I] MOTIVATION for studying (weighted) composition operators

$\mathcal{L}(H) = \{T : H \to H$ linear bounded$\}$

1) Rota ('59-'60) introduced the concept of universal operators

Def 1: Let H be a Hilbert space. An operator $U \in \mathcal{L}(H)$ is universal if for all $T \in \mathcal{L}(H)$,

- $\exists\, \mathcal{M}$ a closed subspace of H, $\mathcal{M} \neq \{0\}$
- $\exists\, J : \mathcal{M} \to H$ an isomorphism
- $\exists\, \lambda \neq 0$, $\lambda \in \mathbb{C}$ st

$$U \mid \begin{cases} U\mathcal{M} \subset \mathcal{M} \\ U|_{\mathcal{M}} = J^{-1}(\lambda T)J \end{cases}$$

In other words, a universal operator is a sort of "model" for all $T \in \mathcal{L}(H)$

Caradus '69: gave a convenient sufficient condition for a Hilbert space operator to be universal

Theorem 2: U is universal, $U \in \mathcal{L}(H)$ if

(i) $\dim \operatorname{Ker} U = +\infty$

(ii) U is surjective

(Necessary)

Consider now the hyperbolic automorphism of $\mathbb{D}$ $=\{z\in\mathbb{C}: |z|<1\}$

$$\varphi_r(z)=\frac{z+r}{1+rz}\qquad r\in(0,1)$$

$\varphi_r(\mathbb{D})=\mathbb{D}$ bijective hol

$\varphi_r(1)=1\ ,\ \varphi_r(-1)=-1$

Nordgren-Rosenthal-Wintrobe '87

proved that, thanks to Thm 2 + a lot of extra material (Carleson sequences, model spaces RK)

that $C_{\varphi_r}-Id$ is universal on the

Hilbert Space $H^2(\mathbb{D})=\{f:\mathbb{D}\to\mathbb{C}$ holomorphic and $\sup\limits_{0\le r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{it})|^2dt<+\infty\}$

$$\left(=\{f\in Hol(\mathbb{D},\mathbb{C}):(\hat{f}(n))_n\in\ell^2\}\right)$$

Another proof of the universality of $C_{\varphi_r}-Id$ relies on the fact that

φ_r can be embedded into a holomorphic flow

and therefore C_{φ_r} can be embedded into a group of composition operators

$\varphi_r(z) = \psi_{t_0}(z)$ $\quad (\psi_t)_{t\in\mathbb{R}}$ is a flow

$r = \dfrac{1-e^{-t_0}}{1+e^{-t_0}}$ $\quad \psi_t(z) := \varphi_{\frac{1-e^{-t}}{1+e^{-t}}}(z)$

The question whether a given $T \in \mathcal{L}(X)$ can be embedded into a semigroup or a group is a project of research in progress with Fida, George, Fares, Ihab

2) Isometries

- Another reason to study weighted composition operators goes back to Banach (1892 – 1945) [' 87 Theory of linear operators)
  Lviv

who showed that every surjective isometry T on $\mathcal{C}(K) = \{f : K \to \mathbb{C} \text{ continuous}\}$ has the
compact metric space

form $Tf = w(f \circ \varphi)$ where $w \in \mathcal{C}(K)$, $|w(z)| = 1$ and φ is a homeomorphism on K

- $H^2(\mathbb{D})$ is a Hilbert space and thus has many linear isometries. However for other p
  $1 \leq p < +\infty$, if one consider

$$H^p(\mathbb{D}) := \left\{ f : \mathbb{D} \to \mathbb{C} : \sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt < +\infty \right\}$$

the isometries on $H^p(\mathbb{D})$ are relatively few and they are expressible as weighted composition operators:

In 1960 de Leeuw, Rudin, Wermer (J) gave a description of isometric surjections of $H^1(\mathbb{D})$ which arise from conformal mapping of $\mathbb{D}$ onto $\mathbb{D}$

This was completed by Forelli 1964 who proved this theorem and did not assume surjectivity:

Theorem 3: (Forelli 1964)

Suppose that $\underset{1 \le p < +\infty}{p \neq 2}$, $T : H^p(\mathbb{D}) \to H^p(\mathbb{D})$ is a linear isometry. Then there is a non constant inner function Φ and $w \in H^p(\mathbb{D})$ such that $Tf = w f \circ \Phi$, $f \in H^p(\mathbb{D})$

$\Phi : \mathbb{D} \to \mathbb{D}$ analytic is called "inner" if

$\lim_{r \to 1^-} \Phi(re^{it})$ exists a.e $/t$ and

is of modulus 1

One knows exactly how to describe such functions:

$$\Phi(z) = \text{Blaschke product} \times e^{-\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t)}$$

$\mu \perp m$ lebesgue

$\mu \geq 0$

"

$$e^{i\theta} \prod \frac{\overline{\alpha_n}}{|\alpha_n|} \frac{\alpha_n - z}{1 - \overline{\alpha_n} z}$$

$(\alpha_n)_n \in \mathbb{D}$

$\sum 1 - |\alpha_n| < +\infty$

3) Change of domains

$\mathbb{C}_+ =$ right half-plane

$$H^2(\mathbb{C}_+) := \left\{ f : \mathbb{C}_+ \to \mathbb{C} \text{ holomorphic } \sup_{x > 0} \int_{-\infty}^{+\infty} |f(x+iy)|^2\, dy < +\infty \right\}$$

It is well-known that composition operators on $H^2(\mathbb{C}^+)$ are unitarily equivalent to <u>weighted</u> composition operators on $H^2(\mathbb{D})$

For example the following explicit formula is given

Prop 6 (I C + J. Partington 2003)

Let M denote the self-inverse bijection from $\mathbb{D}$ to $\mathbb{C}_+$ given by $M(z) = \frac{1-z}{1+z}$ and $\Psi : \mathbb{C}_+ \to \mathbb{C}_+$ be holomorphic. Then the composition operator C_Ψ on $H^2(\mathbb{C}_+)$ is unitarily equivalent to the operator

$L_\Phi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ defined by

$$L_\Phi f(z) = \frac{1+\Phi(z)}{1+z} f(\Phi(z))$$

where $\Phi = M \circ \Psi \circ M$

So for example, C_0-group $(T_t)_{t\in\mathbb{R}}$ on $H^2(\mathbb{C}_+)$ given by $T_t g(z) = g(e^t z)$ $(z \in \mathbb{C}_+)$ for $g \in H^2(\mathbb{C}_+)$ is unitarily equivalent to weighted composition group $(S_t)_{t\in\mathbb{R}}$ given by

$$S_t f(z) = \frac{2}{1+z+e^t(1-z)} f\left(\frac{1+z-e^t(1-z)}{1+z+e^t(1-z)}\right)$$