Higher order generalizations of harmonic maps

At the confluence of geometry, analysis and mathematical physics

Volker Branding

University of Vienna

March 3, 2022

universität wien

FWF Der Wissenschaftsfonds.

Outline

1. Introduction
   - An short introduction to harmonic maps
   - Biharmonic maps between Riemannian manifolds
2. r-harmonic maps between Riemannian manifolds
   - A structure theorem for polyharmonic maps
   - Unique continuation theorems
3. ES-r-harmonic maps between Riemannian manifolds
4. Outlook

Harmonic maps

- Let (M, g_{ij}) and $(N, h_{\alpha\beta})$ be Riemannian manifolds.
- Let $\phi\colon M \to N$ be a map.
- We define the energy functional

$$E(\phi) = \int_M |d\phi|^2 dV = \int_M \frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j} g^{ij} h_{\alpha\beta}(\phi) dV\,.$$

- $E(\phi)$ is invariant under conformal transformations on M if $\dim M = 2$.
- **Harmonic maps** are critical points of $E(\phi)$, which satisfy

$$\tau(\phi) := \mathrm{tr}_g \bar{\nabla} d\phi = 0, \qquad \tau(\phi) \in \Gamma(\phi^* TN),$$

where $\bar{\nabla}$ represents the connection on $\phi^* TN$.
- In terms of local coordinates we have

$$\Delta_M \phi^\alpha + \Gamma^\alpha_{\beta\gamma} g_{ij} \frac{\partial\phi^\beta}{\partial x_i}\frac{\partial\phi^\gamma}{\partial x_j} = 0.$$

Existence of harmonic maps

- Via the L^2-**gradient flow**

$$\frac{\partial \phi_t}{\partial t} = \tau(\phi_t), \qquad \phi(\cdot, 0) = \phi_0 \,. \tag{1}$$

Theorem (Eells - Sampson, 1964)

*Let M and N be closed Riemannian manifolds and assume that the **sectional curvature of N is non-positive**. Then (1) has a unique smooth solution $\phi_t \in C^\infty(M \times [0,\infty), N)$ for arbitrary $\phi_0 \in C^\infty(M,N)$, which for $t \to \infty$, converges to a harmonic map $\phi_\infty \in C^\infty(M,N)$ in $C^2(M,N)$.*

- What happens if we weaken the condition $K_N \leq 0$?
- Eells-Wood have shown: There does not exist a harmonic map $\phi\colon \mathbb{T}^2 \to \mathbb{S}^2$ with $\deg \phi = \pm 1$ regardless of the metrics on M, N.

Biharmonic maps

- For $\phi\colon M \to N$ consider the bienergy

$$E_2(\phi) = \int_M |\tau(\phi)|^2 dV.$$

- Critical points of $E_2(\phi)$ are called (intrinsic) biharmonic maps and are characterized by the fourth order equation

$$0 = \tau_2(\phi) := -\bar{\Delta}\tau(\phi) + R^N(d\phi(e_i), \tau(\phi))d\phi(e_i),$$

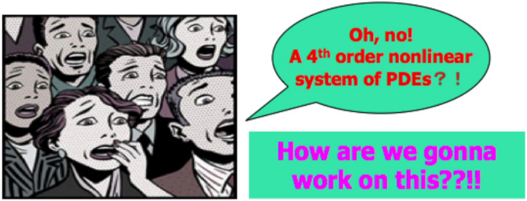
  where $\bar{\Delta}$ is the rough Laplacian on $\phi^* TN$.
- Why study biharmonic maps?
  1. Find interesting maps where $\tau(\phi) \neq 0$.
  2. Scale invariant variational problem on a four-dimensional domain.
  3. Applications in elasticity and string theory.
  4. Similar to the Willmore energy for maps instead of immersions.
- Every harmonic map is biharmonic, a non-harmonic biharmonic map is called *proper biharmonic*.

