Locality, Generalized Evaluators and Group Actions

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Plan

- Meromorphic germs
- Locality structures
- Locality group actions
- Applications to meromorphic germs
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- References

Filtered lattice space

- A filtered lattice Euclidean space is $(\mathbb{R}^\infty, \mathbb{Z}^\infty, Q)$ consisting of
- direct limits

$$\mathbb{R}^\infty = \varinjlim_k \mathbb{R}^k, \quad \mathbb{Z}^\infty = \varinjlim_k \mathbb{Z}^k,$$

  under standard embeddings $i_k : \mathbb{R}^k \to \mathbb{R}^{k+1}$, and
- a family $Q = (Q_k)_{k \geq 1}$ of inner products

$$Q_k : \mathbb{R}^k \otimes \mathbb{R}^k \to \mathbb{R},$$

  such that

$$Q_{k+1}|_{\mathbb{R}^k \times \mathbb{R}^k} = Q_k, \quad Q_k(\mathbb{Z}^k \times \mathbb{Z}^k) \subset \mathbb{Q}.$$

- For a field $\mathbb{K}$ with $\mathbb{Q} \subset \mathbb{K} \subset \mathbb{R}$, we denote by $\mathcal{L}_\mathbb{K}(\mathbb{C}^k) = \mathcal{L}_\mathbb{K}(\mathbb{K}^k \otimes \mathbb{C})$ the space of linear forms on $\mathbb{C}^k$ which take $\mathbb{K}$-values on $\mathbb{K}^k$.

Meromorphic germs with linear poles

- On the filtered lattice space $(\mathbb{R}^\infty, \mathbb{Z}^\infty)$, a meromorphic germ f on $\mathbb{R}^k \otimes \mathbb{C}$ is said to have **$\mathbb{K}$-linear poles** at zero if there are vectors $L_1, \ldots, L_k \in (\mathbb{Z}^k)^* \otimes \mathbb{K}$ (possibly with repetitions) such that

$$f \prod_{i=1}^k L_i$$

  is a holomorphic germ at zero.
- Let $\mathcal{M}_\mathbb{K}(\mathbb{C}^k) = \mathcal{M}_\mathbb{K}(\mathbb{R}^k \otimes \mathbb{C})$ (resp. $\mathcal{M}_{\mathbb{K}+}(\mathbb{C}^k) = \mathcal{M}_{\mathbb{K}+}(\mathbb{R}^k \otimes \mathbb{C})$) denote the space of meromorphic germs at zero with $\mathbb{K}$-linear poles (resp. holomorphic germs).
- There are natural embeddings and direct limits

$$p_k : \mathcal{M}_\mathbb{K}(\mathbb{C}^k) \to \mathcal{M}_\mathbb{K}(\mathbb{C}^{k+1}), \quad p_k : \mathcal{M}_{\mathbb{K}+}(\mathbb{C}^k) \to \mathcal{M}_{\mathbb{K}+}(\mathbb{C}^{k+1}),$$

$$\mathcal{M}_\mathbb{K} = \mathcal{M}_\mathbb{K}(\mathbb{C}^\infty) = \varinjlim_k \mathcal{M}_\mathbb{K}(\mathbb{C}^k), \quad \mathcal{M}_{\mathbb{K}+} = \mathcal{M}_{\mathbb{K}+}(\mathbb{C}^\infty) = \varinjlim_k \mathcal{M}_{\mathbb{K}+}(\mathbb{C}^k)$$

- By restriction, we also let

$$\mathcal{L}_\mathbb{K} := \mathcal{L}_\mathbb{K}(\mathbb{C}^\infty) = \varinjlim_k \mathcal{L}_\mathbb{K}(\mathbb{C}^k)$$

  be the space of $\mathbb{K}$-linear forms.

