\mathfrak{o}$, satisfying

$$\forall x \in F_{\mathbf{a}}^{\times}, \chi_{\mathbf{a}}(x) = |x|^{i\rho}, \tag{6.2}$$

for some $\rho \in \mathbf{R}^{\mathbf{a}}$ such that $\|\rho\| = 0$. This will imply the desired equality for arbitrary systems of multiplicative symbols (see page 805 of [Sh 87]).

We prove (5.4) following the argument in section 7 of [Sh 87]. We start with equation (5.30) from the previous section:

$$Cg_{\tau\lambda}(w) = \int_{\Phi} \theta(z, w; \eta_\lambda) h(z) y^k d_H z. \tag{6.3}$$

Up to a constant, this is the same integral as equation (7.6) of [Sh 87]. Namely, h, φ, $\mathfrak{r}\mathfrak{c}$, and $\Gamma[2\mathfrak{d}^{-1}, 2^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{d}]$ here correspond to f, ψ, $\mathfrak{c}$, and Γ there; η_λ here is $(\varphi_{\mathfrak{r}\mathfrak{c}}/\varphi)(t_\lambda) D_F^{1/2} \mathrm{N}(2^{-1}\mathfrak{c}\mathfrak{t}_\lambda)$ times the function of [Sh 87], equations (7.7) and (7.8); and $2^{-1}\mathfrak{r}^{-1}\mathfrak{t}_\lambda$ here corresponds to $\mathfrak{x}$ there. Note that Shimura assumes in [Sh 87] that $\mathfrak{x}$ is prime to $\mathfrak{c}$ there, which in our case translates to $2^{-1}\mathfrak{r}^{-1}\mathfrak{t}_\lambda$ being prime to $\mathfrak{r}\mathfrak{c}$. This can be arranged by a suitable choice of the representatives $\{\mathfrak{t}_\lambda\}$ of the ideal classes mod $\mathbf{a}$. In that case, $(\varphi_{\mathfrak{r}\mathfrak{c}}/\varphi)(t_\lambda)$ becomes equal to $\varphi^*(2\mathfrak{r}\mathfrak{t}_\lambda^{-1})$.

The main point is to compute the Dirichlet series of $\mathbf{g}_\tau$, by taking the Mellin transform of the $g_{\tau\lambda}$. To this end, recall from (3.14) that $K = \{y \in \mathbf{R}^{\mathbf{a}} \mid y \gg 0\}/\mathfrak{o}_+^{\times}$, and consider the integral

$$\int_K g_{\tau\lambda}(ir) r^{m+su-u+i\rho+i\mu}\, dr. \tag{6.4}$$

This integral converges (for the real part of s sufficiently large) if and only if the constant term $c_0(y; g_{\tau\lambda})$ vanishes. In that case, since $\mathbf{g}_\tau$ is "even," we invoke Proposition 3.2 and Proposition 3a.1 to conclude that the expression in (6.4) is equal to

$$(6.5)\qquad 2^{-\|u+2m\|}\pi^{-\|su+m+u/2\|}\sum_{\xi\in F^\times/\mathfrak{o}_+^\times}\mathrm{c}(\xi\mathfrak{t}_\lambda^{-1}, u; \mathbf{g}_\tau)\mathrm{M}(\xi\mathfrak{o})X(s, \operatorname{sgn}\xi).$$

Here, for $s\in\mathbf{C}$ and $\sigma\in\{\pm1\}^{\mathbf{a}}$, we define

$$(6.6)\ X(s,\sigma) = \sum_{0\le q\le m}\binom{m}{q}2^{\|q\|}\sigma^q(su+u+i\mu+i\rho+q-m)_{m-q}$$
$$\times\Gamma((su+i\mu+i\rho+m+q)/2+\beta)\Gamma((su+i\mu+i\rho+m+q)/2+\gamma).$$

Here q runs over elements of $\mathbf{Z}^{\mathbf{a}}$, and the symbols $\binom{m}{q}$, $\Gamma(x)$, and $(x)_n$ for $x\in\mathbf{C}^{\mathbf{a}}$ and $0\le n\in\mathbf{Z}^{\mathbf{a}}$ are defined as in (0.13). This means that X is the product over all $v\in\mathbf{a}$ of the local Gamma-factors Y, after removing the powers of 2 and π. Recall that $\mathbf{g}_\tau$ has parameters 2β and 2γ.

Replace $g_{\tau\lambda}$ in (6.4) with its value as given in (6.3). An ingenious manipulation of $\theta(z, w; \eta_\lambda)$, as in equations (7.17) through (7.19) of [Sh 87], then yields that the expression in (6.4) is equal to

$$(6.7)\ \sum_{0\le n\le m}\sum_{\mathfrak{a}_\beta}\binom{m}{n}2^{\|n-2m+su-2u\|/2}\pi^{-\|m+su+u\|/2}$$
$$\times\Gamma((su+u+n+i\mu+i\rho)/2)\mathrm{M}(\mathfrak{r}^{-1}\mathfrak{t}_\lambda)L^+_{\mathfrak{rc}}(s+1, \varphi\chi, 2\mathfrak{r}\mathfrak{t}_\lambda^{-1}\mathfrak{a}_\beta)$$
$$\times\sum_{\xi\in F^\times/\mathfrak{o}_+^\times}\mu_h(1,\xi\mathfrak{a}_\beta^{-1})\mathrm{M}(\xi\mathfrak{a}_\beta^{-1})$$
$$\times\int_{y\gg0}H_{m-n}(\sqrt{4\pi y}\operatorname{sgn}\xi)W_{\beta,\gamma}(y/2)\mathbf{e}_{\mathbf{a}}(iy/2)y^{(su-2u+m+i\mu+i\rho)/2}\,dy.$$

Here $\mathfrak{a}_\beta$ runs over a set of representatives for the ideal classes mod $\mathbf{a}$, in such a way that each $\mathfrak{a}_\beta$ is prime to $\mathfrak{rc}$. (See page 802 and (6.12b) of [Sh 87]: we are essentially summing over a set of cusps.) Also, L^+ is the L-series defined in equation (6.17b) of [Sh 87]. (We give the definition of L^+ below, in (6.10).) We will rewrite $\mu_h(1,\xi\mathfrak{a}_\beta^{-1})$ as $\mu_f(\tau, \mathfrak{q}^{-1}\mathfrak{r}^{-1}\cdot\xi\mathfrak{a}_\beta^{-1})$. $H_n(x)$ is the product of the Hermite polynomials $\prod_{v\in\mathbf{a}}H_{n_v}(x_v)$, which are defined in equation (2.1) of [Sh 87]; see also (6a.5) and (6a.6) in the appendix to this section. The derivation of (6.7) proceeds by showing that $\int_K\theta(z, ir; \eta_\lambda)r^{m+su-u+i\rho+i\mu}\,dr$ is a linear combination of products of Eisenstein series with theta-functions of the form $\theta_l(z) = y^{-l/2}\sum_{b\in\mathfrak{o}}H_l(\sqrt{4\pi y}b)\mathbf{e}_{\mathbf{a}}(b^2z/2)$ (see (7.14) of [Sh 87]). Thus the Mellin transform of $g_{\tau\lambda}$ is the sum of Rankin-Selberg integrals involving h and θ_l, which ultimately yield (6.7).

As is shown in the appendix to this section, the sum over n in (6.7) simplifies miraculously:

$$(6.8)\ \sum_{0\le n\le m}\binom{m}{n}2^{\|n\|/2}\Gamma((su+u+n+i\mu+i\rho)/2)$$
$$\times\int_{y\gg0}H_{m-n}(\sqrt{4\pi y}\operatorname{sgn}\xi)W_{\beta,\gamma}(y/2)\mathbf{e}_{\mathbf{a}}(iy/2)y^{(su-2u+m+i\mu+i\rho)/2}\,dy$$
$$= (2\pi)^{-\|su+m\|/2}2^{-\|m\|/2}X(s, \operatorname{sgn}\xi),$$

with the same X as before (!). Replace this in (6.7), and equate it with (6.5), so

$$\tag{6.9} \begin{aligned} &\sum_{\xi \in F^\times/\mathfrak{o}_+^\times} \mathrm{c}(\xi \mathfrak{t}_\lambda^{-1}, u; \mathbf{g}_\tau) \mathrm{M}(\xi \mathfrak{t}_\lambda^{-1}) X(s, \operatorname{sgn} \xi) \\ &\quad = \sum_{\mathfrak{a}_\beta} L^+_{\mathfrak{r}\mathfrak{c}}(s+1, \varphi\chi, 2\mathfrak{r}\mathfrak{t}_\lambda^{-1}\mathfrak{a}_\beta) \\ &\qquad \times \sum_{\xi \in F^\times/\mathfrak{o}_+^\times} \mu_f(\tau, \mathfrak{q}^{-1}\mathfrak{r}^{-1} \cdot \xi \mathfrak{a}_\beta^{-1}) \mathrm{M}(\mathfrak{r}^{-1} \cdot \xi \mathfrak{a}_\beta^{-1}) X(s, \operatorname{sgn} \xi). \end{aligned}$$

Moreover, the L-series can be rewritten as

$$\tag{6.10} \begin{aligned} L^+_{\mathfrak{r}\mathfrak{c}}(s+1, \varphi\chi, 2\mathfrak{r}\mathfrak{t}_\lambda^{-1}\mathfrak{a}_\beta) &= \sum_{\mathfrak{n} \sim 2\mathfrak{r}\mathfrak{t}_\lambda^{-1}\mathfrak{a}_\beta} (\varphi\chi)^*(\mathfrak{n}) \mathrm{N}(\mathfrak{n})^{-s-1} \\ &= \sum_{\mathfrak{n} \sim 2\mathfrak{r}\mathfrak{t}_\lambda^{-1}\mathfrak{a}_\beta} \varphi^*(\mathfrak{n}) \mathrm{N}(\mathfrak{n})^{-1} \mathrm{M}(\mathfrak{n}), \end{aligned}$$

where $\mathfrak{n}$ ranges through the integral ideals prime to $\mathfrak{r}\mathfrak{c}$ that are in the same ideal class mod $\mathbf{a}$ as $(2)\mathfrak{r}\mathfrak{t}_\lambda^{-1}\mathfrak{a}_\beta$. Now $\mathfrak{a}_\beta$ varies through the ideal classes mod $\mathbf{a}$. Thus each integral ideal $\mathfrak{n}$ that is prime to $\mathfrak{r}\mathfrak{c}$ can be written as $\alpha\mathfrak{r}\mathfrak{t}_\lambda^{-1}\mathfrak{a}_\beta$, with a totally positive element $\alpha \in F^\times$ and a unique $\mathfrak{a}_\beta$. Rearrange the right-hand side of (6.9) by replacing the sum over ξ by one over $\xi' = \alpha\xi$. Since $\operatorname{sgn} \xi = \operatorname{sgn} \xi'$ and $\mathfrak{r}^{-1} \cdot \xi\mathfrak{a}_\beta^{-1} = \xi'\mathfrak{n}^{-1}\mathfrak{t}_\lambda^{-1}$, this yields

$$\tag{6.11} \begin{aligned} &\sum_{\xi \in F^\times/\mathfrak{o}_+^\times} \mathrm{c}(\xi \mathfrak{t}_\lambda^{-1}, u; \mathbf{g}_\tau) \mathrm{M}(\xi \mathfrak{t}_\lambda^{-1}) X(s, \operatorname{sgn} \xi) \\ &\quad = \sum_{\mathfrak{n}+\mathfrak{r}\mathfrak{c}=\mathfrak{o}} \varphi^*(\mathfrak{n}) \mathrm{N}(\mathfrak{n})^{-1} \mathrm{M}(\mathfrak{n}) \\ &\qquad \times \sum_{\xi' \in F^\times/\mathfrak{o}_+^\times} \mu_f(\tau, \mathfrak{q}^{-1} \cdot \xi'\mathfrak{n}^{-1}\mathfrak{t}_\lambda^{-1}) \mathrm{M}(\xi'\mathfrak{n}^{-1}\mathfrak{t}_\lambda^{-1}) X(s, \operatorname{sgn} \xi'). \end{aligned}$$

Add up this equation over all $\mathfrak{t}_\lambda$ to obtain

$$\tag{6.12} \begin{aligned} &\sum_{\mathfrak{m}} \mathrm{c}(\mathfrak{m}, u; \mathbf{g}_\tau) \mathrm{M}(\mathfrak{m}) \sum_{\substack{\xi \in F^\times/\mathfrak{o}_+^\times,\, \mathfrak{t}_\lambda \\ \xi\mathfrak{t}_\lambda^{-1} = \mathfrak{m}}} X(s, \operatorname{sgn} \xi) \\ &\quad = \sum_{\mathfrak{n}+\mathfrak{r}\mathfrak{c}=\mathfrak{o}} \varphi^*(\mathfrak{n}) \mathrm{N}(\mathfrak{n})^{-1} \mathrm{M}(\mathfrak{n}) \\ &\qquad \times \sum_{\mathfrak{m}} \mu_f(\tau, \mathfrak{q}^{-1}\mathfrak{m}) \mathrm{M}(\mathfrak{m}) \sum_{\substack{\xi' \in F^\times/\mathfrak{o}_+^\times,\, \mathfrak{t}_\lambda \\ \xi'\mathfrak{n}^{-1}\mathfrak{t}_\lambda^{-1} = \mathfrak{m}}} X(s, \operatorname{sgn} \xi'). \end{aligned}$$

Finally, note that the innermost sum on both sides of the above equation is merely $\sum_{\sigma \in \{\pm 1\}^{\mathbf{a}}} X(s, \sigma)$, since for a given ideal $\mathfrak{m}$ and a given signature, there is exactly one $\mathfrak{t}_\lambda$ and exactly one ξ with the given signature (up to a totally positive unit), such that $\xi\mathfrak{t}_\lambda^{-1} = \mathfrak{m}$; similarly for the innermost sum on the right-hand side. Fortunately, $\sum_{\sigma\in\{\pm1\}^{\mathbf{a}}} X(s,\sigma)$ is a nonzero function of s, as observed in the appendix to section 3, and can be cancelled. This completes the proof of (5.4).