Local form of the biharmonic map equation

$$
\begin{aligned}
\Delta^2\phi^\theta = &-\big(4\langle\nabla\Delta\phi^\alpha, d\phi^\beta\rangle + 2\langle d\phi^\alpha(\mathrm{Ric}^M), d\phi^\beta\rangle + 2\langle\nabla d\phi^\alpha, \nabla d\phi^\beta\rangle\big)\Gamma^\theta_{\alpha\beta} \\
&- 4g^{ij}\langle\nabla_{\frac{\partial}{\partial x^i}} d\phi^\alpha, d\phi^\beta\rangle\phi^\delta_j A^\theta_{\alpha\beta\delta} - 2\langle d\phi^\alpha, d\phi^\beta\rangle(\Delta\phi^\delta)A^\theta_{\alpha\beta\delta} \\
&- \langle d\phi^\alpha, d\phi^\beta\rangle\langle d\phi^\delta, d\phi^\gamma\rangle B^\theta_{\alpha\beta\delta\gamma} - (\Delta\phi^\alpha)(\Delta\phi^\beta)\Gamma^\theta_{\alpha\beta},
\end{aligned}
$$

$$
A^\theta_{\alpha\beta\delta} := \frac{\partial\Gamma^\theta_{\alpha\beta}}{\partial y^\delta} + \Gamma^\gamma_{\alpha\beta}\Gamma^\theta_{\gamma\delta},
$$

$$
B^\theta_{\alpha\beta\delta\gamma} := \frac{\partial^2\Gamma^\theta_{\alpha\beta}}{\partial y^\delta\partial y^\gamma} + \frac{\partial\Gamma^\omega_{\delta\gamma}}{\partial y^\alpha}\Gamma^\theta_{\omega\beta} + \frac{\partial\Gamma^\omega_{\delta\gamma}}{\partial y^\beta}\Gamma^\theta_{\omega\alpha} + \Gamma^\omega_{\delta\gamma}\frac{\partial\Gamma^\theta_{\alpha\beta}}{\partial y^\omega} + \Gamma^\omega_{\delta\gamma}\Gamma^\sigma_{\alpha\beta}\Gamma^\theta_{\omega\sigma}.
$$



Picture due to Yelin Ou

When are biharmonic maps harmonic?

- Quick answer: If the target has non-positive curvature!

1. Consider solutions of $\Delta^2 f = 0, f = \mathbb{R}^m \to \mathbb{R}$. If f is bounded from above and below, then f is constant, Huilgol (1971).
2. Biharmonic maps from compact Riemannian manifolds and the target having negative curvature must be harmonic, Jiang (1986).
3. If M is a complete non-compact Riemannian manifold with $\mathrm{Ric}^M > 0$ and N a Riemannian manifold with $K^N \leq 0$ then every biharmonic map with $E_2(\phi) < \infty$ must be harmonic, Baird et al. (2010).

Examples of proper biharmonic maps

- Need to consider a target with positive curvature!
- The inverse stereographic projection $\phi\colon \mathbb{R}^4 \to \mathbb{S}^4$ is proper biharmonic.

Theorem (Caddeo - Loubeau - Montaldo - Oniciuc, 2001)

Proper-biharmonic hypersurfaces of $\mathbb{S}^{m+1}$ are for example

1. *hyperspheres $\mathbb{S}^m(1/\sqrt{2})$*
2. *products of spheres $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, where $m_1 + m_2 = m$ and $m_1 \neq m_2$.*

- Conjecture: These are the only proper biharmonic hypersurfaces of $\mathbb{S}^{m+1}$.