Polar germs

- An inner product Q on $(\mathbb{R}^\infty, \mathbb{Z}^\infty)$ induces an inner product in $\mathcal{L}_{\mathbb{K}}(\mathbb{C}^\infty)$ which we still denote by Q.
- A germ of meromorphic functions at zero is called a **polar germ** in $\mathbb{C}^k$ with $\mathbb{K}$-coefficients if it is of the form

$$\frac{h(\ell_1, \dots, \ell_m)}{L_1^{s_1} \cdots L_n^{s_n}},$$

where

- h lies in $\mathcal{M}_{\mathbb{K}+}(\mathbb{C}^m)$,
- $\ell_1, \dots, \ell_m, L_1, \dots, L_n$ lie in $\mathcal{L}_{\mathbb{K}}(\mathbb{C}^k)$, with $L_1, \dots, L_n$ linearly independent, such that

$$Q(\ell_i, L_j) = 0 \quad \forall i \in \{1, \dots, m\}, j \in \{1, \dots, n\},$$

- $s_1, \dots, s_n$ are positive integers.
- Let $\mathcal{M}_{\mathbb{K}-}^Q$ denote the space spanned by polar germs.
- $\mathcal{M}_{\mathbb{Q}}^Q$ has a rich structure: Laurent expansion, residues, gradations, etc.

Minimal subtraction

- There is a decomposition

$$\mathcal{M}^{Q}_{\mathbb{K}} = \mathcal{M}^{Q}_{\mathbb{K}+} \oplus \mathcal{M}^{Q}_{\mathbb{K}-}.$$

- The minimal subtraction scheme on $(\mathcal{M}, \mathcal{M}_+)$ is the projection:

$$\pi_+ : \mathcal{M}^{Q}_{\mathbb{K}} \to \mathcal{M}^{Q}_{\mathbb{K}+}$$

  alone $\mathcal{M}^{Q}_{\mathbb{K}-}$.
- In the one variable case,

$$\pi_+ = \pi_+^{Q} : \mathcal{M}(\mathbb{C}) = \mathbb{C}[z^{-1}, z]] \longrightarrow \mathcal{M}_+(\mathbb{C}) = \mathbb{C}[[z]],$$

$$f(z) = \sum_{k=-K}^{\infty} a_k\, z^k \mapsto \sum_{k=0}^{\infty} a_k\, z^k.$$

- This is used in the one variable regularization/renormalization, as in the algebraic approach of Connes and Kreimer.

Simplex fractions

- For any subset U of $\mathcal{M}_{\mathbb{K}}$, let $\mathbb{Q}U$ denote the $\mathbb{Q}$-subspace of $\mathcal{M}_{\mathbb{K}}$ spanned by U. For any $\mathbb{Q}$-subspace V of $\mathcal{M}$, let $\overline{V}$ denote $\mathbb{Q}+V$ where $\mathbb{Q}$ stands for the constant functions.
- A **simplex fraction** is a fraction of the form $\frac{1}{L_1^{s_1}\cdots L_k^{s_k}}$, where $L_1,\ldots,L_k \in \mathcal{L}_{\mathbb{Q}}$ are linearly independent and $s_i \in \mathbb{Z}_{>0}$, $i=1,\ldots,k$.
- Let $\mathcal{F}$ be the set of all simplex fractions over $\mathbb{Q}$. Then for any inner product Q in $(\mathbb{R}^\infty, \mathbb{Z}^\infty)$, we have $\mathbb{Q}\mathcal{F} \subset \mathcal{M}^Q_{\mathbb{Q}-}$.
- Let $\mathcal{E} := \{e_1, e_2, \ldots\}$ be an orthonormal basis of $\mathbb{R}^\infty$ with respect to Q. Let z_i be the coordinate function corresponding to e_i.
- A **Chen fraction** is of the form

$$\mathfrak{f}\left(\begin{matrix} s_1,\ldots,s_k \\ u_1,\ldots,u_k \end{matrix}\right) := \frac{1}{z_{u_1}^{s_1}(z_{u_1}+z_{u_2})^{s_2}\cdots(z_{u_1}+z_{u_2}+\cdots+z_{u_k})^{s_k}},$$

  $u_i, s_i \in \mathbb{Z}_{>0}, k \in \mathbb{N}, u_i \neq u_j \text{ if } i \neq j$.
- Let $\mathcal{F}^{\mathrm{Ch}} = \mathcal{F}^{\mathrm{Ch},Q,\mathcal{E}}$ denote the set of Chen fractions.