We conclude this section by remarking that the proof in section 7 of [Sh 87] (in particular, equations (7.24) through (7.25)) shows that the possible constant terms of $g_{\tau\lambda}(w)$ are all linear combinations of expressions $\langle h(z), \theta_m(z)\rangle v^{u-m}$, where $\theta_m(z)$ is a weight k theta-function of the type defined in [Sh 87], (4.5); these amount to inner products of f with the theta-series of Theorem 5.1, (4). One can show that for any $v \in \mathbf{a}$,

$$L_v^k \theta_m = (m_v(m_v - 1)/4)\theta_m. \tag{6.13}$$

Therefore, since different eigenspaces for L_v^k are mutually orthogonal, the only way in which we could get a constant term would be if h (hence f) and θ_m had the same eigenvalues for L_v^k, for all $v \in \mathbf{a}$. In this case, the β and γ parameters of f are $-m/2$ and $(1-m)/2$, and thus f is essentially the same as the holomorphic form

$$f_{hol} = \left(\prod_{v\in\mathbf{a}} (\epsilon_v)^{\lfloor m/2\rfloor}\right) f. \tag{6.14}$$

We see that f_{hol} has weight 1/2 or 3/2 at all archimedean places. By Theorem 6.4 of [Sh 87], there can exist nonzero f_{hol} orthogonal to the theta-series only if f_{hol} has weight $3u/2$. In fact, a stronger result holds:

Proposition 6.1. *Let $f \in \mathcal{S}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi)$ be holomorphic and of weight* 1/2 *at* one *archimedean place $v \in \mathbf{a}$, so $m_v = 0$ and $\lambda_v = 0$. Assume that $\mu_f(\tau, \mathfrak{m}) \neq 0$ for some $\tau \gg 0$ and some $\mathfrak{m}$ (this can be arranged by applying flipping operators to f at some other archimedean places, if necessary; refer to the discussion at the end of section 1). Then f is a theta-series of the form $\theta_m(bz)$, with $b \gg 0$, as in Theorem 5.1 (4). In particular, f is essentially the same as a holomorphic form, and f_{hol} (as defined by (6.14)) is a holomorphic theta-series of the above type.*

Proof: If f is orthogonal to the space of all $\theta_m(bz)$, then the corresponding form $\mathbf{g}_\tau$, as given by Theorem 5.1, is a cusp form which is holomorphic and of weight 0 at the v-place. Since $\epsilon_v \mathbf{g}_\tau = 0$, we see from (2.13) that $\mathrm{c}(\mathfrak{m}, \sigma; \mathbf{g}_\tau) = 0$ unless $\sigma_v > 0$. Since $\mathbf{g}_\tau$ is even, however, $\mathrm{c}(\mathfrak{m}, (-1)_v\sigma; \mathbf{g}_\tau) = \mathrm{c}(\mathfrak{m}, \sigma; \mathbf{g}_\tau)$, so all of the Fourier coefficients of $\mathbf{g}_\tau$ vanish. This contradicts the fact that $\mathbf{g}_\tau \neq 0$. ∎

Appendix: Proof of (6.8).

The identity (6.8) is the product over all places $v \in \mathbf{a}$ of the following identity, where we will drop the v-subscripts and absorb $i\mu_v + i\rho_v$ into s to simplify the notation:

$$\begin{aligned} &\sum_{n=0}^{m} \binom{m}{n} 2^{n/2}\Gamma((s+1+n)/2) \\ &\qquad \times \int_0^\infty H_{m-n}(\sqrt{4\pi y}\,\mathrm{sgn}\,\xi) V(2\pi y; \beta, \gamma) e^{-\pi y} y^{(s+m)/2-1}\, dy \\ &= (2\pi)^{-(s+m)/2} 2^{-m/2} \sum_{q=0}^{m} \binom{m}{q} 2^q (\mathrm{sgn}\,\xi)^q (s-m+q+1)_{m-q} \\ &\quad \times \Gamma((s+m+q)/2+\beta)\Gamma((s+m+q)/2+\gamma). \end{aligned} \tag{6a.1}$$

After we rescale y by 2π, this equation will follow from the two identities

$$\int_0^\infty V(y;\beta,\gamma)e^{-y/2}y^{s/2-1}\,dy = \frac{\Gamma(s/2+\beta)\Gamma(s/2+\gamma)}{\Gamma(s/2+\beta+\gamma)} \tag{6a.2}$$

(where we can replace $\beta+\gamma$ by $1/2-m$), and

$$\begin{aligned}\sum_{n=0}^{m}\binom{m}{n}2^{n/2}\Gamma((s+1+n)/2)H_{m-n}(T) \\ = \sum_{q=0}^{m}\binom{m}{q}2^{(q-m)/2}\Gamma((s-m+q+1)/2)(s-m+q+1)_{m-q}T^q,\end{aligned} \tag{6a.3}$$

where T will be replaced with $\sqrt{2y}\,\operatorname{sgn}\xi$.

We need only prove (6a.2) under the assumption that β and s have sufficiently large real parts, since we can then invoke analytic continuation. In that case, we obtain from (1.13) that the integral in (6a.2) equals

$$\Gamma(s/2+\beta)\Gamma(\beta)^{-1}\int_0^\infty (1+t)^{-s/2-\beta-\gamma}t^{\beta-1}\,dt. \tag{6a.4}$$

As in the appendix to section 3, the above integral is a Beta-integral. This yields (6a.2).

We prove (6a.3) by a tedious brute-force argument, which boils down to computing the coefficient of T^q in the left-hand side (which is a polynomial in T of degree m). To do this, we expand

$$H_n(T) = \sum_{k=\lceil n/2\rceil}^{n}\frac{n!\,(-2)^{k-n}}{(2k-n)!\,(n-k)!}T^{2k-n}. \tag{6a.5}$$

This can be shown by induction from the definition in [Sh 87], (2.1), or by writing

$$\begin{aligned}H_n(T) &= \exp(T^2/2)\left(-\frac{d}{dT}\right)^n\exp(-T^2/2) \\ &= \exp(T^2/2)\left(\frac{-1}{T}\frac{d}{da}\right)^n\bigg|_{a=1}\exp(-a^2T^2/2) \\ &= T^{-n}\left(-\frac{d}{da}\right)^n\bigg|_{a=1}\exp((1-a^2)T^2/2) \\ &= \sum_{k\geq 0}\left(\left(-\frac{d}{da}\right)^n\bigg|_{a=1}(1-a^2)^k\right)\frac{T^{2k-n}}{2^k k!}.\end{aligned} \tag{6a.6}$$

(Expand the exponential into its power series.) We can then calculate by Leibniz' rule that

$$\left(-\frac{d}{da}\right)^n\bigg|_{a=1}(1-a)^k(1+a)^k = \frac{k!\,n!\,(-1)^{n-k}2^{2k-n}}{(n-k)!\,(2k-n)!} \tag{6a.7}$$

provided $\lceil n/2 \rceil \le k \le n$; otherwise, the value is 0. Thus for $0 \le q \le m$, the coefficient of T^q in the left side of (6a.3) is

$$\sum_{\substack{n,k \\ 2k-m+n=q}} \frac{m!}{n!\,(m-n)!} 2^{n/2}\Gamma((s+n+1)/2)\frac{(m-n)!\,(-2)^{k-m+n}}{(2k-m+n)!\,(m-n-k)!} \tag{6a.8}$$

where n and k range over integers, under the conditions $0 \le n \le m$, $\lceil (m-n)/2 \rceil \le k \le m-n$, and (of course) $2k-m+n=q$. We wish to show that the above sum is equal to $\binom{m}{q}2^{(q-m)/2}\Gamma((s-m+q+1)/2)(s-m+q+1)_{m-q}$. Write $p=m-k$; thus $n=2p+q-m$, and p ranges over integers between $(m-q)/2$ and $m-q$. The expression in (6a.8) then becomes

$$\begin{aligned} &\binom{m}{q}2^{(q-m)/2}\Gamma((s-m+q+1)/2) \\ &\quad\times \sum_{p=\lceil (m-q)/2 \rceil}^{m-q} \frac{(m-q)!\,2^{2p+q-m}(-1)^{p+q-m}}{(2p+q-m)!\,(m-q-p)!}((s-m+q+1)/2)_p. \end{aligned} \tag{6a.9}$$

Now the sum over p in the above expression is $(s-m+q+1)_{m-q}$. To see this, observe that it can be written as

$$\sum_{p \ge 0} \left(-\frac{d}{da}\right)^{m-q}\bigg|_{a=1} \frac{(1-a^2)^p}{p!}((s-m+q+1)/2)_p. \tag{6a.10}$$

By the binomial expansion, this becomes

$$\left(-\frac{d}{da}\right)^{m-q}\bigg|_{a=1} (1-(1-a^2))^{-(s-m+q+1)/2}. \tag{6a.11}$$

This last expression is just $(s-m+q+1)_{m-q}$, as desired. This concludes the calculation.

7. More on Theorem 5.2.

In this section, we will look more closely at the map from forms of even weight to forms of half-integral weight. Let $\mathbf{g} = (g_\lambda)_\lambda \in \mathcal{S}_{2m,4\lambda}(\mathfrak{o}, 2^{-1}\mathfrak{c}, \psi^2)$, and define

$$\tilde{h}(z) = \sum_{\lambda} \langle \theta(z,w;\eta_\lambda), g_\lambda(w) \rangle = \langle \Theta(z,p;\eta), \mathbf{g}(p) \rangle. \tag{7.1}$$

By the preliminary version of Theorem 5.2, $\tilde{h} \in \mathcal{S}_{k,\lambda}(\mathfrak{o}, 4^{-1}\mathfrak{rc}, \varphi)$, so $\mu_{\tilde{h}}(\xi, \mathfrak{m}) = 0$ unless $\xi \in \mathfrak{m}^{-2}$. We wish to prove the stronger result that

$$\mu_{\tilde{h}}(\xi, \mathfrak{m}) = 0 \quad \text{unless } \xi \in \mathfrak{rm}^{-2}. \tag{7.2}$$

Then, by Proposition 4.2, $\tilde{h}$ comes from a form $\tilde{f} \in \mathcal{S}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi)$, as in (5.1). This will complete the proof of Theorem 5.2.

Lemma 7.1. *Let $\tilde{\eta} \in \mathcal{S}(V_{\mathbf{f}})$ satisfy*

$$\tilde{\eta}(\alpha) = 0, \quad \text{unless } (\det \alpha)\mathfrak{o} \subset \mathfrak{r}, \tag{7.3}$$

for some ideal $\mathfrak{r}$. Put $\tilde{h}(z) = \langle \Theta(z, p; \tilde{\eta}), \mathbf{g}(p) \rangle$, for $\mathbf{g}$ as above. Then

$$\mu_{\tilde{h}}(\xi, \mathfrak{m}) = 0, \quad \text{unless } \xi \in \mathfrak{r}\mathfrak{m}^{-2}. \tag{7.4}$$

Proof: For a given $\mathfrak{m}$, Lemma 6.4 of [Sh 88] (or equation (5.22) of [Sh 93a]) implies

$$\sum_{\xi \in F^\times} \mu_{\tilde{h}}(\xi, \mathfrak{m}) W_{\beta,\gamma}(\xi y/2) \mathbf{e}_{\mathbf{a}}(\xi x/2) = \langle \Theta(x + iy, p; \eta_{\mathfrak{m}}), \mathbf{g}(p) \rangle, \tag{7.5}$$

where we define $\eta_{\mathfrak{m}}(\alpha) = \varphi(s)^{-1} \mathrm{N}(\mathfrak{m})^{-1} \tilde{\eta}(s\alpha)$ for some $s \in F_{\mathbf{f}}^\times$ with $\mathfrak{m} = s\mathfrak{o}$. By our assumption (7.3), $\eta_{\mathfrak{m}}(\alpha) = 0$, unless $(\det \alpha)\mathfrak{o} \subset \mathfrak{r}\mathfrak{m}^{-2}$. Hence the desired conclusion follows by expanding the theta-function into a Fourier series in x (or from Lemma 5.2 of [Sh 93a], in the holomorphic case). ∎

As it turns out, η as defined by (5.22) does not quite satisfy (7.3). However, we will find an $\tilde{\eta}$ satisfying (7.3), but which yields the same $\tilde{h}$ as in (7.1). To this end, notice that $\Theta(z, p; \eta)$ is automorphic in p with respect to $\tilde{D}[\mathfrak{r}\mathfrak{d}^{-1}, 2^{-1}\mathfrak{c}\mathfrak{d}]$, but $\mathbf{g}$ is automorphic with respect to the slightly larger group $\tilde{D}[\mathfrak{d}^{-1}, 2^{-1}\mathfrak{c}\mathfrak{d}]$. Thus, if $w \in \tilde{G}_{\mathbf{f}} \cap \tilde{D}[\mathfrak{d}^{-1}, 2^{-1}\mathfrak{c}\mathfrak{d}]$, then

$$\tilde{h}(z) = \left\langle \Theta(z, pw^{-1}; \eta), \mathbf{g}(pw^{-1}) \right\rangle = \left\langle \overline{\psi}^2_{2^{-1}\mathfrak{c}}(d_{w^{-1}}) \varphi(\det w)^{-1} \Theta(z, p; \eta^w), \mathbf{g}(p) \right\rangle. \tag{7.6}$$

(This uses (2.16) and (5.16).) Of course, if $w \in \tilde{D}[\mathfrak{r}\mathfrak{d}^{-1}, 2^{-1}\mathfrak{c}\mathfrak{d}]$, then we have gained nothing, since then $\overline{\psi}^2_{2^{-1}\mathfrak{c}}(d_{w^{-1}}) \varphi(\det w)^{-1} \eta^w = \eta$. So take a finite set $W \subset \tilde{G}_{\mathbf{f}}$ of coset representatives for $\tilde{D}[\mathfrak{r}\mathfrak{d}^{-1}, 2^{-1}\mathfrak{c}\mathfrak{d}] \backslash \tilde{D}[\mathfrak{d}^{-1}, 2^{-1}\mathfrak{c}\mathfrak{d}]$, and define

$$\tilde{\eta}(\alpha) = |W|^{-1} \sum_{w \in W} \overline{\psi}^2_{2^{-1}\mathfrak{c}}(d_{w^{-1}}) \varphi(\det w)^{-1} \eta^w(\alpha). \tag{7.7}$$

Each term in this sum is independent of the choice of w in its coset. $|W|$ is the cardinality of W.