Polyharmonic maps between Riemannian manifolds

- How can we obtain a general higher order version of harmonic maps?
- If $k = 2s, s \geq 1$, we set

$$E_{2s}(\phi) = \int_M |\bar{\Delta}^{s-1}\tau(\phi)|^2 dV \,.$$

- In the case that $k = 2s+1, s \geq 1$, we set

$$E_{2s+1}(\phi) = \int_M |\bar{\nabla}\bar{\Delta}^{s-1}\tau(\phi)|^2 dV.$$

- A harmonic map will always be a critical point of $E_k(\phi)$.
- The critical points of $E_{2s}(\phi), E_{2s+1}(\phi)$ can be calculated explicitly and are given by semilinear elliptic PDE's with prinicpal part $\bar{\Delta}^{k-1}\tau(\phi)$.
- A general existence theory is out of reach.

The Euler-Lagrange equations

- First variation of $E_{2s}(\phi), E_{2s+1}(\phi)$ was calculated by Wang/Maeta.
- We set $\bar{\Delta}^{-1} = 0$.
- The critical points of $E_{2s}(\phi)$ are those which satisfy

$$\begin{aligned} 0 = \tau_{2s}(\phi) :=& \bar{\Delta}^{2s-1}\tau(\phi) - R^N(\bar{\Delta}^{2s-2}\tau(\phi), d\phi(e_j))d\phi(e_j) \\ & - \sum_{\ell=1}^{s-1}\Big(R^N(\bar{\nabla}_{e_j}\bar{\Delta}^{s+\ell-2}\tau(\phi), \bar{\Delta}^{s-\ell-1}\tau(\phi))d\phi(e_j) \\ & \qquad - R^N(\bar{\Delta}^{s+\ell-2}\tau(\phi), \bar{\nabla}_{e_j}\bar{\Delta}^{s-\ell-1}\tau(\phi))d\phi(e_j)\Big). \end{aligned}$$

- A similar equation holds for the critical points of $E_{2s+1}(\phi)$.
- Solutions of $\tau_r(\phi) = 0$ are called *polyharmonic maps of order r* or *r-harmonic maps*. They are called *proper* if they are non-harmonic.
- In general, an r-harmonic map is not s-harmonic if $r < s < 1$.

A structure theorem for polyharmonic maps

Theorem (Branding, 2021)

Let (M,g) be a complete non-compact Riemannian manifold that admits a Euclidean type Sobolev inequality and let $\phi\colon M\to N$ be a polyharmonic map of even order k and $\dim =: n > 2k-2$.

1. *Suppose that, for some $\epsilon > 0$ small enough, we have*

$$\int_M |\bar\nabla^q d\phi|^{\frac{n}{q+1}} dV < \epsilon, \qquad \textit{for all} \quad 0\leq q\leq k-2.$$

2. *In addition, assume that*

$$\sum_{q=0}^{k-2}\int_M |\bar\nabla^q\bar\Delta^{\frac{k}{2}-1}\tau(\phi)|^2 dV < \infty.$$

Then ϕ must be harmonic.

- A similar results also holds for k being odd.

Some ideas of the proof I

- An *Euclidean type Sobolev inequality* is an inequality of the form

$$\Big(\int_M |u|^{2m/(m-2)} dV\Big)^{\frac{m-2}{m}} \leq C_2 \int_M |\nabla u|^2 dV$$

  for all $u \in W^{1,2}(M)$ with compact support, where C_2 is a positive constant that depends on the geometry of M.
- Holds in $\mathbb{R}^m$: *Gagliardo-Nirenberg inequality*.
- However, not every complete, non-compact manifold admits such an inequality!

Some ideas of the proof II

- For $r \in \mathbb{N}$ and $X \in \Gamma(\phi^* TN)$ we have

$$
\begin{aligned}
[\bar{\Delta}, \bar{\nabla}^r]X = \sum_{\sum_{l_i} + \sum_{m_j} = r} & \bar{\nabla}^{l_1} R^N \star \underbrace{\bar{\nabla}^{m_1} d\phi \star \ldots \bar{\nabla}^{m_{l_1}} d\phi}_{l_1 - \text{times}} \\
& \star \bar{\nabla}^{l_2} d\phi \star \bar{\nabla}^{l_3} d\phi \star \bar{\nabla}^{l_4} X.
\end{aligned}
$$

Here, a $\star$ refers to various contractions between the objects involved.