Locality Sets

- A **locality set** is a set X with a binary *symmetric* relation

$$\top := X \times_\top X \subseteq X \times X,$$

called a **locality relation**.
- For $x_1, x_2 \in X$, denote $x_1 \top x_2$ if $(x_1, x_2) \in \top$.
- For $U \subset X$, denote its **polar subset**

$$U^\top := \{x \in X \,|\, (x, U) \subseteq \top\}.$$

- For locality sets $(X, \top_X)$ and $(Y, \top_Y)$, a map $f : X \to Y$ is called a **locality map** if

$$x_1 \top_X x_2 \Longrightarrow f(x_1) \top_Y f(x_2), \quad \forall x_1, x_2 \in X.$$

Examples of locality sets

- For any nonempty set X, being distinct $x_1 \top x_2 \Leftrightarrow x_1 \neq x_2$ defines a locality relation on X.
- $f, f' \in \mathcal{M}_{\mathbb{Q}}$ are ***Q*-orthogonal** (also called locality independent), denoted $f \perp^Q f'$, if $f = f(L_1, \ldots, L_k), f' = f'(L'_1, \ldots, L'_{k'})$ such that
$$Q(L_i, L'_j) = 0, \quad \forall 1 \leq i \leq k, 1 \leq j \leq k'.$$
- This makes $\mathcal{M}_{\mathbb{Q}}$ into a locality set.
- **Example.** Let $(e_1, e_2, \ldots)$ be an orthonormal basis of $(\mathbb{R}^\infty, \mathbb{Z}^\infty, Q)$.
$$\left((z_1, z_2) \mapsto \frac{1}{z_1 + z_2}\right) \perp^Q ((z_1, z_2) \mapsto z_1 - z_2).$$
- The map
$$f : X \to \mathcal{M}_{\mathbb{Q}}, \quad n \mapsto \frac{1}{z_n}, n > 1,$$
is a locality map.

Locality algebras

- A **locality vector space** is a vector space V equipped with a locality relation $\top$ which is compatible with the linear structure on V:

$$X \subseteq V \Longrightarrow X^{\top} \leq V.$$

- A **locality algebra** over K is a locality vector space $(A, \top)$ over K together with a map

$$m_A : A \times_{\top} A \to A, (x, y) \mapsto x \cdot y = m_A(x, y) \quad \text{for all } (x, y) \in A \times_{\top} A$$

having the following variations of the associativity and distributivity.

(a) For $x, y, z \in A$,

$$x \top y, x \top z, y \top z \Longrightarrow (x \cdot y) \top z, \quad x \top (y \cdot z), \quad (x \cdot y) \cdot z = x \cdot (y \cdot z).$$

(b)

$$x \top z, y \top z \Longrightarrow (x + y) \cdot z = x \cdot z + y \cdot z, \quad z \cdot (x + y) = z \cdot x + z \cdot y,$$
$$(kx) \cdot z = k(x \cdot z), \quad x \cdot (kz) = k(x \cdot z), \; k \in K.$$

- A **unitary locality algebra** is a nonunitary locality algebra $(A, \top, m_A)$ with a unit 1_A such that, for each $u \in A$, we have $1_A \top u$ and

$$1_A \cdot x = x \cdot 1_A = x.$$

Locality algebra homomorphisms

- Given locality algebras $(A_i, \top_i), i = 1, 2$, a (resp. **unitary**) **locality algebra homomorphism** is a linear map $\varphi : A_1 \longrightarrow A_2$ such that

$$\text{if } a \top_1, b \text{ then } \varphi(a) \top_2 \varphi(b) \quad \text{and} \quad \varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$$

(resp. and $\varphi(1_{A_1}) = 1_{A_2}$).
- The pair $(\mathcal{M}_{\mathbb{Q}}, \perp^Q)$ (resp. $(\mathcal{M}_{\mathbb{Q}+}, \perp^Q)$) is a unitary locality algebra and the projection

$$\pi_+^Q : \mathcal{M}_{\mathbb{Q}} = \mathcal{M}_{\mathbb{Q}+}^Q \oplus \mathcal{M}_{\mathbb{Q}-}^Q \to \mathcal{M}_{\mathbb{Q}+}^Q$$

along $\mathcal{M}_{\mathbb{Q}-}^Q$ is a unitary locality algebra morphism.
- For a unital locality algebra $(A, \top)$, a unitary locality endomorphism of A is a **locality automorphism** if it is invertible, preserves the unit, and the inverse map is a locality algebra homomorphism. Let $\mathrm{Aut}^\top(A)$ denote the set of locality automorphisms of A.
- $\mathrm{Aut}^\top(A)$ forms a group for the composition.