Proposition 7.1.1. *$\tilde{\eta}$ as above satisfies (7.3), and $\tilde{h}(z) = \langle \Theta(z, p; \tilde{\eta}), \mathbf{g}(p) \rangle$, with the same $\tilde{h}$ as in (7.1). Combined with Lemma 7.1, this completes the proof of (7.2) and hence of Theorem 5.2.*

Proof: The second assertion is clear from (7.6). We prove (7.3) locally at each $v \in \mathbf{f}$. This is sufficient, since we can pick the representatives in W independently at each finite prime v, and since η is a product of local Schwartz functions η_v. If $v \nmid \mathfrak{r}$, our assertion follows trivially from the definition of η_v.

We will thus assume that $v \mid \mathfrak{r}$. Since $\mathfrak{r}$ is integral and squarefree, $\mathfrak{r}_v$ is the maximal ideal of $\mathfrak{o}_v$. Although we shall be working locally, we shall for notational convenience drop all the v-subscripts. For instance, we shall write $\mathbf{e}(x)$ instead of $\mathbf{e}_v(x_v)$ and φ instead of φ_v. We recall the local definition of η in this notation. η is supported on $S = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \;\middle|\; a \in \mathfrak{o}, b \in 2\mathfrak{c}^{-1}\mathfrak{d}^{-1}, c \in 2^{-1}\mathfrak{c}\mathfrak{d} \right\}$. For an element of S, we defined in (5.22) that

$$\eta\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right) = \sum_{t \in (\mathfrak{o}/\mathfrak{r}\mathfrak{c})^\times} \varphi(2^{-1}r^{-1}t)^{-1}\mathbf{e}(2^{-1}r^{-1}bt). \tag{7.8}$$

Here we have picked once and for all a uniformizer r; that is, $r\mathfrak{o} = \mathfrak{r}$. We shall take δ to be the local v-component of the global idele δ (so $\delta\mathfrak{o} = \mathfrak{d}$). We now pick our (local) representatives for W. If $\mathfrak{r} \mid 2^{-1}\mathfrak{c}$ (equivalently, $\mathfrak{r} \mid \mathfrak{c}$), then a set of representatives for $\tilde{D}[\mathfrak{r}\mathfrak{d}^{-1}, 2^{-1}\mathfrak{c}\mathfrak{d}]\backslash\tilde{D}[\mathfrak{d}^{-1}, 2^{-1}\mathfrak{c}\mathfrak{d}]$ is just the set $\left\{\begin{pmatrix} 1 & \delta^{-1}\lambda \\ 0 & 1 \end{pmatrix}\right\}$, as λ runs through a set of representatives for $\mathfrak{o}/\mathfrak{r}$. If $\mathfrak{r} \dagger \mathfrak{c}$, then we must include one other representative, namely $\begin{pmatrix} 0 & -\delta^{-1} \\ \delta & 0 \end{pmatrix}$. With this choice of representatives, the factors in (7.7) involving ψ and φ become 1. (Note especially that if $\mathfrak{r} \dagger \mathfrak{c}$, then the factor involving $\psi^2_{2^{-1}\mathfrak{c}}$ makes no local contribution.) Another point is that conjugating by any of these representatives preserves S, whence $\tilde{\eta}$ is also supported on S. So we need to prove (7.3) only for $\alpha = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in S$. Using our choice of representatives, we see that we must show that for $a \in \mathfrak{o}$, $b \in 2\mathfrak{c}^{-1}\mathfrak{d}^{-1}$, and $c \in 2^{-1}\mathfrak{c}\mathfrak{d}$,

$$\begin{aligned} &\sum_{\lambda \in \mathfrak{o}/\mathfrak{r}} \sum_{t \in (\mathfrak{o}/\mathfrak{r}\mathfrak{c})^\times} \varphi(t)^{-1}\mathbf{e}(2^{-1}r^{-1}t(b - 2a\delta^{-1}\lambda - c\delta^{-2}\lambda^2)) \\ &\quad +_{\text{if } \mathfrak{r}\dagger\mathfrak{c}} \sum_{t \in (\mathfrak{o}/\mathfrak{r}\mathfrak{c})^\times} \varphi(t)^{-1}\mathbf{e}(-2^{-1}r^{-1}t\delta^{-2}c) \\ &= 0, \quad \text{unless } a^2 + bc \in \mathfrak{r}. \end{aligned} \tag{7.9}$$

(We have eliminated the irrelevant factor $|W|^{-1}\varphi(2r)$.) We divide the proof of (7.9) into several cases.

Case I: $c \in 2\mathfrak{r}\mathfrak{d}$. Here $\mathbf{e}(-2^{-1}r^{-1}tc\delta^{-2}\lambda^2) = \mathbf{e}(-2^{-1}r^{-1}t\delta^{-2}c) = 1$, so we can reduce the calculation considerably. The term in (7.9) corresponding to $\begin{pmatrix} 0 & -\delta^{-1} \\ \delta & 0 \end{pmatrix}$ (which is present only when $\mathfrak{r} \dagger \mathfrak{c}$) becomes $\sum_t \varphi(t)^{-1} = 0$, since φ is ramified if $\mathfrak{r} \dagger \mathfrak{c}$. (Indeed, we then have $\mathfrak{c} = \mathfrak{o}$, so ψ is unramified, and φ coincides on $\mathfrak{o}^\times$ with ϵ_τ. Now $\tau\mathfrak{b} = \mathfrak{q}^2\mathfrak{r}$ and $\mathfrak{b} = \mathfrak{o}$, so the local field extension by $\sqrt{\tau}$ is ramified; hence ϵ_τ is just the quadratic character on $(\mathfrak{o}/\mathfrak{r})^\times$. Note that $\mathfrak{o}/\mathfrak{r}$ has odd characteristic, since $\mathfrak{r} \dagger \mathfrak{c} = 4\mathfrak{b}\mathfrak{b}'$.) So whether or not $\mathfrak{r}$ divides $\mathfrak{c}$, we are left with the sum

$$\sum_{t \in (\mathfrak{o}/\mathfrak{r}\mathfrak{c})^\times} \varphi(t)^{-1}\mathbf{e}(2^{-1}r^{-1}tb) \sum_{\lambda \in \mathfrak{o}/\mathfrak{r}} \mathbf{e}(-r^{-1}ta\delta^{-1}\lambda). \tag{7.10}$$

The inner sum is 0 if $a \notin \mathfrak{r}$, and is $\mathrm{N}(\mathfrak{r})$ if $a \in \mathfrak{r}$. We thus need to show that the Gauss sum $G(b) = \sum_{t \in (\mathfrak{o}/\mathfrak{r}\mathfrak{c})^\times} \varphi(t)^{-1}\mathbf{e}(2^{-1}r^{-1}tb)$ vanishes unless $bc \in \mathfrak{r}$. Recall that $b \in 2\mathfrak{c}^{-1}\mathfrak{d}^{-1}$ and $c \in 2^{-1}\mathfrak{c}\mathfrak{d} \cap 2\mathfrak{r}\mathfrak{d}$.

Case Ia: φ is ramified, with conductor $\mathfrak{h}$. Then $\mathfrak{h} \mid \mathfrak{c} \cap 4\mathfrak{r}$, so $c \in 2^{-1}\mathfrak{d}\mathfrak{h}$. By a standard property of Gauss sums, $G(b) = 0$ unless $b\mathfrak{o} = 2\mathfrak{r}\mathfrak{h}^{-1}\mathfrak{d}^{-1}$. Then $bc \in \mathfrak{r}$, as desired.

Case Ib: φ is unramified. In this case $G(b) = 0$ unless $b \in 2\mathfrak{d}^{-1}$, so $bc \in 4\mathfrak{r}$.

Case II: $c \in 2^{-1}\mathfrak{c}\mathfrak{d}$ but $c \notin 2\mathfrak{r}\mathfrak{d}$. Since we are working in a local field, the ideals are totally ordered by inclusion. Thus $2\mathfrak{r}\mathfrak{d} \subsetneq 2^{-1}\mathfrak{c}\mathfrak{d}$, so $\mathfrak{r} \subsetneq 4^{-1}\mathfrak{c}$. Now $4^{-1}\mathfrak{c} = \mathfrak{b}\mathfrak{b}'$ is integral, and $\mathfrak{r}$ is the maximal ideal in $\mathfrak{o}$, so $\mathfrak{c} = 4\mathfrak{o}$, and $c\mathfrak{o} = 2^{-1}\mathfrak{c}\mathfrak{d} = 2\mathfrak{d}$. Write $c = 2\delta\hat{c}$, with $\hat{c} \in \mathfrak{o}^\times$.

Case IIa: $\mathfrak{r} \dagger 2$, so $\mathfrak{c} = \mathfrak{o}$ and φ restricted to $(\mathfrak{o}/\mathfrak{r})^\times$ is the quadratic character ϵ_τ. Then the left hand side of (7.9) becomes

$$\begin{aligned}(7.11)\qquad &\sum_{\lambda \in \mathfrak{o}/\mathfrak{r}} \sum_{t \in (\mathfrak{o}/\mathfrak{r})^\times} \epsilon_\tau(t)^{-1}\mathbf{e}(-4^{-1}r^{-1}t\delta^{-1}\hat{c}^{-1}((2\hat{c}\lambda + a)^2 - a^2 - bc)) \\ &\quad + \sum_{t \in (\mathfrak{o}/\mathfrak{r})^\times} \epsilon_\tau(t)^{-1}\mathbf{e}(-r^{-1}t\delta^{-1}\hat{c}).\end{aligned}$$

Since $2\hat{c} \in \mathfrak{o}^\times$, we can replace the sum over λ with one over $x = 2\hat{c}\lambda + a \in \mathfrak{o}/\mathfrak{r}$. It then follows from standard facts about Gauss sums that

$$(7.12)\qquad \epsilon_\tau(t)^{-1} \sum_{x \in \mathfrak{o}/\mathfrak{r}} \mathbf{e}(-4^{-1}r^{-1}t\delta^{-1}\hat{c}^{-1}x^2) = \sum_{x \in \mathfrak{o}/\mathfrak{r}} \mathbf{e}(-r^{-1}\delta^{-1}\hat{c}^{-1}x^2)$$

(remember that $4^{-1}t\hat{c}^{-1}$ is a unit), and

$$(7.13)\qquad \sum_{t \in (\mathfrak{o}/\mathfrak{r})^\times} \epsilon_\tau(t)^{-1}\mathbf{e}(-r^{-1}t\delta^{-1}\hat{c}) = \sum_{x \in \mathfrak{o}/\mathfrak{r}} \mathbf{e}(-r^{-1}\delta^{-1}\hat{c}^{-1}x^2).$$

Apply (7.12) and (7.13) to the first and second terms, respectively, of (7.11). The expression in (7.11) then becomes the following sum over all $t \in \mathfrak{o}/\mathfrak{r}$ (not just units):

$$(7.14)\qquad \sum_{t \in \mathfrak{o}/\mathfrak{r}} \sum_{x \in \mathfrak{o}/\mathfrak{r}} \mathbf{e}(-r^{-1}\delta^{-1}\hat{c}^{-1}x^2 + 4^{-1}r^{-1}t\delta^{-1}\hat{c}^{-1}(a^2 + bc)).$$

Sum over t first. The expression then vanishes unless $a^2 + bc \in \mathfrak{r}$, as desired.