Unique continuation theorems for r-harmonic maps

- Assume that M is connected.

Theorem (Branding - Montaldo - Oniciuc - Ratto, 2021)

Let $\phi\colon M \to N$ be an r-harmonic map, $r \geq 3$. If ϕ is harmonic on an open set U, then ϕ is harmonic everywhere.

Theorem (Branding - Montaldo - Oniciuc - Ratto, 2021)

Let $\phi, \tilde{\phi}\colon M \to N$ be two r-harmonic maps, $r \geq 2$. If they agree on an open subset U of M, then they are identical.

- We also have the following geometric application:

Theorem (Branding - Montaldo - Oniciuc - Ratto, 2021)

Let $\phi\colon M \to \mathbb{S}^n$ be an r-harmonic map. If an open subset of M is mapped into the equator $\mathbb{S}^{n-1}$, then all of M is mapped into $\mathbb{S}^{n-1}$.

The mother of all unique continuation results

Theorem (Aronszajn, 1957)

Let A be a linear elliptic second-order differential operator defined on an open subset D of $\mathbb{R}^m$. Let $u = (u^1, \ldots, u^m)$ be functions in D satisfying the inequality

$$|Au^a| \leq C\left(\sum_{b,i}\left|\frac{\partial u^b}{\partial x^i}\right| + \sum_b \left|u^b\right|\right).$$

If $u = 0$ in an open subset of D, then $u = 0$ throughout D.

- How can we apply Aronszajn's result to r-harmonic maps?

Recursively defined variables

- Let $\phi, \tilde{\phi}$ be two r-harmonic maps which coincide on an open subset.
- In a local chart we define a vector-valued function u for ϕ via

$$u = \begin{pmatrix} \phi \\ d\phi \\ u_0 := \Delta\phi \\ v_0 := \nabla\Delta\phi \\ u_1 := \Delta u_0 \\ v_1 := \nabla\Delta u_0 \\ \vdots \\ v_{k-3} := \nabla\Delta u_{k-4} \\ u_{k-2} := \Delta u_{k-3} \end{pmatrix}.$$

- We define a similar variable $\tilde{u}$ for the map $\tilde{\phi}$.

Reduction to a second order problem

- After a long but straightforward calculation we find the inequality

$$
\begin{aligned}
|\Delta(u-\tilde{u})| \leq C\Big(&|\phi-\tilde{\phi}|+|d\phi-d\tilde{\phi}|+|\nabla d\phi-\nabla d\tilde{\phi}| \\
&+\sum_{\ell=0}^{k-2}|u_\ell-\tilde{u}_\ell|+\sum_{\ell=0}^{k-3}(|v_\ell-\tilde{v}_\ell|+|\nabla(v_\ell-\tilde{v}_\ell)|) \\
&+|\nabla(u_{k-2}-\tilde{u}_{k-2})|\Big),
\end{aligned}
$$

  where we used that ϕ and $\tilde{\phi}$ are r-harmonic maps.
- Hence, we can apply Aronszajn's Theorem and deduce that $u=\tilde{u}$ and in particular $\tilde{\phi}=\phi$.

A higher order energy functional of Eells and Sampson

- The study of polyharmonic maps was first proposed by Eells and Sampson in 1964 and later again by Eells and Lemaire in 1983.
- From the book "Selected topics in harmonic maps":

(8.7) A *polyharmonic map of order* k is an extremal of

$$F_k(\phi) = \int_M |(d + d^*)^k \phi|^2 v_g.$$

If $k > m/2$, then F_k satisfies Condition (C) of Palais-Smale; therefore, there is a polyharmonic map of order k in every homotopy class.

(8.8) *Problem. Study the existence of polyharmonic maps in the critical dimension* $m = 2k$. More precisely, what are the existence and nonexistence results analogous to those for harmonic maps in dimension 2?