Locality algebras of meromorphic germs

- Given a set $\mathcal{S}$ of simplex fractions, let

$$\Pi^Q(\mathcal{S}) = \left\{ \prod_i s_i \,\middle|\, s_i \in \mathcal{S}, \forall i, s_i \perp^Q s_j, \forall i \neq j \right\}$$

be the set of simplex fractions locality generated by $\mathcal{S}$.

- The subspace of $\mathbb{Q}\mathcal{F}$

$$\mathbb{Q}\Pi^Q(\mathcal{S}) := \left\{ \sum_i c_i S_i \,\middle|\, c_i \in \mathbb{Q}, S_i \in \Pi^Q(\mathcal{S}) \right\}$$

spanned by $\Pi^Q(\mathcal{S})$ is a nonunital locality subalgebra of $\mathbb{Q}\mathcal{F}$.

- The set

$$\mathcal{M}^Q_{\mathbb{Q}+}(\Pi^Q(\mathcal{S})) := \left\{ h_0 + \sum_i h_i S_i, \,\middle|\, h_0, h_i \in \mathcal{M}_{\mathbb{Q}+}, h_i \perp^Q S_i \right\}$$

is a unital locality subalgebra of $\mathcal{M}_{\mathbb{Q}}$.

Examples

- A **Chen fraction** is of the form

$$\mathfrak{f}\left(\begin{smallmatrix} s_1,\ldots,s_k \\ u_1,\ldots,u_k \end{smallmatrix}\right) := \frac{1}{z_{u_1}^{s_1}(z_{u_1}+z_{u_2})^{s_2}\cdots(z_{u_1}+z_{u_2}+\cdots+z_{u_k})^{s_k}},$$

$u_i, s_i \in \mathbb{Z}_{>0}, k \in \mathbb{N}, u_i \neq u_j \text{ if } i \neq j$.

- The set $\mathcal{F}^{\rm Ch} = \mathcal{F}^{{\rm Ch},Q,\mathcal{E}}$ of Chen fractions generates the locality subalgebra $\mathcal{M}_{\mathbb{Q}}^{\rm Ch} := \mathcal{M}_{\mathbb{Q}+}^{Q}\left(\Pi^Q(\mathcal{F}^{\rm Ch})\right)$.
- For a subset J of $\mathbb{Z}_{>0}$, denote $z_J := \sum_{j\in J} z_j$.
- A **Feynman fraction** is a simplex fraction

$$\frac{1}{\prod_{J\in\mathcal{J}} z_J^{s_i}}, s_i > 0,$$

for a finite collection $\mathcal{J}$ of subsets of $\mathbb{N}$.

- The set $\mathcal{F}^{\rm Fe} = \mathcal{F}^{{\rm Fe},Q,\mathcal{E}}$ of Feynman fractions generates the locality subalgebra $\mathcal{M}_{\mathbb{Q}}^{\rm Fe} := \mathcal{M}_{\mathbb{Q}+}^{Q}\left(\Pi^Q(\mathcal{F}^{\rm Fe})\right)$.

Locality polynomial algebras

- We shall use the following locality version of polynomial algebras.
- Let $(A, \top)$ be a locality algebra. Let X be a subset of A.
- A **locality monomial** from X is a product $x_1 \cdots x_r$ where $x_i \top x_j$ for $i \neq j$.
- The set X is called **locality algebraically independent** if the locality monomials from X are linearly independent.
- The set X is called a **locality generating set** of $(A, \top)$ if the only locality subalgebra of $(A, \top)$ containing X is A itself.
- The locality algebra $(A, \top)$ is called a **locality polynomial algebra** generated by X if X is locality algebraically independent, and is a locality generating set of $(A, \top)$.