Case IIb: $\mathfrak{r} \mid 2$. As mentioned above, $\mathfrak{c} = 4\mathfrak{o}$, and $c = 2\delta\hat{c}$, with $\hat{c} \in \mathfrak{o}^\times$. The left side of (7.9) is now

$$(7.15)\qquad \sum_{\lambda \in \mathfrak{o}/\mathfrak{r}} \sum_{t \in (\mathfrak{o}/4\mathfrak{r})^\times} \psi(t)^{-1}\epsilon_\tau(t)^{-1}\mathbf{e}(2^{-1}r^{-1}tb - r^{-1}ta\delta^{-1}\lambda - r^{-1}t\hat{c}\delta^{-1}\lambda^2).$$

We can ignore the case where $a \in \mathfrak{r}$, since then $\mathbf{e}(-r^{-1}ta\delta^{-1}\lambda) = 1$, and we get an inner sum $\sum_{\lambda \in \mathfrak{o}/\mathfrak{r}} \mathbf{e}(-r^{-1}t\hat{c}\delta^{-1}\lambda^2)$, which vanishes. (Recall that $\mathfrak{o}/\mathfrak{r}$ has characteristic 2, so $\lambda \mapsto \lambda^2$ is an automorphism.) Thus we may as well assume that $a \in \mathfrak{o}^\times$. Now we have no control over ψ, except that it has conductor dividing $4\mathfrak{o}$. Let us therefore split up the sum over t into the cosets of $1 + 4\mathfrak{o}$. This means that we will write $t = t_0(1 + 4u)$ as t_0 ranges over a set of representatives for $(\mathfrak{o}/4\mathfrak{o})^\times$, and u runs through $\mathfrak{o}/\mathfrak{r}$. We will show that the portion of (7.15) corresponding to each t_0

vanishes separately, unless $a^2 + bc \in \mathfrak{r}$. In other words, after removing the factor $\psi(t_0)^{-1}\epsilon_\tau(t_0)^{-1}$, we will show that

$$\begin{aligned}(7.16)\qquad & \sum_{\lambda\in\mathfrak{o}/\mathfrak{r}} \sum_{u\in\mathfrak{o}/\mathfrak{r}} \epsilon_\tau(1+4u)^{-1} \\ & \quad \times \mathbf{e}\Big(-4^{-1}r^{-1}\delta^{-1}\hat{c}^{-1}t_0(1+4u)((2\hat{c}\lambda+a)^2 - a^2 - bc)\Big) \\ & = 0 \quad \text{unless } a^2+bc\in\mathfrak{r}.\end{aligned}$$

We prove (7.16) by proving a formula analogous to (7.12) in characteristic 2. Write $x = a^{-1}\hat{c}\lambda$, which runs over $\mathfrak{o}/\mathfrak{r}$, since $a, \hat{c} \in \mathfrak{o}^\times$. Then we shall show below that

$$\begin{aligned}(7.17)\qquad & \epsilon_\tau(1+4u)^{-1} \sum_{x\in\mathfrak{o}/\mathfrak{r}} \mathbf{e}(-4^{-1}r^{-1}\delta^{-1}\hat{c}^{-1}a^2t_0(1+4u)(1+2x)^2) \\ & = \sum_{x\in\mathfrak{o}/\mathfrak{r}} \mathbf{e}(-4^{-1}r^{-1}\delta^{-1}\hat{c}^{-1}a^2t_0(1+2x)^2),\end{aligned}$$

and hence (7.16) becomes

$$\begin{aligned}(7.18)\ \sum_{x\in\mathfrak{o}/\mathfrak{r}} \sum_{u\in\mathfrak{o}/\mathfrak{r}} & \mathbf{e}(-4^{-1}r^{-1}\delta^{-1}\hat{c}^{-1}a^2t_0(1+2x)^2 \\ & + 4^{-1}r^{-1}\delta^{-1}\hat{c}^{-1}t_0(1+4u)(a^2+bc)).\end{aligned}$$

The inner sum over u then vanishes unless $a^2 + bc \in \mathfrak{r}$. Except for (7.17), this concludes the proof of (7.9) and hence of Proposition 7.1.1. ∎

It still remains to show (7.17). We first prove the following

Lemma 7.2. *Assuming Case IIb above, let $u \in \mathfrak{o}$ (although we only care about its value mod $\mathfrak{r}$). Then $\epsilon_\tau(1+4u) = 1$ if and only if $1+4u = (1+2y)^2$ for some $y \in \mathfrak{o}$.*

Proof: Since ϵ_τ is a quadratic character, $\epsilon_\tau((1+2y)^2) = 1$. Conversely, if $\epsilon_\tau(1+4u) = 1$, then $1+4u$ is a norm from the quadratic extension $F(\sqrt{\tau})/F$. Recall that $\tau\mathfrak{b} = \mathfrak{q}^2\mathfrak{r}$ and that $\mathfrak{b} = \mathfrak{o}$ (since $\mathfrak{c} = 4\mathfrak{b}\mathfrak{b}' = 4\mathfrak{o}$). Hence $F(\sqrt{\tau}) = F(\sqrt{r})$ for some uniformizer r, which, without loss of generality, is the same as our old r. Thus we can find $y, z \in F$, such that $1 + 4u$ is the norm of $1 + 2y + 2z\sqrt{r}$; equivalently,

$$(7.19)\qquad u = y + y^2 - z^2 r.$$

Since u is an integer, we can play around with valuations and conclude that y and z must also be integers. Then $1+4u \equiv (1+2y)^2 \pmod{4\mathfrak{r}}$. We can then use Hensel's Lemma to modify y by an element of $\mathfrak{r}$ so that this congruence will actually become an equality. ∎

Note that the multiplicative group of squares of the form $(1+2y)^2$ (with $y \in \mathfrak{o}$) is of index 2 in the multiplicative group $1 + 4\mathfrak{o}$, since the former group is the kernel of ϵ_τ, restricted to the latter group. This restriction is not trivial, otherwise we would always be able to find y such that $u = y + y^2$ in $\mathfrak{o}/\mathfrak{r}$. But the linear map $y \in \mathfrak{o}/\mathfrak{r} \mapsto y + y^2 \in \mathfrak{o}/\mathfrak{r}$ has kernel $\{0, 1\}$. This also implies that as y ranges over

$\mathfrak{o}/\mathfrak{r}$, the image of $(1+2y)^2$ in $(1+4\mathfrak{o})/(1+4\mathfrak{r})$ hits each element of the kernel of ϵ_τ (restricted to $1+4\mathfrak{o}$) exactly twice, once for y and once for $y+1$.

We can now prove (7.17). First, if $1+4u=(1+2y)^2$ as above, then $\epsilon_\tau(1+4u)=1$. Then for any unit l,

$$\begin{aligned}\sum_{x\in\mathfrak{o}/\mathfrak{r}} &\mathbf{e}(4^{-1}r^{-1}\delta^{-1}l(1+4u)(1+2x)^2)\\ &=\sum_{x\in\mathfrak{o}/\mathfrak{r}} \mathbf{e}(4^{-1}r^{-1}\delta^{-1}l(1+2(x+y))^2)\\ &=\sum_{x\in\mathfrak{o}/\mathfrak{r}} \mathbf{e}(4^{-1}r^{-1}\delta^{-1}l(1+2x)^2),\end{aligned}\tag{7.20}$$

proving the equality. On the other hand, if $1+4u$ is not of the form $(1+2y)^2$, then $\epsilon_\tau(1+4u)=-1$. In this case, we must show that

$$\sum_{x\in\mathfrak{o}/\mathfrak{r}} \mathbf{e}(4^{-1}r^{-1}\delta^{-1}l(1+4u)(1+2x)^2)+\sum_{x\in\mathfrak{o}/\mathfrak{r}} \mathbf{e}(4^{-1}r^{-1}\delta^{-1}l(1+2x)^2)=0.\tag{7.21}$$

By the remark after Lemma 7.2, the left-hand side of (7.21) equals

$$2\sum_{z\in\mathfrak{o}/\mathfrak{r}} \mathbf{e}(4^{-1}r^{-1}\delta^{-1}l(1+4z)).\tag{7.22}$$

This duly vanishes, as required. This completes the proof of Proposition 7.1.1. Theorem 5.2 is now completely proved.

8. Generalization of Theorems 3.2 and 3.4 of [Sh 93a].

Using Theorem 5.1 and Theorem 5.2 in the nonholomorphic case, we can now proceed as in [Sh 93a] to derive relations between the Fourier coefficients of $f\in\mathcal{S}_{k,\lambda}(\mathfrak{b},\mathfrak{b}',\psi)$, and the special values of Dirichlet series for the corresponding normalized Hecke eigenform $\mathbf{g}\in\mathcal{S}_{2m,4\lambda}(\mathfrak{o},2^{-1}\mathfrak{c},\psi^2)$.

We begin with some notation. The symbols τ, $\mathfrak{q}$, $\mathfrak{r}$, and φ are the same as before; $\mathfrak{h}$ is the conductor of φ; and $\mathfrak{j}=\prod_{\mathfrak{p}}\mathfrak{p}$, where the product is over prime ideals $\mathfrak{p}$ dividing $\mathfrak{r}\mathfrak{c}$ but not dividing $\mathfrak{h}$. We define the Dirichlet series that interest us as follows: take an "even" cusp form $\mathbf{f}\in\mathcal{S}_{2m,4\lambda}(\mathfrak{o},\mathfrak{z},\Psi)$. Let ω be a Hecke character, $\mathfrak{x}$ an integral ideal, and χ a system of Hecke eigenvalues. Then define

$$D(s,\mathbf{f},\omega,\mathfrak{x})=\sum_{\mathfrak{m}}\omega^*(\mathfrak{m})\,\mathrm{c}(\mathfrak{x}\mathfrak{m},u;\mathbf{f})\mathrm{N}(\mathfrak{m})^{-s},\tag{8.1}$$

$$D(s,\chi,\omega)=\sum_{\mathfrak{m}}\omega^*(\mathfrak{m})\chi(\mathfrak{m})\mathrm{N}(\mathfrak{m})^{-s-1}.\tag{8.2}$$

(These are equations (3.15a,b) of [Sh 93a].) The series in (8.2) is normalized so that 0 is the center of its critical strip. We also let $\mu(\mathfrak{m})$ be the Moebius function on the integral ideals of F, and denote by h_F the class number of F. We define $\gamma(\varphi)$ to be the Gauss sum of φ; that is,

$$\gamma(\varphi)=\sum_{0\neq t\in\mathfrak{h}^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1}}\varphi_{\mathbf{a}}(t)\varphi^*(t\mathfrak{h}\mathfrak{d})\mathbf{e}_{\mathbf{a}}(t).\tag{8.3}$$

This assumes that $\mathfrak{h}$ is nontrivial; if $\mathfrak{h}=\mathfrak{o}$, we define the Gauss sum to be $\varphi^*(\mathfrak{d})$.

The following theorem generalizes Theorem 3.4 of [Sh 93a].

Theorem 8.1. *Let $f \in \mathcal{S}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi)$ be an eigenfunction of all the Hecke operators T_v, with eigenvalues ω_v. Assume that f is orthogonal to the relevant theta-series as in Theorem 5.1, and that it has some nonzero Fourier coefficient $\mu_f(\xi, \mathfrak{m})$ with $\xi \gg 0$. Also assume that any form $\tilde{f} \in \mathcal{S}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi)$, such that $T_v \tilde{f} = \omega_v \tilde{f}$ for almost all $v \in \mathbf{f}$, is a constant multiple of f. Let $\mathbf{g} \in \mathcal{S}_{2m,4\lambda}(\mathfrak{o}, 2^{-1}\mathfrak{c}, \psi^2)$ be the normalized Hecke eigenform corresponding to f. Let $f^* \in \mathcal{S}_{k,\lambda}(\mathfrak{b}', \mathfrak{b}, \overline{\psi})$ be the "inversion" of f, as in (4.19), and $\mathbf{g}^* = J_{2^{-1}\mathfrak{c}}\mathbf{g} \in \mathcal{S}_{2m,4\lambda}(\mathfrak{o}, 2^{-1}\mathfrak{c}, \overline{\psi}^2)$ be the inversion of $\mathbf{g}$, as in (2.19). Then*

$$\begin{aligned}(8.4)\qquad &\overline{\mu(\tau, \mathfrak{q}^{-1}; f, \psi)}\mu(\tau, \mathfrak{q}^{-1}\mathfrak{b}; f^*, \overline{\psi})\langle \mathbf{g}, \mathbf{g}\rangle / \langle f, f\rangle \\ &= Q \sum_{\mathfrak{o}\supset\mathfrak{t}\supset\mathfrak{j}} \mu(\mathfrak{t})\overline{\varphi}^*(\mathfrak{t})\mathrm{N}(\mathfrak{t})^{-1} D(0, \mathbf{g}^*, \varphi, \mathfrak{t}^{-1}\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}),\end{aligned}$$

where

$$\begin{aligned}(8.5)\qquad Q = {} & h_F[\mathfrak{o}_+^\times : (\mathfrak{o}^\times)^2] 2^{\|u/2-m\|-1}\pi^{-\|m\|}\tau^m \overline{\psi_{\mathbf{a}}(\tau)}\mathrm{N}(\mathfrak{h})^{-1}\overline{\gamma(\varphi)} \\ & \times \pi^{-\|u/2\|}\Gamma(\beta+m)\Gamma(\gamma+m).\end{aligned}$$

If furthermore $\mathbf{g}^*$ is an eigenform of all the Hecke operators $\mathfrak{T}(\mathfrak{n})$, with the system of eigenvalues χ', then the expression in (8.4) is

$$(8.6)\qquad Q\mathrm{N}(\mathfrak{h}\mathfrak{r}^{-1}\mathfrak{c}^{-1})\chi'(\mathfrak{j}^{-1}\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}) \prod_{\mathfrak{p}|\mathfrak{j}} (\chi'(\mathfrak{p}) - \overline{\varphi}^*(\mathfrak{p}))\, \mathrm{c}(\mathfrak{o}, u; \mathbf{g}^*) D(0, \chi', \varphi).$$

Proof: Before doing anything, we explain why the factor $\Gamma(\beta+m)\Gamma(\gamma+m)$ is finite, which amounts to explaining why we are not evaluating the Gamma-functions at their poles. Indeed, if, say, $\beta_v + m_v$ is a nonpositive integer for some $v \in \mathbf{a}$, then $\gamma_v = 1/2 - \beta_v - m_v$ is strictly positive. Since L_v^k is positive, its eigenvalue $\lambda_v = \beta_v\gamma_v$ is nonnegative. This implies that $\beta_v = m_v = 0$, which is a contradiction by Proposition 6.1. If f is holomorphic, then $\beta = 0$ and $\gamma = u/2 - m$. In that case, $\pi^{-\|u/2\|}\Gamma(\beta+m)\Gamma(\gamma+m) = \Gamma(m)$, and we recover the statement of Theorem 3.4 of [Sh 93a].