- Are these r-harmonic maps? Or something different?

How to make sense of this functional?

- Recall $E_r^{ES}(\phi) = \int_M |(d+d^*)^r\phi|^2 dV$.
- We have $d^*\phi = 0, d^2\phi = 0$.
- The tension field can be written as $\tau(\phi) = -d^*d\phi$.
- Then we obtain

$$
\begin{aligned}
E_1^{ES}(\phi) &= \int_M |(d+d^*)\phi|^2 dV = \int_M |d\phi|^2 dV = E(\phi),\\
E_2^{ES}(\phi) &= \int_M |(d+d^*)^2\phi|^2 dV = \int_M |\tau(\phi)|^2 dV = E_2(\phi),\\
E_3^{ES}(\phi) &= \int_M |(d+d^*)^3\phi|^2 dV = \int_M |\bar{\nabla}\tau(\phi)|^2 dV = E_3(\phi).
\end{aligned}
$$

- This gives us the energy functionals that we have studied so far!

Curvature makes the difference!

- The 4-energy of Eells and Sampson has the form

$$\begin{aligned} E_4^{ES}(\phi) &= \int_M |(d+d^*)^2\tau(\phi)|^2 dV \\ &= \int_M |d^2\tau(\phi) + d^*d\tau(\phi)|^2 dV \\ &= \int_M |d^2\tau(\phi)|^2 + |d^*d\tau(\phi)|^2 dV. \end{aligned}$$

- Note that $d^2\tau(\phi) = R^N(d\phi(e_i), d\phi(e_j))\tau(\phi) \neq 0$ in general!
- Taking into account the curvature term this yields

$$\begin{aligned} E_4^{ES}(\phi) = \frac{1}{2}\int_M |R^N(d\phi(e_i), d\phi(e_j))\tau(\phi)|^2 dV + \int_M |\bar{\Delta}\tau(\phi)|^2 dV \\ \neq E_4(\phi). \end{aligned}$$

ES-r-harmonic maps

Definition

1. We call critical points of

$$E_r^{ES}(\phi) = \int_M |(d + d^*)^r \phi|^2 dV$$

ES-r-harmonic maps.

2. A ES-r-harmonic map is called **proper** if it is **not harmonic**.
3. A ES-r-harmonic submanifold, that is a ES-r-harmonic isometric immersion, is called **proper** if it is **not minimal**.

- Harmonic maps are always ES-r-harmonic.
- For $r \geq 5$ it becomes difficult to explicitly compute the critical points of $E_r^{ES}(\phi)$.

ES-r-harmonic hyperspheres

Theorem (Branding - Montaldo - Oniciuc - Ratto, 2020)

Assume that $r \geq 2, m \geq 2$.
The hypersphere $\iota \colon \mathbb{S}^m(R) \hookrightarrow \mathbb{S}^{m+1}$ is a proper $ES - r$-harmonic submanifold of $\mathbb{S}^{m+1}$ if and only if the radius R is equal to $1/\sqrt{r}$.

- A similar existence result can be obtained for the **generalized Clifford torus** $\iota \colon \mathbb{S}^p(R_1) \times \mathbb{S}^q(R_2) \hookrightarrow \mathbb{S}^{p+q+1}$, where $R_1^2 + R_2^2 = 1$.
  More precisely, we can give conditions on

$$p, q, R_1, R_2$$

  under which ι is a **proper** $ES - r$**-harmonic submanifold of** $\mathbb{S}^{p+q+1}$.

Ideas of the proof

- $E_r^{ES}(\phi)$ is invariant under isometries of both target and domain.
- Using the *principle of symmetric criticality* of Palais we obtain:

Let M, N be two Riemannian manifolds and assume that G is a compact Lie group which acts by isometries on both M and N.
If ϕ is a G-equivariant map, then ϕ is a critical point of $E_r^{ES}(\phi)$ if and only if it is stationary with respect to G-equivariant variations.