Locality Galois groups

- Let
$$A := \mathcal{M}_{\mathbb{Q}+}\left(\Pi^Q(\mathcal{S})\right)$$
be the $\mathcal{M}^Q_{\mathbb{Q}+}$-subalgebra of $\mathcal{M}_{\mathbb{Q}}$ generated by a set $\mathcal{S}$ of simplex fractions.
- The set
$$\mathrm{Gal}^Q(A/\mathcal{M}_{\mathbb{Q}+}) := \left\{\varphi \in \mathrm{Aut}^Q(A) \;\middle|\; \varphi \text{ restricts to the identity on } \mathcal{M}_{\mathbb{Q}+}\right\}$$
defines a group, which we call the **locality Galois group** of A over $\mathcal{M}_{\mathbb{Q}+}$.
- Define a subset of $\mathrm{Gal}^Q(A/\mathcal{M}_{\mathbb{Q}+})$ by
$$\mathrm{Gal}^Q_{\mathrm{Res}}(A/\mathcal{M}_{\mathbb{Q}+}) := \left\{\varphi \in \mathrm{Gal}^Q(A/\mathcal{M}_{\mathbb{Q}+}) \;\middle|\; \begin{array}{l}\varphi \text{ preserves the p-residue,}\\ \text{d-residue and } B\end{array}\right\}$$
for $B = \overline{\mathbb{Q}(\Pi^Q(\mathcal{S}))}$.

Group structure

- The following theorem shows that an element of $\mathrm{Aut}^Q_{\mathrm{Res}}(B)$ can be extended to an element of $\mathrm{Gal}^Q_{\mathrm{Res}}(A/\mathcal{M}_{\mathbb{Q}+})$.
- **Theorem.** Let $B = \overline{\mathbb{Q}\Pi^Q(\mathcal{S})}$.
  1. Any element $\varphi \in \mathrm{Aut}^Q_{\mathrm{Res}}(B)$ uniquely extends to an element of $\mathrm{Gal}^Q_{\mathrm{Res}}(A/\mathcal{M}_{\mathbb{Q}+})$ defined by

  $$\tilde{\varphi}\left(h_0 + \sum_i h_i S_i\right) := h_0 + \sum_i h_i \varphi(S_i)$$

  for any

  $$f = h_0 + \sum_i h_i S_i \in A, h_i \in \mathcal{M}_{\mathbb{Q}+}, S_i \in \Pi^Q(\mathcal{S}).$$

  2. The subset $\mathrm{Gal}^Q_{\mathrm{Res}}(A/\mathcal{M}_{\mathbb{Q}+}) \subseteq \mathrm{Gal}^Q(A/\mathcal{M}_{\mathbb{Q}+})$ is a subgroup. Taking restriction to B gives rise to a group isomorphism

  $$\mathrm{Gal}^Q_{\mathrm{Res}}(A/\mathcal{M}_{\mathbb{Q}+}) \cong \mathrm{Aut}^Q_{\mathrm{Res}}(B).$$

Locality generalised evaluators

- We now consider the action of the locality Galois group on generalised evaluators. Let A be a locality subalgebra of the algebra $\mathcal{M}_{\mathbb{Q}}$ equipped with the locality relation $\perp^Q$.
- A **locality generalised evaluator** $\mathcal{E}$ on the locality algebra $(A, \perp^Q)$ is a linear form $\mathcal{E} : A \to \mathbb{C}$, such that

$$f_1 \perp^Q f_2 \Longrightarrow \mathcal{E}(f_1 \cdot f_2) = \mathcal{E}(f_1) \cdot \mathcal{E}(f_2),$$

$$\mathcal{E}(h) = h(0),$$

  for all $h \in \mathcal{M}_{\mathbb{Q}+}$ and $f_1, f_2 \in A$. We use $E(A) = E^Q(A)$ to denote the set of locality generalized evaluators on $(A, \perp^Q)$.
- Let $\mathrm{ev}_0 : \mathcal{M}_{\mathbb{Q}+} \to \mathbb{Q}$ be the evaluation at 0 defined as $\mathrm{ev}_0(h) = h(0)$. Then for the map $\pi_+^Q : \mathcal{M}_{\mathbb{Q}} \to \mathcal{M}_{\mathbb{Q}+}$, the composition

$$\mathcal{E}_{\mathrm{MS}}^Q := \mathrm{ev}_0 \circ \pi_+^Q \tag{1}$$

  is a locality generalised evaluator on A, called the **locality minimal subtraction scheme.**