To prove (8.4), we follow the arguments of sections 6 through 8 of [Sh 93a], making modifications to allow for nonholomorphic forms. We have seen that

$$(8.7)\qquad C\mu_f(\tau, \mathfrak{q}^{-1})\mathbf{g}(p) = C\mathbf{g}_\tau(p) = \int_\Phi \Theta(z, p; \eta)h(z)y^k\, d_H z,$$

with h, Φ, and C as in Theorem 5.1. On the other hand, we have seen in Theorem 5.2 that $\tilde{h}(z) = \langle \Theta(z, p; \eta), \mathbf{g}(p)\rangle$ is a form corresponding to an $\tilde{f} \in \mathcal{S}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi)$, and that for almost all $v \in \mathbf{f}$, $\tilde{f}$ is an eigenform of T_v, with the same eigenvalue as f. Hence $\tilde{f} = Af$ for a constant A, and $\tilde{h} = Ah$. It then follows that

$$(8.8)\qquad A\,\mathrm{vol}(\Phi)\langle h, h\rangle = \int_\Phi \overline{h(z)}\langle \Theta(z, p; \eta), \mathbf{g}(p)\rangle y^k\, d_H z = \overline{C\mu_f(\tau, \mathfrak{q}^{-1})}\langle \mathbf{g}, \mathbf{g}\rangle.$$

Since $\langle h, h\rangle = \tau^k \mathrm{N}(\mathfrak{q}\mathfrak{r})^{-1}\langle f, f\rangle$, by (4.12), we get

$$(8.9)\qquad A = \frac{2^{1+\|m+u\|} i^{-\|m\|}\tau^{-k}\mathrm{N}(\mathfrak{q}\mathfrak{r}^2\mathfrak{c})\langle \mathbf{g}, \mathbf{g}\rangle}{\mathrm{vol}(\Gamma[2\mathfrak{d}^{-1}, 2^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{d}]\backslash\mathcal{H}^{\mathbf{a}})\langle f, f\rangle}\,\overline{\mu_f(\tau, \mathfrak{q}^{-1})}.$$

This is the same as Proposition 5.8 of [Sh 93a].

Let $h^* \in \mathcal{S}_{k,\lambda}(4^{-1}\mathfrak{r}\mathfrak{c}, \mathfrak{o}, \overline{\varphi})$ be the "inversion" of h. Use Proposition 4.3 and (4.6), and remember that $\overline{\psi}$ satisfies (4.2) with $-\mu$ instead of μ, to obtain

$$\mu_{h^*}(\xi, \mathfrak{m}) = \tau^k \mathrm{N}(\mathfrak{q}\mathfrak{r})^{-1}\mu_{f^*}(\xi/\tau, \mathfrak{q}\mathfrak{r}\mathfrak{m}) = \tau^{u/2+i\mu}\mathrm{N}(\mathfrak{q}\mathfrak{r})^{-1}\mu_{f^*}(\xi\tau, \mathfrak{q}^{-1}\mathfrak{b}\mathfrak{m}). \tag{8.10}$$

Write $\mathbf{g}^* = (g'_\lambda)_\lambda$. As shown at the beginning of section 6 of [Sh 93a], $Ah(z) = \langle \Theta(z, p; \eta), \mathbf{g}(p)\rangle$ implies that

$$Ah^*(z) = \langle \Theta(z, p; \sigma), \mathbf{g}^*(p)\rangle = \sum_\lambda \langle \theta(z, w; \sigma_\lambda), g'_\lambda(w)\rangle \tag{8.11}$$

for a particular $\sigma \in \mathcal{S}(V_\mathbf{f})$, for which an exact expression is given in equation (6.5) of [Sh 93a]. (This is basically a matter of computing ${}^\gamma(\eta^\alpha)$ for suitable γ and α, as in (5.16) and (5.15).) Recall that $\sigma_\lambda(y) = \overline{\varphi}(t_\lambda)^{-1}\sigma(x_\lambda^{-1} y x_\lambda)$, analogously to (5.10).

As in Proposition 6.1 of [Sh 93a], we can express $\sum_\mathfrak{m} \mu_{h^*}(1, \mathfrak{m})\mathrm{N}(\mathfrak{m})^{-2s}$ (which is essentially $\sum_\mathfrak{m} \mu_{f^*}(\tau, \mathfrak{q}^{-1}\mathfrak{b}\mathfrak{m})\mathrm{N}(\mathfrak{m})^{-2s}$) as a Rankin-Selberg integral of h^* with a holomorphic theta-function θ_0, defined below. Let $\Gamma = \Gamma[\mathfrak{x}, \mathfrak{y}\,]$ be a congruence group whose level $\mathfrak{x}\mathfrak{y}$ is divisible by $\mathfrak{r}\mathfrak{c}$ (and hence by $\mathfrak{h}$). We define the following Eisenstein series, which are a special case of the ones used in section 4 of [Sh 93a]:

$$E(z, s; \Gamma) = \sum_{\alpha \in R} \varphi_\mathbf{a}(d_\alpha)\varphi^*(d_\alpha \mathfrak{a}_\alpha^{-1})\mathrm{N}(\mathfrak{a}_\alpha)^{2s} y^{su+(i\mu-m)/2}\|_m \alpha, \tag{8.12}$$

$$C(z, s; \Gamma) = L_{\mathfrak{x}\mathfrak{y}}(2s, \varphi)E(z, s; \Gamma). \tag{8.13}$$

Here R is a set of representatives for $P\backslash(G \cap P_\mathbf{A} D[\mathfrak{x}, \mathfrak{y}\,])$; for $\alpha \in R$, we define $\mathfrak{a}_\alpha$ by writing $\alpha = pw$ with $p \in P_\mathbf{A}$ and $w \in D[\mathfrak{x}, \mathfrak{y}\,]$, and setting $\mathfrak{a}_\alpha = d_p\mathfrak{o}$. We also define the L-series

$$L_{\mathfrak{x}\mathfrak{y}}(s, \varphi) = \sum_{\mathfrak{m}+\mathfrak{x}\mathfrak{y}=\mathfrak{o}} \varphi^*(\mathfrak{m})\mathrm{N}(\mathfrak{m})^{-s}. \tag{8.14}$$

We note that E transforms by

$$E(z, s; \Gamma)\|_m \gamma = \overline{\varphi}_{\mathfrak{x}\mathfrak{y}}(a_\gamma)E(z, s; \Gamma),\ \forall \gamma \in \Gamma. \tag{8.15}$$

Lemma 8.2. *Let* $\Gamma = \Gamma[2^{-1}\mathfrak{d}^{-1}\mathfrak{r}\mathfrak{c}, 2\mathfrak{d}\,]$*, and let* $\theta_0(z) = \sum_{b\in\mathfrak{o}} \mathbf{e_a}(b^2 z/2)$*. Then*

$$\begin{aligned}\int_{\Gamma\backslash\mathcal{H}^\mathbf{a}} & h^*(z)\overline{\theta_0(z)E(z, \overline{s}+1/2; \Gamma)}y^k\, d_H z \\ &= D_F^{-1/2} 2^{1-\|u\|}\tau^{u/2+i\mu}\mathrm{N}(\mathfrak{q}^{-1}\mathfrak{c})(2\pi)^{\|-su-m/2\|} \\ &\times \frac{\Gamma(su + (m - i\mu)/2 + \beta)\Gamma(su + (m - i\mu)/2 + \gamma)}{\Gamma(su + (m - i\mu)/2 + \beta + \gamma)} \\ &\times \sum_\mathfrak{m} \mu_{f^*}(\tau, \mathfrak{q}^{-1}\mathfrak{b}\mathfrak{m})\mathrm{N}(\mathfrak{m})^{-2s}.\end{aligned} \tag{8.16}$$

Proof: We first note that the integrand is invariant under Γ, since $\theta_0(z)$ is of weight $u/2$ with respect to $\Gamma[2\mathfrak{d}^{-1}, 2\mathfrak{d}\,]$. Note also that the Gamma-function in the denominator can be rewritten as $\Gamma(su+(u-m-i\mu)/2)$. The proof of this equality proceeds as in the proof of Proposition 6.1 of [Sh 93a], by unfolding the integral using the Eisenstein series E. The only difference is that since h^* is not necessarily holomorphic, we must write its Fourier expansion using Whittaker functions. For each $v \in \mathbf{a}$, we then get a local Gamma-factor $\int_0^\infty W_{\beta_v,\gamma_v}(y/2)e^{-\pi y}y^{s+(m_v-i\mu_v)/2-1}\,dy$, which we evaluate using (6a.2). ∎

We return to the proof of Theorem 8.1. Our expression (8.11), giving h^* in terms of a theta-integral, implies that A times the integral in (8.16) is equal to

$$\sum_{\lambda}\left\langle \int_{\Gamma\backslash\mathcal{H}^{\mathbf{a}}} \theta_0(z)\theta(z,w;\sigma_\lambda)E(z,\overline{s}+1/2;\Gamma)y^k\,d_Hz, g'_\lambda(w)\right\rangle. \tag{8.17}$$

We can again unfold the integral over the variable z, and equate the result with A times the right-hand side of (8.16). This yields equation (6.20) of [Sh 93a], slightly modified to allow for the nonholomorphy of h^*:

$$\begin{aligned} &A\cdot 2^{\|-2su-u/2-m\|}\pi^{\|u/2\|}\mathrm{N}(\mathfrak{q}\mathfrak{r})^{-1}\tau^{u/2+i\mu} \\ &\quad\times\frac{\Gamma(su+(m-i\mu)/2+\beta)\Gamma(su+(m-i\mu)/2+\gamma)}{\Gamma(su+(u-m-i\mu)/2)\Gamma(su+(u+m-i\mu)/2)} \\ &\quad\times\sum_{\mathfrak{m}}\mu_{f^*}(\tau,\mathfrak{q}^{-1}\mathfrak{b}\mathfrak{m})\mathrm{N}(\mathfrak{m})^{-2s} \\ &\quad=\sum_{\lambda}\left\langle\sum_{\beta\in B}\overline{\varphi}^*(\mathfrak{a}_\beta)\mathrm{N}(\mathfrak{a}_\beta)^{2\overline{s}+1}S_{\beta\lambda}(w,\overline{s}), g'_\lambda(w)\right\rangle. \end{aligned} \tag{8.18}$$

Here the set B is determined by the disjoint decomposition $G\cap P_{\mathbf{A}}D[2^{-1}\mathfrak{d}^{-1}\mathfrak{r}\mathfrak{c},2\mathfrak{d}\,]=\bigsqcup_{\beta\in B}P\beta\Gamma$. The ideals $\mathfrak{a}_\beta$ are as in (8.12), and run through a set of representatives for the ideal class group of F (see section 4 of [Sh 93a]). We can arrange for these representatives to all be prime to $\mathfrak{r}\mathfrak{c}$. The functions $S_{\beta\lambda}$ are given in equation (6.18) of [Sh 93a]; namely,

$$S_{\beta\lambda}(w,s)=\sum_{\xi,b}\sigma_\lambda(r\xi)[\xi,w]^{-m}|[\xi,w]/v|^{-2su-u+m-i\mu}, \tag{8.19}$$

where v is the imaginary part of w, and the sum is over pairs $(\xi,b)\in(V\times\mathfrak{a}_\beta)/\mathfrak{o}^\times$ such that $\xi\neq0$ and $\det\xi=-b^2$. In this expression, we have chosen $r\in F_{\mathbf{f}}^\times$ such that $r\mathfrak{o}=\mathfrak{a}_\beta$ and $r_v=1$ for $v\mid\mathfrak{r}\mathfrak{c}$.

Proposition 8.3. *Let q range through a set of representatives for $2^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{d}\mathfrak{t}_\lambda/\mathfrak{r}\mathfrak{c}\mathfrak{d}\mathfrak{t}_\lambda$, and let $\Gamma^\lambda=\Gamma[2\mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1},\mathfrak{r}\mathfrak{c}\mathfrak{d}\mathfrak{t}_\lambda\,]$. Then there exist functions $T_{\beta\lambda}(w,s)$, which are essentially squares of Eisenstein series of weight m in w, such that*

$$S_{\beta\lambda}(w,s)=(-1)^{\|m\|}2^{\|u\|}\sum_q T_{\beta\lambda}(w,s)\|_{2m}\begin{pmatrix}1&0\\q&1\end{pmatrix}; \tag{8.20}$$

$$\begin{aligned}&\sum_{\beta\in B}\overline{\varphi}^*(\mathfrak{a}_\beta)\mathrm{N}(\mathfrak{a}_\beta)^{2s}T_{\beta\lambda}(w,s-1/2)\\ &\quad=\mathrm{N}(2^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{d}\mathfrak{t}_\lambda)^{2s}C(w,s;\Gamma^\lambda)E(w,s;\Gamma^\lambda).\end{aligned} \tag{8.21}$$

Proof: This is the result of a delicate calculation in section 7 of [Sh 93a], which is stated there in equations (7.14a,b). ∎

We are now in a position to combine (8.18), (8.20), and (8.21) in order to express $\sum_{\mathfrak{m}} \mu_{f^*}(\tau, \mathfrak{q}^{-1}\mathfrak{b}\mathfrak{m})\mathrm{N}(\mathfrak{m})^{-2s}$ in terms of a Rankin-Selberg integral for $\mathbf{g}^*$, following section 8 of [Sh 93a]. We know that

$$g'_\lambda\|_{2m}\gamma = \overline{\psi}^2_{2^{-1}\mathfrak{c}}(a_\gamma)g'_\lambda,\ \forall\gamma \in \Gamma[\mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1}, 2^{-1}\mathfrak{c}\mathfrak{d}\mathfrak{t}_\lambda]; \tag{8.22}$$

in particular, $g'_\lambda\|_{2m}\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} = g'_\lambda$ for q as in Proposition 8.3. Hence the expression in (8.18) is equal to the value at $s = t$ of

$$\begin{aligned}(8.23)\quad &(-1)^{\|m\|}2^{\|2u\|} \\ &\times \sum_\lambda \left\langle \mathrm{N}(2^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{d}\mathfrak{t}_\lambda)^{\overline{s}+\overline{t}+1}C(z, \overline{s}+1/2; \Gamma^\lambda)E(z, \overline{t}+1/2; \Gamma^\lambda), g'_\lambda(z)\right\rangle.\end{aligned}$$

We have introduced the parameter t to allow us to evaluate (8.23) for the real part of $t - s$ sufficiently large, where everything converges nicely; we will then invoke analytic continuation. We have also written z instead of w, to avoid notational confusion when we expand everything into Fourier series.