- Now, we can easily determine R and R_1, R_2!
- It turns out that the maps given by the last two theorems are critical points of both $E_r(\phi)$ and $E_r^{ES}(\phi)$.
  Is there a difference regarding their stability?

Polyharmonic maps from the punctured disk

- We consider maps from the punctured Euclidean disc of the form

$$\begin{aligned} \phi_{\alpha^*} : B^m \backslash \{O\} &\to \mathbb{S}^m \subset \mathbb{R}^m \times \mathbb{R} \\ w &\mapsto \left(\sin \alpha^* \frac{w}{|w|}, \cos \alpha^* \right), \end{aligned} \tag{2}$$

where $\alpha^* \in (0, \pi/2)$ is a constant.

Theorem (Branding - Montaldo - Oniciuc - Ratto, 2020)

There exists a map $\phi_{\alpha^*} : B^m \setminus \{O\} \to \mathbb{S}^m$ of the type (2) which is both $ES-4$-harmonic and 4-harmonic if and only if $m = 8, 9$.

- In the proof one gets an algebraic equation for α^* which is only solvable if $m = 8, 9$.

A remark on Condition C

Theorem (Branding - Montaldo - Oniciuc - Ratto, 2020)

Let $\mathbb{T}^2$ denote the flat 2-torus. Then the following holds true:

(i)

$$\operatorname{Inf}\left\{E_4^{ES}(\phi)\colon \phi \in C^\infty\left(\mathbb{T}^2, \mathbb{S}^2\right),\ \phi \text{ has degree one}\right\} = 0\,.$$

(ii) *The functional $E_4^{ES}(\phi)$ does not admit a minimum in the homotopy class of maps $\phi\colon \mathbb{T}^2 \to \mathbb{S}^2$ of degree one.*

- The proof is inspired by a result of Lemaire for biharmonic maps.
- Should also be true for $r \geq 5$.
- The relation between Condition C and ES-r-harmonic maps is far from being well-understood...

A difference between 4-harmonic and ES-4-harmonic maps

Theorem (Branding, 2021)

1. *Let* $\phi\colon \mathbb{R}^m \to N$ *be a smooth 4-harmonic map,* $m \neq 8$. *Assume that*

$$F_{L4}(\phi) := \int_{\mathbb{R}^m} (|d\phi|^2 + |\bar{\nabla} d\phi|^2 + |\bar{\nabla}^2 d\phi|^2 + |\bar{\nabla}^3 d\phi|^2) dV < \infty.$$

   If $m = 2$ *then* ϕ *must be harmonic, if* $m > 2$ *then* ϕ *must be constant.*

2. *Let* $\phi\colon \mathbb{R}^m \to N$ *be a smooth ES-4-harmonic map,* $m \neq 8$ *and suppose that* $|R^N|_{L^\infty} < \infty$. *Assume that*

$$F_{L4}(\phi) + \int_{\mathbb{R}^m} (|d\phi|^4 |\bar{\nabla} d\phi|^2 + |d\phi|^6) dV < \infty.$$

   If $m = 2$ *then* ϕ *must be harmonic, if* $m > 2$ *then* ϕ *must be constant.*

Sketch of the proof

- Let $u\colon M \to M$ be a diffeomorphism. Then

$$E_4(\phi\circ u, u^*g) = E_4(\phi, g), \qquad E_4^{ES}(\phi\circ u, u^*g) = E_4^{ES}(\phi, g).$$