Transitive group action

- The group $\mathrm{Gal}^Q(A/\mathcal{M}_{\mathbb{Q}+})$ acts on $E^Q(A)$:

$$\mathrm{Gal}^Q(A/\mathcal{M}_{\mathbb{Q}+}) \times E^Q(A) \to E^Q(A) : (g, \mathcal{E}) \mapsto \mathcal{E}^g := \mathcal{E} \circ g^{-1}.$$

- **Theorem.** Let $A = \mathcal{M}^Q_{\mathbb{Q}+}(\Pi^Q(\mathcal{S}))$ be a simplex locality subalgebra of $\mathcal{M}$. Provided $\overline{\mathbb{Q}\Pi^Q(\mathcal{S})}$ is a locality polynomial subalgebra of $\overline{\mathbb{Q}\mathcal{F}}$, then a locality generalised evaluator $\mathcal{E}$ on A factorises through the minimimal subtraction scheme $\mathcal{E}^Q_{\mathrm{MS}}$, i.e, there is $\tilde{\varphi} \in \mathrm{Gal}^Q_{\mathrm{Res}}(A/\mathcal{M}_{\mathbb{Q}+})$ such that

$$\mathcal{E}^Q_{\mathrm{MS}} \circ \tilde{\varphi} = \mathcal{E}.$$

- In this respect, $\mathrm{Gal}^Q_{\mathrm{Res}}(A/\mathcal{M}_{\mathbb{Q}+})$ can be regarded as a renormalisation group on the generalised locality evaluators (regarded as regulators).

Ordered fractions

- Let $(U, \leq)$ be a countable well-ordered set equipped with an irreflexive locality relation $\top$. let

$$L : U \to \mathcal{L}_{\mathbb{Q}}, \quad u \mapsto L_u, u \in U,$$

be a map, parameterizing a family of linear forms parameterised by U.

- For u_i in U, $s_i \geq 1, 1 \leq i \leq k$, define the **ordered fraction** (with respect to L)

$$\mathfrak{f}^L\left(\begin{array}{c} s_1,\ldots,s_k \\ u_1,\ldots,u_k \end{array}\right) := \frac{1}{L_{u_1}^{s_1}(L_{u_1}+L_{u_2})^{s_2}\cdots(L_{u_1}+\cdots+L_{u_k})^{s_k}}.$$

- Define the set of **ordered fractions** (with respect to L)

$$\mathcal{F}^L := \left\{ \mathfrak{f}^L\left(\begin{array}{c} s_1,\ldots,s_k \\ u_1,\ldots,u_k \end{array}\right) \middle| s_i \geq 1, u_i \in U, 1 \leq i \leq k, k \geq 0 \right\} \subseteq \mathcal{F}. \qquad (2)$$

- Define $L : \mathbb{Z}_{>0} \to \mathcal{L}_{\mathbb{Q}}$ by $L(i) = z_i$. The fraction $\frac{1}{(z_1+z_2)(z_2+z_3)}$ is not an ordered fraction.

Locality polynomial algebras

- Let $\mathbb{Q}\mathcal{F}^L$ be the $\mathbb{Q}$-subspace spanned in $\mathbb{Q}\mathcal{F}^L$.
- When $L = L_{\rm wCh} : \mathbb{Z}_{>0} \to \mathcal{L}_{\mathbb{Q}}$ is given by $L(u) = z_u$, then

$$\mathfrak{f}^L\left(\begin{array}{c} s_1,\ldots,s_k \\ u_1,\ldots,u_k \end{array}\right) = \mathfrak{f}^{\rm wCh}\left(\begin{array}{c} s_1,\ldots,s_k \\ u_1,\ldots,u_k \end{array}\right)$$