Note first that for any ideal class λ,

$$\mathrm{vol}(\Gamma^\lambda\backslash\mathcal{H}^{\mathbf{a}}) = 2^{\|u\|}\,\mathrm{vol}(\Gamma[2\mathfrak{d}^{-1}, 2^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{d}]\backslash\mathcal{H}^{\mathbf{a}}). \tag{8.24}$$

Thus the expression in (8.23) becomes

$$\begin{aligned}(8.25)\quad &(-1)^{\|m\|}2^{\|u\|}\,\mathrm{vol}(\Gamma[2\mathfrak{d}^{-1}, 2^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{d}]\backslash\mathcal{H}^{\mathbf{a}})^{-1} \\ &\times \sum_\lambda \mathrm{N}(2^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{d}\mathfrak{t}_\lambda)^{s+t+1}\int_{\Gamma^\lambda\backslash\mathcal{H}^{\mathbf{a}}} \overline{C(z, \overline{s}+1/2; \Gamma^\lambda)E(z, \overline{t}+1/2; \Gamma^\lambda)}g'_\lambda(z)y^{2m}\,d_Hz;\end{aligned}$$

when $s = t$, this equals the quantity of (8.18). Also, we can find a finite set $A \subset G$ such that for all λ, $G \cap P_{\mathbf{A}}D[2\mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1}, \mathfrak{r}\mathfrak{c}\mathfrak{d}\mathfrak{t}_\lambda] = \bigsqcup_{\alpha\in A} P\alpha\Gamma^\lambda$. This involves taking each α in $G \cap \begin{pmatrix} r^{-1} & 0 \\ 0 & r \end{pmatrix}U$, with $r \in F_{\mathbf{f}}^\times$ and a sufficiently small open subgroup U of $G_{\mathbf{A}}$, in such a way that the ideals $\mathfrak{a}_\alpha = r\mathfrak{o}$ range over a set of representatives for the ideal classes of F. This follows by the same argument as that in Lemmas 1.3 through 1.5 of [Sh 83]. We can then unfold the integrals in (8.25) into

$$\begin{aligned}(8.26)\quad &\int_{\Gamma^\lambda\backslash\mathcal{H}^{\mathbf{a}}} \overline{C(z, \overline{s}+1/2; \Gamma^\lambda)E(z, \overline{t}+1/2; \Gamma^\lambda)}g'_\lambda(z)y^{2m}\,d_Hz \\ &\quad= \sum_{\alpha\in A} \overline{\varphi}_{\mathbf{a}}(d_\alpha)\overline{\varphi}^*(d_\alpha\mathfrak{a}_\alpha^{-1})\mathrm{N}(\mathfrak{a}_\alpha)^{2t+1} \\ &\quad\quad\times \int_{\Psi_\lambda^\alpha} g_\lambda^\alpha(z)\overline{C_\lambda^\alpha(z)}y^{tu+(u+3m-i\mu)/2}\,d_Hz,\end{aligned}$$

where we define $\Psi_\lambda^\alpha = (P \cap \alpha\Gamma^\lambda\alpha^{-1})\backslash\mathcal{H}^{\mathbf{a}}$, $g_\lambda^\alpha = g'_\lambda\|_{2m}\alpha^{-1}$, and $C_\lambda^\alpha(z) = C(z, \overline{s}+1/2; \Gamma^\lambda)\|_m\alpha^{-1}$. We can take Ψ_λ^α to be $(\mathbf{R}^{\mathbf{a}}/2\mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1}\mathfrak{a}_\alpha^{-2}) \times \{y \in \mathbf{R}^{\mathbf{a}} \mid y \gg 0\}/(\mathfrak{o}^\times)^2$. As in section 8 of [Sh 93a], the Fourier expansions of g_λ^α and $\overline{C_\lambda^\alpha}$ are

$$\begin{aligned}(8.27)\quad g_\lambda^\alpha(z) =&\varphi_{\mathbf{a}}(d_\alpha)^2\varphi^*(d_\alpha\mathfrak{a}_\alpha^{-1})^2 \\ &\times \sum_{0\neq\xi\in\mathfrak{t}_\lambda\mathfrak{a}_\alpha^2} \mathrm{c}(\xi\mathfrak{t}_\lambda^{-1}\mathfrak{a}_\alpha^{-2}, \mathrm{sgn}\,\xi; \mathbf{g}^*)|\xi|^{m-i\mu}W_{2\beta,2\gamma}(\xi y)\mathbf{e}_{\mathbf{a}}(\xi x)\end{aligned}$$

(recall that $\psi^2 = \varphi^2$), and

$$\begin{aligned}(8.28)\quad &\varphi_{\mathbf{a}}(d_\alpha)\varphi^*(d_\alpha \mathfrak{a}_\alpha^{-1})\mathrm{N}(\mathfrak{a}_\alpha)^{-2s-1}y^{-su-(u-m-i\mu)/2}\overline{C_\lambda^\alpha(z)} \\ &= L_{\mathfrak{rc}}(2s+1,\overline{\varphi}) + D_F^{-1/2}\overline{\gamma(\varphi)}\mathrm{N}(\mathfrak{h})^{-1}\sum_{\mathfrak{o}\supset\mathfrak{t}\supset 2\mathfrak{h}^{-1}\mathfrak{rc}}\mu(\mathfrak{t})\overline{\varphi}^*(\mathfrak{t})\mathrm{N}(\mathfrak{t})^{-2s-1} \\ &\quad\times\sum_{\mathfrak{y}}\mathrm{N}(\mathfrak{y})^{2s}\sum_{h,b}\varphi_{\mathbf{a}}(b)|b|^{-2su}\varphi_{\mathbf{a}}(h)\varphi^*(h\mathfrak{h}\mathfrak{d}\mathfrak{y})\mathbf{e}_{\mathbf{a}}(-bhx) \\ &\quad\times\overline{\xi(y,bh;\overline{s}u+(u+m+i\mu)/2,\overline{s}u+(u-m+i\mu)/2)}.\end{aligned}$$

Here $\mathfrak{y}$ runs through a set of representatives for the ideal class group of F, h runs through $\mathfrak{h}^{-1}\mathfrak{d}^{-1}\mathfrak{y}^{-1}$, and b runs through $(F^\times \cap \mathfrak{t}^{-1}\mathfrak{rc}\mathfrak{d}\mathfrak{t}_\lambda\mathfrak{a}_\alpha^2\mathfrak{y})/\mathfrak{o}^\times$. ξ is a confluent hypergeometric function closely related to the Whittaker function. We shall discuss it further in the appendix to this section; for now, note only that (8a.2) implies that

$$\begin{aligned}(8.29)\quad &\xi(y,bh;\overline{s}u+(u+m+i\mu)/2,\overline{s}u+(u-m+i\mu)/2) \\ &= |bh|^{2\overline{s}u+i\mu}\xi(y|bh|,\operatorname{sgn} bh;\overline{s}u+(u+m+i\mu)/2,\overline{s}u+(u-m+i\mu)/2).\end{aligned}$$

We also note that $\mathbf{g}^*$ is an "even" form, and hence that

$$\begin{aligned}(8.30)\qquad &\mathrm{c}(\xi\mathfrak{t}_\lambda^{-1}\mathfrak{a}_\alpha^{-2},\operatorname{sgn}\xi;\mathbf{g}^*) \\ &= \left(\prod_{v\in\mathbf{a},\xi_v<0}(-1)^{m_v}(2\beta_v)_{m_v}(2\gamma_v)_{m_v}\right)\mathrm{c}(\xi\mathfrak{t}_\lambda^{-1}\mathfrak{a}_\alpha^{-2},u;\mathbf{g}^*).\end{aligned}$$

Putting together these identities, we find that the quantity of (8.26) is equal to

$$\begin{aligned}(8.31)\quad &\overline{\gamma(\varphi)}\mathrm{N}(2\mathfrak{h}^{-1}\mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1})\sum_\alpha\mathrm{N}(\mathfrak{a}_\alpha)^{2s+2t}\sum_{\mathfrak{t},\mathfrak{y}}\mu(\mathfrak{t})\overline{\varphi}^*(\mathfrak{t})\mathrm{N}(\mathfrak{t})^{-2s-1}\mathrm{N}(\mathfrak{y})^{2s} \\ &\times\sum_{a\in F^\times/(\mathfrak{o}^\times)^2}\sum_{bh=a}\mathrm{c}(a\mathfrak{t}_\lambda^{-1}\mathfrak{a}_\alpha^{-2},u;\mathbf{g}^*)|a|^{su-tu}|b|^{-2su}\varphi^*(ab^{-1}\mathfrak{h}\mathfrak{d}\mathfrak{y})M(s,t,\operatorname{sgn} a),\end{aligned}$$

where

$$\begin{aligned}(8.32)\quad &M(s,t,\operatorname{sgn} a) = \left(\prod_{v\in\mathbf{a},a_v<0}(2\beta_v)_{m_v}(2\gamma_v)_{m_v}\right) \\ &\times\int_{y\gg 0}W_{2\beta,2\gamma}(y\operatorname{sgn} a)\overline{\xi(y,\operatorname{sgn} a;\overline{s}u+(u+m+i\mu)/2,\overline{s}u+(u-m+i\mu)/2)} \\ &\qquad\times y^{su+tu+m-i\mu-u}\,dy.\end{aligned}$$

We will evaluate M in the appendix to this section.

As b and $\mathfrak{y}$ vary in their respective ranges, the ideal $\mathfrak{n} = b\mathfrak{t}\mathfrak{r}^{-1}\mathfrak{c}^{-1}\mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1}\mathfrak{a}_\alpha^{-2}\mathfrak{y}^{-1}$ ranges through all integral ideals. The condition that $h = a/b \in \mathfrak{h}^{-1}\mathfrak{d}^{-1}\mathfrak{y}^{-1}$ means that $0 \neq a \in \mathfrak{z}\mathfrak{t}_\lambda/(\mathfrak{o}^\times)^2$, where $\mathfrak{z} = \mathfrak{t}^{-1}\mathfrak{h}^{-1}\mathfrak{rc}\mathfrak{n}\mathfrak{a}_\alpha^2$. Since we only care about the value of a up to a unit in $\mathfrak{o}_+^\times$, we can insert an extra factor $[\mathfrak{o}_+^\times : (\mathfrak{o}^\times)^2]$ and sum instead over $a \in \mathfrak{z}\mathfrak{t}_\lambda/\mathfrak{o}_+^\times$. Combining these observations, we see that the expression in (8.25) becomes

$$\begin{aligned}(8.33)\quad &(-1)^{\|m\|}2^{\|u\|}\operatorname{vol}(\Gamma[2\mathfrak{d}^{-1},2^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{d}]\backslash\mathcal{H}^{\mathbf{a}})^{-1}\overline{\gamma(\varphi)}[\mathfrak{o}_+^\times:(\mathfrak{o}^\times)^2]\mathrm{N}(\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c})\\ &\times \mathrm{N}(2\mathfrak{o})^{-s-t}\mathrm{N}(\mathfrak{d}\mathfrak{h})^{t-s}\sum_{\lambda,\alpha,\mathfrak{t},\mathfrak{n}}\ \sum_{0\neq a\in\mathfrak{z}\mathfrak{t}_\lambda/\mathfrak{o}_+^\times}\mu(\mathfrak{t})\overline{\varphi}^*(\mathfrak{t})\mathrm{N}(\mathfrak{t})^{t-s-1}\\ &\quad\times \mathrm{c}(\mathfrak{t}^{-1}\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{n}\cdot a\mathfrak{z}^{-1}\mathfrak{t}_\lambda^{-1},u;\mathbf{g}^*)\mathrm{N}(\mathfrak{n})^{-s-t}\varphi^*(a\mathfrak{z}^{-1}\mathfrak{t}_\lambda^{-1})\mathrm{N}(a\mathfrak{z}^{-1}\mathfrak{t}_\lambda^{-1})^{s-t}\\ &\qquad\times M(s,t,\operatorname{sgn}a).\end{aligned}$$