- This yields a conserved quantity (similarly we obtain $S_4^{ES}(X,Y)$)

$$\begin{aligned}
S_4(X,Y) :=& g(X,Y)\big(-\frac{1}{2}|\bar\Delta\tau(\phi)|^2 - \langle\tau(\phi),\bar\Delta^2\tau(\phi)\rangle \\
&\qquad - \langle d\phi, \bar\nabla\bar\Delta^2\tau(\phi)\rangle + \langle\bar\nabla\tau(\phi),\bar\nabla\bar\Delta\tau(\phi)\rangle\big) \\
&- \langle\bar\nabla_X\tau(\phi),\bar\nabla_Y\bar\Delta\tau(\phi)\rangle - \langle\bar\nabla_Y\bar\Delta\tau(\phi),\bar\nabla_X\bar\Delta\tau(\phi)\rangle \\
&+ \langle d\phi(X),\bar\nabla_Y\bar\Delta^2\tau(\phi)\rangle + \langle d\phi(Y),\bar\nabla_X\bar\Delta^2\tau(\phi)\rangle.
\end{aligned}$$

- For $R>0$ let $\eta\in C_0^\infty(\mathbb{R})$ be a smooth cut-off function satisfying $\eta = 1$ for $|z|\leq R$, $\eta = 0$ for $|z|\geq 2R$ and $|\eta^l(z)|\leq \frac{C}{R^l}$, $l = 1,\ldots,4$.
- Now, we calculate

$$0 = -\int_{\mathbb{R}^m}\langle x\eta(|x|), \operatorname{div} S_4\rangle dV.$$

- The result follows after a lengthy calculation taking the limit $R\to\infty$.

Polyharmonic curves on the sphere

- If $\dim M = 1$ then $E_r^{ES}(\phi) = E_r(\phi)$.

Theorem (Branding, 2021)

The curve $\gamma \colon I \to \mathbb{S}^n$ given by

$$\gamma(s) = \cos(\sqrt{r}s)e_1 + \sin(\sqrt{r}s)e_2 + e_3,$$

where $e_i, i = 1, 2, 3$ are mutually perpendicular and satisfy

$$|e_1|^2 = |e_2|^2 = \frac{1}{r}, |e_3|^2 = \frac{r-1}{r}$$

is a proper r-harmonic curve which is parametrized by arclength.

- These curves have constant geodesic curvature.

Idea of the proof

- Let $\iota\colon \mathbb{S}^n \to \mathbb{R}^{n+1}$ be the inclusion map.
  The Levi-Civita connection ∇ on the sphere along a curve γ satisfies
  $$d\iota(\nabla_{\gamma'}X) = X' + \langle X, \gamma' \rangle \gamma,$$
  where X is a vector field on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$.
- Consider the curve $\gamma\colon I \to \mathbb{S}^n$ given by
  $$\gamma(s) = \cos(as)e_1 + \sin(as)e_2 + e_3,$$
  where $e_i, i = 1,2,3$ are mutually perpendicular and
  $$|e_1|^2 = |e_2|^2 = \alpha^2, |e_1|^2 + |e_3|^2 = 1, \qquad a \in \mathbb{R}.$$
- For such a curve the r-energy is given by
  $$E_r(\gamma) = |I| a^{2r} \alpha^2 (1-\alpha^2)^{r-1}.$$
- Now, just solve $\frac{d}{d\alpha} E_r(\gamma) = 0$.

Outlook

There are many questions on r-harmonic maps / ES-r-harmonic maps as for example:

- Existence via geometric or analytic methods?
- Are k-harmonic/ES-k-harmonic maps to targets with negative curvature harmonic?
- Are k-harmonic/ES-k-harmonic maps stable?
- Can we find a general difference between k-harmonic and ES-k-harmonic maps?

References

V. Branding, S. Montaldo, C. Oniciuc, and A. Ratto.
Higher order energy functionals.
Adv. Math., 370:107236, 60, 2020.

V. Branding.
On Finite Energy Solutions of 4-harmonic and ES-4-harmonic Maps.
J. Geom. Anal., 31(8):8666–8685, 2021.

V. Branding.
A structure theorem for polyharmonic maps between Riemannian manifolds.
J. Differential Equations, 273:14–39, 2021.

V. Branding, S. Montaldo, C. Oniciuc, and A. Ratto.
Unique continuation properties for polyharmonic maps between Riemannian manifolds.
Canad. J. Math., 2021.

The end

Thank you for your attention!