  and $\mathcal{F}^L = \mathcal{F}^{\rm wCh}$ is the set of weak Chen fractions for applications to multiple zeta values.
- Consider

$$\mathcal{F}^L_{\rm loc} := \mathbb{Q}\left\{ \mathfrak{f}^L\left(\begin{array}{c} s_1,\ldots,s_k \\ u_1,\ldots,u_k \end{array}\right) \middle| \begin{array}{c} s_i \geq 1, u_i \in U, u_i \top u_j \in U, \\ 1 \leq i \neq j \leq k, k \geq 0 \end{array} \right\} \subseteq \mathbb{Q}\mathcal{F}$$

  and the $\mathbb{Q}$-subspace $\mathbb{Q}\mathcal{F}^L_{\rm loc}$ of $\mathbb{Q}\mathcal{F}$.
- **Theorem.** Let $(U, \leq, \top)$ be a countable well-ordered set with an irreflexive locality relation $\top$. If the map $L : (U, \top) \to (\mathcal{L}_{\mathbb{Q}}, \perp^Q)$ is a locality map:

$$x \top y \Rightarrow L_x \perp^Q L_y, \quad \forall x, y \in U,$$

  then the locality algebra $\mathbb{Q}\mathcal{F}^L_{\rm loc}$ is a locality polynomial algebra.

Examples: Chen fractions

- Let U be $\mathbb{Z}_{>0}$ with the natural order and the locality relation

$$n \top m \Leftrightarrow n \neq m.$$

For $L : \mathbb{Z}_{>0} \to \mathcal{L}_{\mathbb{Q}}, i \mapsto z_i, i \in \mathbb{Z}_{>0}$, The set $\mathcal{F}^L_{\mathrm{loc}}$ of ordered fractions is the set $\mathcal{F}^{\mathrm{Ch}}$ of Chen fractions.

Examples: Speer fractions

- Let U be the set $\mathcal{P}_{\text{fin}}(\mathbb{Z}_{>0})$ of nonempty finite subsets of $\mathbb{Z}_{>0}$. The order is the lexicographic order: for elements

$$I := \{i_1 > i_2 > \cdots > i_r\}, \quad J := \{j_1 > j_2 > \cdots > j_s\}$$

in $\mathcal{P}_{\text{fin}}(\mathbb{Z}_{>0})$, define $I \geq J$ if either the first nonzero element in the sequence

$$i_1 - j_1, i_2 - j_2, \ldots, i_{\min\{r,s\}} - j_{\min\{r,s\}}$$

is positive, or the above sequence of numbers are all zero and $r > s$. The locality relation in $\mathcal{P}_{\text{fin}}(\mathbb{Z}_{>0})$ is:

$$I \top J \Leftrightarrow I \cap J = \emptyset.$$

- Define $L : \mathcal{P}_{\text{fin}}(\mathbb{Z}_{>0}) \to \mathcal{L}_{\mathbb{Q}}, I \mapsto z_I := \sum_{i \in I} z_i, \quad \forall I \in \mathcal{P}_{\text{fin}}(\mathbb{Z}_{>0})$.
- Then the fractions $\mathfrak{f}^L\left(\begin{array}{c} s_1, \ldots, s_k \\ I_1, \ldots, I_k \end{array}\right)$, called **Speer fractions**, are of the form

$$\frac{1}{z_{I_1}^{s_1}(z_{I_1} + z_{I_2})^{s_2} \cdots (z_{I_1} + \cdots + z_{I_k})^{s_k}}.$$

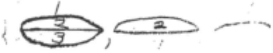
- Let $\mathcal{F}^{\text{Sp}}$ denote the set formed by the Speer fractions.