As a and $\mathfrak{t}_\lambda$ run through their respective ranges, the pair $(a\mathfrak{z}^{-1}\mathfrak{t}_\lambda^{-1},\operatorname{sgn}a)$ ranges through all possible pairs $(\mathfrak{m},\sigma)$, with $\mathfrak{m}$ an integral ideal, and $\sigma\in\{\pm1\}^{\mathbf{a}}$. Also, there are h_F possible choices of $\mathfrak{a}_\alpha$. Therefore, the expression in (8.25) becomes

$$\begin{aligned}(8.34)\quad &(-1)^{\|m\|}2^{\|u\|}\operatorname{vol}(\Gamma[2\mathfrak{d}^{-1},2^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{d}]\backslash\mathcal{H}^{\mathbf{a}})^{-1}h_F\overline{\gamma(\varphi)}[\mathfrak{o}_+^\times:(\mathfrak{o}^\times)^2]\mathrm{N}(\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c})\\ &\times \mathrm{N}(2\mathfrak{o})^{-s-t}\mathrm{N}(\mathfrak{d}\mathfrak{h})^{t-s}\sum_{\mathfrak{o}\supset\mathfrak{t}\supset 2\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}}\mu(\mathfrak{t})\overline{\varphi}^*(\mathfrak{t})\mathrm{N}(\mathfrak{t})^{t-s-1}\\ &\quad\times\sum_{\mathfrak{m},\mathfrak{n}\subset\mathfrak{o}}\mathrm{c}(\mathfrak{t}^{-1}\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{m}\mathfrak{n},u;\mathbf{g}^*)\mathrm{N}(\mathfrak{n})^{-s-t}\varphi^*(\mathfrak{m})\mathrm{N}(\mathfrak{m})^{s-t}\\ &\qquad\times\sum_{\sigma\in\{\pm1\}^{\mathbf{a}}}M(s,t,\sigma).\end{aligned}$$

Note that the sum over $\mathfrak{t}$ may be rewritten as a sum over $\mathfrak{o}\supset\mathfrak{t}\supset\mathfrak{j}$, because $\mu(\mathfrak{t})\overline{\varphi}^*(\mathfrak{t})=0$ unless $\mathfrak{t}$ is squarefree and prime to $\mathfrak{h}$. Put

$$(8.35)\qquad Y_{\mathfrak{t}}(s,t)=\sum_{\mathfrak{m},\mathfrak{n}\subset\mathfrak{o}}\mathrm{c}(\mathfrak{t}^{-1}\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{m}\mathfrak{n},u;\mathbf{g}^*)\mathrm{N}(\mathfrak{n})^{-s-t}\varphi^*(\mathfrak{m})\mathrm{N}(\mathfrak{m})^{s-t}.$$

By Lemma 3.9 of [Sh 93a], $Y_{\mathfrak{t}}$ has a meromorphic continuation to all of $\mathbf{C}^2$, and $Y_{\mathfrak{t}}(s,s)$ is holomorphic if the real part of s is positive; moreover, if s tends to $+\infty$, then $Y_{\mathfrak{t}}(s,s)$ tends to $D(0,\mathbf{g}^*,\varphi,\mathfrak{t}^{-1}\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c})$. We shall see in the appendix to this section that $\sum_{\sigma\in\{\pm1\}^{\mathbf{a}}}M(s,t,\sigma)$ can be meromorphically continued to all of $\mathbf{C}^2$, and that its value at $s=t$ is

$$\begin{aligned}(8.36)\ &i^{\|m\|}2^{\|-m\|}\pi^{\|-m\|}\\ &\times\frac{\Gamma(su+(m-i\mu)/2+\beta)\Gamma(su+(m-i\mu)/2+\gamma)\Gamma(\beta+m)\Gamma(\gamma+m)}{\Gamma(su+(u-m-i\mu)/2)\Gamma(su+(u+m-i\mu)/2)}.\end{aligned}$$

Since the value of (8.34) at $s=t$ is equal to the value of (8.18), we get

$$\begin{aligned}(8.37)\qquad &\Bigl(\langle\mathbf{g},\mathbf{g}\rangle/\langle f,f\rangle\Bigr)\overline{\mu_f(\tau,\mathfrak{q}^{-1})}\sum_{\mathfrak{m}}\mu_{f^*}(\tau,\mathfrak{q}^{-1}\mathfrak{b}\mathfrak{m})\mathrm{N}(\mathfrak{m})^{-2s}\\ &=Q\sum_{\mathfrak{t}\supset\mathfrak{j}}\mu(\mathfrak{t})\overline{\varphi}^*(\mathfrak{t})\mathrm{N}(\mathfrak{t})^{-1}Y_{\mathfrak{t}}(s,s).\end{aligned}$$

Letting s tend to $+\infty$ yields (8.4). The formula (8.6) follows from equations (8.6a) and (8.6b) of [Sh 93a]. We note that if χ is the system of eigenvalues corresponding to $\mathbf{g}$, then $\chi'(\mathfrak{m})=\overline{\chi}(\mathfrak{m})$, for $\mathfrak{m}$ prime to $2^{-1}\mathfrak{c}$. This follows from the remark immediately preceding Proposition 3.1 of [Sh 93a]. This concludes the proof of Theorem 8.1. ∎

We shall now drop the "multiplicity one" hypothesis on f that was made in Theorem 8.1. We shall derive a generalization of Theorem 3.2 of [Sh 93a], where we still assume that f is an eigenform of all the Hecke operators, with $T_v f = \omega_v f$. Define the spaces

$$X = \{\tilde{f} \in \mathcal{S}_{k,\lambda}(\mathfrak{b}, \mathfrak{b}', \psi) \mid T_v \tilde{f} = \omega_v \tilde{f} \text{ for almost all } v \in \mathbf{f}\}, \tag{8.38}$$

$$Y = \{\mathbf{g} \in \mathcal{S}_{2m,4\lambda}(\mathfrak{o}, 2^{-1}\mathfrak{c}, \psi^2) \mid \mathfrak{T}(\mathfrak{p})\mathbf{g} = \mathrm{N}(\mathfrak{p})\omega_v \mathbf{g} \text{ for almost all } v \in \mathbf{f}\}, \tag{8.39}$$

where $\mathfrak{p}$ is the prime corresponding to $v \in \mathbf{f}$.

Theorem 8.4. *Let $\{f_i \mid i \in I\}$ and $\{f_i' \mid i \in I\}$ be dual bases for X with respect to the inner product, and let $\{\mathbf{g}_j \mid j \in J\}$ and $\{\mathbf{g}_j' \mid j \in J\}$ be dual bases for Y. Let f_i^* be the inversion of f_i, and $\mathbf{g}_j^*$ be the inversion of $\mathbf{g}_j$, in the notation of Theorem 8.1. Assume that the forms in X are all orthogonal to the relevant theta-series, as in Theorem 5.1 (by (6.13), this holds if $\lambda_v \neq m_v(m_v-1)/4$ for some $v \in \mathbf{a}$). Then*

$$\begin{aligned}\sum_{i\in I} \overline{\mu(\tau, \mathfrak{q}^{-1}; f_i', \psi)}\,\mu(\tau, \mathfrak{q}^{-1}\mathfrak{b}; f_i^*, \overline{\psi}) \\ = Q \sum_{\mathfrak{o}\supset\mathfrak{t}\supset\mathfrak{j}} \mu(\mathfrak{t})\overline{\varphi}^*(\mathfrak{t})\mathrm{N}(\mathfrak{t})^{-1} \sum_{j\in J} \overline{\mathrm{c}(\mathfrak{o}, u; \mathbf{g}_j')}\,D(0, \mathbf{g}_j^*, \varphi, \mathfrak{t}^{-1}\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}),\end{aligned} \tag{8.40}$$

with Q and the other symbols as in Theorem 8.1.

Proof: We note that both quantities in (8.40) are independent of the choice of dual bases. Our proof follows by the same method as in section 9 of [Sh 93a]. Without loss of generality, we take the $\{f_i\}$ and $\{\mathbf{g}_j\}$ to be orthonormal bases of X and Y, respectively. Let h_i correspond to f_i as in Theorem 5.1, and define $\tilde{\mathbf{g}}_i$ by

$$C\tilde{\mathbf{g}}_i(p) = \int_\Phi \Theta(z, p; \eta) h_i(z) y^k d_H z. \tag{8.41}$$

By Theorem 5.1, $\tilde{\mathbf{g}}_i \in Y$ and $\mathrm{c}(\mathfrak{o}, u; \tilde{\mathbf{g}}_i) = \mu(\tau, \mathfrak{q}^{-1}; f_i, \psi)$. Similarly, $\tilde{h}_j(z) = \langle \Theta(z, p; \eta), \mathbf{g}_j(p)\rangle$ comes from a form $\tilde{f}_j \in X$, by Theorem 5.2. By the same reasoning as in Theorem 8.1, we get the following identities, which are equivalent to (9.5) and (9.6) of [Sh 93a]:

$$\begin{aligned}\overline{C^{-1}}\,\mathrm{vol}(\Phi)\tau^k \mathrm{N}(\mathfrak{q}\mathfrak{r})^{-1}\mu(\tau, \mathfrak{q}^{-1}\mathfrak{b}; \tilde{f}_j^*, \overline{\psi}) \\ = Q \sum_{\mathfrak{t}\supset\mathfrak{j}} \mu(\mathfrak{t})\overline{\varphi}^*(\mathfrak{t})\mathrm{N}(\mathfrak{t})^{-1} D(0, \mathbf{g}_j^*, \varphi, \mathfrak{t}^{-1}\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}),\end{aligned} \tag{8.42}$$

$$\overline{C^{-1}}\,\mathrm{vol}(\Phi)\tau^k \mathrm{N}(\mathfrak{q}\mathfrak{r})^{-1}\Big\langle f_i, \tilde{f}_j \Big\rangle = \overline{C^{-1}}\,\mathrm{vol}(\Phi)\Big\langle h_i, \tilde{h}_j \Big\rangle = \langle \tilde{\mathbf{g}}_i, \mathbf{g}_j \rangle. \tag{8.43}$$

Now $\tilde{f}_j = \sum_{i\in I} \Big\langle f_i, \tilde{f}_j \Big\rangle f_i$, so $\tilde{f}_j^* = \sum_{i\in I} \Big\langle f_i, \tilde{f}_j \Big\rangle f_i^*$. We similarly obtain that $\tilde{\mathbf{g}}_i = \sum_{j\in J} \langle \mathbf{g}_j, \tilde{\mathbf{g}}_i \rangle \mathbf{g}_j$, so $\mu(\tau, \mathfrak{q}^{-1}; f_i, \psi) = \mathrm{c}(\mathfrak{o}, u; \tilde{\mathbf{g}}_i) = \sum_{j\in J} \langle \mathbf{g}_j, \tilde{\mathbf{g}}_i \rangle\, \mathrm{c}(\mathfrak{o}, u; \mathbf{g}_j)$. Combining all this yields (8.40). ∎

Appendix: Calculation of $M(s,t,\sigma)$.

Given s and t in $\mathbf{C}$ such that the real parts of both s and $t-s$ are sufficiently large, and given a signature $\sigma \in \{\pm 1\}^{\mathbf{a}}$, we will express the function $M(s,t,\sigma)$ (as defined in (8.32)) in terms of Gamma-functions. We first mention a few propoerties of the confluent hypergeometric function ξ, as culled from section 4 of [Sh 93a] and equations (4.19) through (4.21) of [Sh 85b].

Given $h, y \in \mathbf{R}^{\mathbf{a}}$ and $\alpha, \beta \in \mathbf{C}^{\mathbf{a}}$ such that $y \gg 0$ and $\alpha - \beta \in \mathbf{Z}^{\mathbf{a}}$, we define ξ as a product of local functions

$$\xi(y,h;\alpha,\beta) = \prod_{v \in \mathbf{a}} \xi(y_v, h_v; \alpha_v, \beta_v). \tag{8a.1}$$

For $\alpha, \beta \in \mathbf{C}$ in a suitable range, with $\alpha - \beta \in \mathbf{Z}$, and for $h \in \mathbf{R}$ and $y > 0$, the following integral converges and defines the local ξ-function:

$$\xi(y,h;\alpha,\beta) = \int_{-\infty}^{+\infty} e^{-2\pi i h x}(x+iy)^{-\alpha}(x-iy)^{-\beta}\, dx. \tag{8a.2}$$

(We write $(x+iy)^{-\alpha}(x-iy)^{-\beta}$ to denote $|x+iy|^{-2\alpha}(x-iy)^{\alpha-\beta}$, where there is no ambiguity in the meaning of the powers.) We can analytically continue $\xi(y,h;\alpha,\beta)$ in α and β; in fact, equation (4.21b) of [Sh 85b] (or equation (4.14) of [Sh 93a]) tells us that if $h > 0$, then

$$y^{\beta}\xi(y,h;\alpha,\beta) = (-2i)^{\alpha-\beta}\pi^{\alpha}|h|^{\alpha-1}V(4\pi h y; 1-\alpha, \beta)\Gamma(\alpha)^{-1}. \tag{8a.3}$$

(If $h < 0$, the factor $V(4\pi h y; 1-\alpha,\beta)\Gamma(\alpha)^{-1}$ in the above expression should be replaced with $(4\pi|h|y)^{\beta-\alpha}V(4\pi|h|y;\alpha,1-\beta)\Gamma(\beta)^{-1}$.) The integral appearing in the definition of $M(s,t,\sigma)$ involves the complex conjugate of ξ; however, (8a.2) implies that

$$\overline{\xi(y,h;\alpha,\beta)} = \xi(y,-h;\overline{\beta},\overline{\alpha}) = (-1)^{\beta-\alpha}\xi(y,h;\overline{\alpha},\overline{\beta}). \tag{8a.4}$$

Therefore,

$$M(s,t,\sigma) = \frac{i^{\|m\|}\pi^{\|su-tu+u-m\|}2^{\|-2tu-2m+u\|}}{\Gamma(su+(u-m-i\mu)/2)} \tag{8a.5}$$
$$\times \prod_{v\in\mathbf{a}} K(2\beta_v, 2\gamma_v, s+(1-m_v-i\mu_v)/2, -s+(1-m_v+i\mu_v)/2; t+(-1+m_v-i\mu_v)/2, \sigma_v),$$

where we define K as follows: given an integer $m \geq 0$ and $\beta, \gamma, \delta, \epsilon, T \in \mathbf{C}$ such that $2\beta + 2\gamma = 1-2m$ and $\delta + \epsilon = 1 - m$, let

$$K(2\beta, 2\gamma, \delta, \epsilon; T, 1) = ((\delta)_m)^{-1}\int_0^{\infty} V(y; 2\beta, 2\gamma)V(y;\delta,\epsilon)y^{T+m-1}\, dy, \tag{8a.6}$$

$$\begin{aligned} &K(2\beta, 2\gamma, \delta, \epsilon; T, -1) \\ &\quad = (2\beta)_m(2\gamma)_m \int_0^{\infty} V(y; 1-2\beta, 1-2\gamma)V(y;1-\delta,1-\epsilon)y^{T-2m-1}\, dy. \end{aligned} \tag{8a.7}$$

These integrals converge if the real part of T is sufficiently large in comparison to β, γ, δ, and ϵ.

Proposition 8a.1. *For $\sigma \in \{\pm 1\}$,*

$$(8a.8) K(2\beta, 2\gamma, \delta, \epsilon; T, \sigma) = 2^{2T-1}\pi^{-1}\Gamma(T+1)^{-1}$$
$$\times \sum_{q=0}^{m} \binom{m}{q} \frac{\sigma^q (T+\delta+2\epsilon+m-q)_q (T+2\delta+\epsilon+q)_{m-q}}{(\delta-\epsilon-m+q)_q (\delta-\epsilon-m+2q+1)_{m-q}}$$
$$\times \Gamma((T+2\beta+m+\delta+q)/2)\Gamma((T+2\gamma+m+\delta+q)/2)$$
$$\times \Gamma((T+2\beta+m+\epsilon+m-q)/2)\Gamma((T+2\gamma+m+\epsilon+m-q)/2).$$

Proof: As in the proof of Proposition 3a.1, we shall reduce the above formula to the case $m = 0$. Equation (10.7) of [Sh 85b] tells us that for any β and γ,

$$V(y;\beta,\gamma) = V(y;\beta+1,\gamma) + \gamma y^{-1} V(y;\beta+1,\gamma+1). \tag{8a.9}$$

(This follows easily from integrating by parts in (1.13).) Since $V(y;\beta,\gamma)$ is symmetric in β and γ, or by writing $(1+t)^{-\gamma}t^{\beta-1} = (1+t)^{-\gamma-1}t^{\beta-1} + (1+t)^{-\gamma-1}t^{\beta}$ in (1.13), we get

$$V(y;\beta,\gamma) = V(y;\beta,\gamma+1) + \beta y^{-1} V(y;\beta+1,\gamma+1). \tag{8a.10}$$

We can thus express either $V(y;\beta,\gamma)$ or $V(y;\beta+1,\gamma+1)$ in terms of both $V(y;\beta+1,\gamma)$ and $V(y;\beta,\gamma+1)$. Combining this observation with (3a.4), we obtain

$$(8a.11) \qquad \frac{\partial}{\partial y} V(y;\delta,\epsilon) = (-1/2)V(y;\delta,\epsilon)$$
$$+ \delta\epsilon y^{-1}(\delta-\epsilon)^{-1}\Big(V(y;\delta+1,\epsilon) - V(y;\delta,\epsilon+1)\Big),$$

$$V(y;\delta,\epsilon) = (\delta-\epsilon)^{-1}\Big(\delta V(y;\delta+1,\epsilon) - \epsilon V(y;\delta,\epsilon+1)\Big). \tag{8a.12}$$

Similarly, by the above observation and (3a.3), we have

$$(8a.13) \qquad \frac{\partial}{\partial y} V(y;1-\delta,1-\epsilon) = (1/2)V(y;1-\delta,1-\epsilon)$$
$$+ (\delta-\epsilon)^{-1}\Big((1-\delta)V(y;-\delta,1-\epsilon) - (1-\epsilon)V(y;1-\delta,-\epsilon)\Big),$$

$$V(y;1-\delta,1-\epsilon) = y(\delta-\epsilon)^{-1}\Big(V(y;-\delta,1-\epsilon) - V(y;1-\delta,-\epsilon)\Big). \tag{8a.14}$$

We now use (3a.3) on $V(y;2\beta,2\gamma)$ and integrate by parts to obtain

$$(8a.15) \quad K(2\beta,2\gamma,\delta,\epsilon;T,1)$$
$$= ((\delta)_m)^{-1}\int_0^\infty V(y;2\beta+1,2\gamma+1)$$
$$\times \Big(\Big(1/2 + (2-2m)y^{-1} + \partial/\partial y\Big)\Big(V(y;\delta,\epsilon)y^{T+m-1}\Big)\Big)\,dy$$
$$= (\delta-\epsilon)^{-1}\Big((T+\delta+2\epsilon)K(2\beta+1,2\gamma+1,\delta+1,\epsilon;T,1)$$
$$+ (T+2\delta+\epsilon)K(2\beta+1,2\gamma+1,\delta,\epsilon+1;T,1)\Big).$$

Here we have used (8a.11) and (8a.12) and the fact that $\delta+\epsilon = 1-m$. Similarly, by (3a.4), (8a.13), and (8a.14),

$$\text{(8a.16)}\quad \begin{aligned} &K(2\beta,2\gamma,\delta,\epsilon;T,-1) \\ &\quad = (\delta-\epsilon)^{-1}\Big(-(T+\delta+2\epsilon)K(2\beta+1,2\gamma+1,\delta+1,\epsilon;T,-1) \\ &\qquad\qquad + (T+2\delta+\epsilon)K(2\beta+1,2\gamma+1,\delta,\epsilon+1;T,-1)\Big). \end{aligned}$$

By induction, it follows that for any k,

$$\text{(8a.17)}\quad \begin{aligned} &K(2\beta,2\gamma,\delta,\epsilon;T,\sigma) \\ &= \sum_{q=0}^{k}\binom{k}{q}\frac{\sigma^q(T+\delta+2\epsilon+k-q)_q(T+2\delta+\epsilon+q)_{k-q}}{(\delta-\epsilon-k+q)_q(\delta-\epsilon-k+2q+1)_{k-q}} \\ &\qquad \times K(2\beta+k,2\gamma+k,\delta+q,\epsilon+k-q;T,\sigma). \end{aligned}$$

Taking $k=m$, we have thus reduced (8a.8) to the case $m=0$.

If $m=0$, then $V(y;2\beta,2\gamma)$ and $V(y;\delta,\epsilon)$ are essentially K-Bessel functions, and the value of the integral in the definition of $K(2\beta,2\gamma,\delta,\epsilon;T,\sigma)$ can be found in [E], page 93, equation (36). Alternatively, we can first assume that all our parameters are in a suitable range, and use the integral representation for V to express $\int_0^\infty V(y;2\beta,2\gamma)V(y;\delta,\epsilon)y^{T-1}\,dy$ as

$$\text{(8a.18)}\quad \frac{\Gamma(T+2\beta+\delta)}{\Gamma(2\beta)\Gamma(\delta)}\int_0^\infty\int_0^\infty t^{2\beta-1}(1+t)^{-2\gamma}u^{\delta-1}(1+u)^{-\epsilon}(1+t+u)^{-T-2\beta-\delta}\,dt\,du.$$

After the substitutions $t=\xi/(1-\xi)$ and $u=\lambda/(1-\lambda)$, we get

$$\text{(8a.19)}\quad \begin{aligned} &\int_0^\infty V(y;2\beta,2\gamma)V(y;\delta,\epsilon)y^{T-1}\,dy = \frac{\Gamma(T+2\beta+\delta)}{\Gamma(2\beta)\Gamma(\delta)} \\ &\times\int_0^1\int_0^1 \xi^{2\beta-1}(1-\xi)^{T+2\gamma+\delta-1}\lambda^{\delta-1}(1-\lambda)^{T+2\beta+\epsilon-1}(1-\xi\lambda)^{-T-2\beta-\delta}\,d\xi\,d\lambda. \end{aligned}$$

We now expand $(1-\xi\lambda)^{-T-2\beta-\delta}$ into $\sum_{k=0}^\infty (T+2\beta+\delta)_k\xi^k\lambda^k/k!$ and evaluate the Beta-integrals in ξ and λ to get the expression

$$\text{(8a.20)}\quad \begin{aligned} &\frac{\Gamma(T+2\beta+\delta)\Gamma(T+2\gamma+\delta)\Gamma(T+2\beta+\epsilon)}{\Gamma(T+2\beta+2\gamma+\delta)\Gamma(T+2\beta+\delta+\epsilon)} \\ &\quad\times {}_3F_2(T+2\beta+\delta,2\beta,\delta;T+2\beta+2\gamma+\delta,T+2\beta+\delta+\epsilon;1), \end{aligned}$$

where ${}_3F_2$ is the hypergeometric function defined in [H], (8.6-4). If $2\beta+2\gamma$ and $\delta+\epsilon$ are both equal to 1, then Dixon's formula (Theorem 8.6d of [H]) gives the value of ${}_3F_2$ in terms of Gamma-functions. Using Legendre's duplication relation, we conclude that if $2\beta+2\gamma=\delta+\epsilon=1$, then

$$\text{(8a.21)}\quad \begin{aligned} &\int_0^\infty V(y;2\beta,2\gamma)V(y;\delta,\epsilon)y^{T-1}\,dy \\ &= 2^{2T-1}\pi^{-1}\frac{\Gamma((T+2\beta+\delta)/2)\Gamma((T+2\gamma+\delta)/2)\Gamma((T+2\beta+\epsilon)/2)\Gamma((T+2\gamma+\epsilon)/2)}{\Gamma(T+1)}, \end{aligned}$$

as desired. This concludes the proof of Proposition 8a.1. ∎

Corollary 8a.2. *The meromorphic continuation of $M(s,t,\sigma)$ is*

$$\begin{aligned}(8a.22)\quad M(s,t,\sigma) &= \frac{i^{\|m\|}\pi^{\|su-tu-m\|}2^{\|-m-u\|}}{\Gamma(su+(u-m-i\mu)/2)\Gamma(tu+(u+m-i\mu)/2)}\\ &\times \sum_{0\le q\le m}\binom{m}{q}\frac{\sigma^q(tu-su+u-q)_q(tu+su-i\mu+u-m+q)_{m-q}}{(2su-i\mu-m+q)_q(2su-i\mu-m+2q+u)_{m-q}}\\ &\times\Gamma((tu+su-i\mu+2\beta+m+q)/2)\Gamma((tu+su-i\mu+2\gamma+m+q)/2)\\ &\times\Gamma((tu-su+2\beta+2m-q)/2)\Gamma((tu-su+2\gamma+2m-q)/2),\end{aligned}$$

where q runs over elements of $\mathbf{Z}^{\mathbf{a}}$. Furthermore, if $t=s$, and we sum over all signatures σ, then only the term $q=0$ survives. In that case,

$$\begin{aligned}(8a.23)\quad \sum_{\sigma\in\{\pm1\}^{\mathbf{a}}} M(s,s,\sigma) &= i^{\|m\|}\pi^{\|-m\|}2^{\|-m\|}\\ &\times\frac{\Gamma(su+(m-i\mu)/2+\beta)\Gamma(su+(m-i\mu)/2+\gamma)\Gamma(\beta+m)\Gamma(\gamma+m)}{\Gamma(su+(u-m-i\mu)/2)\Gamma(su+(u+m-i\mu)/2)}.\end{aligned}$$

Proof: This uses (8a.5) and (8a.8). If $q_v>0$ for some $v\in\mathbf{a}$, then $(t-s+1-q_v)_{q_v}$ has a zero at $t=s$, and we expect no contribution from that term. We must check, however, that neither $\Gamma((tu-su+2\beta+2m-q)/2)$ nor $\Gamma((tu-su+2\gamma+2m-q)/2)$ has a pole at $t=s$. Since we are summing over all signatures σ, we may assume that $q\in 2\mathbf{Z}^{\mathbf{a}}$. Say that $(2\beta_v+2m_v-q_v)/2$ is a nonpositive integer for some $v\in\mathbf{a}$. Then β_v+m_v is an integer, which we shall call r, and $r\le q_v/2\le m_v/2$. As observed in the beginning of the proof of Theorem 8.1, $r>0$. Since $\gamma_v=1/2-\beta_v-m_v=1/2-r$, we see that $(\beta_v)_r(\gamma_v)_r\ne 0$. (Writing the Pochhammer symbols as products, we see that each factor is strictly negative.) Hence, the form f in Theorem 8.1 is essentially the same as $(\epsilon_v)^r f$. In particular, $\delta_v^{k-(2)_v}\delta_v^{k-(4)_v}\cdots\delta_v^{k-(2r)_v}(\epsilon_v)^r f$ is a nonzero constant times f, so $(\epsilon_v)^r f\ne 0$. Now the eigenvalue of $(\epsilon_v)^r f$ for $L_v^{k-(2r)_v}$ is $(\beta_v+r)(\gamma_v+r)=r-m_v/2\le 0$, but this eigenvalue must be nonnegative. Thus $r=m_v/2$, and $(\epsilon_v)^r f$ satisfies the conditions of Proposition 6.1, contradicting our hypothesis on f. ∎

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