Applications

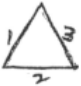
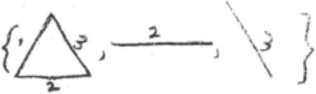
- $\mathbb{Q}\mathcal{F}^{\mathrm{Ch}}$ and $\mathbb{Q}\mathcal{F}^{\mathrm{Sp}}$ are locality polynomial algebras.
- As a direct consequence of the general theorem, we obtain
- The space $E^Q(\mathcal{M}_{\mathbb{Q}}^{\mathrm{Ch}})$ (resp. $E^Q(\mathcal{M}_{\mathbb{Q}}^{\mathrm{Sp}})$) of locality generalized evaluators on the locality algebra $\left(\mathcal{M}_{\mathbb{Q}}^{\mathrm{Ch}}, \perp^Q\right)$ $\left(\text{resp. } \left(\mathcal{M}_{\mathbb{Q}}^{\mathrm{Sp}}, \perp^Q\right)\right)$ is a homogeneous space of $\mathrm{Gal}^Q\left(\mathcal{M}_{\mathbb{Q}}^{\mathrm{Ch}}/\mathcal{M}_{\mathbb{Q}+}\right)$ (resp. $\mathrm{Gal}^Q\left(\mathcal{M}_{\mathbb{Q}}^{\mathrm{Sp}}/\mathcal{M}_{\mathbb{Q}+}\right)$). In other words, these groups act transitively on $\left(\mathcal{M}_{\mathbb{Q}}^{\mathrm{Ch}}, \perp^Q\right)$ $\left(\text{resp. } \left(\mathcal{M}_{\mathbb{Q}}^{\mathrm{Sp}}, \perp^Q\right)\right)$.
- We next show that the latter example is related to the classical work of Speer on analytic renormalization.

Speer's s-families

- For a Feynman graph G, a family $\mathbb{E}$ of subgraphs of G is called a **singularity family** or simply an ***s*-family** if
- (1) every element in $\mathbb{E}$ is either 2-connected or a single line. Let $\mathbb{E}'$ denote the subset of 2-connected elements in $\mathbb{E}$;
- (2) $\mathbb{E}$ is nonoverlapping, that is, for $H_1, H_2 \in \mathbb{E}$, either $H_1 \subset H_2$ or $H_2 \subset H_1$ or $H_1 \cap H_2 = \emptyset$;
- (3) no union of two or more disjoint elements is 2-connected;
- (4) $\mathbb{E}$ is maximal with these properties.

Example 7: If G = , a typical s-family is E = 

with $\mathcal{D}(E) = \{\alpha_3 \geq \alpha_2 \geq \alpha_1\}$. The decomposition (2.16) corresponds to separate integration over such region in which the order of the α_ℓ is completely specified.

If G = , a typical s-family is E = 

with $\mathcal{D}(E) = \{\alpha_1 \geq \alpha_2, \alpha_1 \geq \alpha_3\}$. Here (2.16) is a separation into 3 integrals according to which α_ℓ is largest.

s-families in regularization

- The generalized Feynman amplitude $\mathcal{T}$ of G (regularized by introducing a variable λ_ℓ to every edge ℓ of G) has a decomposition

$$\mathcal{T} = \sum_{\mathtt{E}} \mathcal{T}_{\mathtt{E}}, \text{ the sum is over all s-families of } G.$$

- For an s-family $\mathtt{E}$ of G, and $H \in \mathtt{E}'$, let

$$\Lambda(H) := \sum_{\ell \in L(H)} (\lambda_\ell - 1),$$

  where $L(H)$ is the set of edges of H.

- When renormalizing $\mathcal{T}$ at $\lambda_\ell = 1, \ell \in L(G)$, the singularities are of the form

$$\left(\prod_{H \in \mathtt{E}'} \Lambda(H) \right)^{-1}.$$

s-families and Speer fractions

- By a change of variables $z_i = \lambda_{\ell_i} - 1$, with an ordering $\ell_1, \ldots, \ell_{|L(G)|}$ of $L(G)$, the $\Lambda(H)$ corresponds to the linear form $z_{I(H)}$, for $I(H) = \{i \mid \ell_i \in L(H)\}$.
- So we only need to deal with germs of the form

$$\left(\prod_{H \in \mathbb{E}'} z_{I(H)} \right)^{-1} h$$

  for a holomorphic germ h.
- **Proposition.** For any s-family $\mathbb{E}$ of G, the fraction

$$\left(\prod_{H \in \mathbb{E}'} z_{I(H)} \right)^{-1}$$

  is a linear combination of Speer fractions.

- **Thank You!**

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