Extremal discs and invariant families of curves

Giuseppe Della Sala (American University of Beirut)
joint work with Florian Bertrand and Bernhard Lamel

At the confluence of Geometry, Analysis and Mathematical Physics
March 3, 2022

The equivalence problem

One of the basic problems in Complex Analysis is to understand whether two objects can be identified via an invertible holomorphic map (*biholomorphism*).

The equivalence problem

One of the basic problems in Complex Analysis is to understand whether two objects can be identified via an invertible holomorphic map (*biholomorphism*).

Typical objects of interest are open domains:

- two simply connected domains in $\mathbb{C}$ are always biholomorphic
- instead, the ball $\{|z_1|^2 + |z_2|^2 < 1\} \subset \mathbb{C}^2$ and the bidisc $\{|z_1| < 1, |z_2| < 1\} \subset \mathbb{C}^2$ are *not* biholomorphic

The equivalence problem

One of the basic problems in Complex Analysis is to understand whether two objects can be identified via an invertible holomorphic map (*biholomorphism*).

Typical objects of interest are open domains:

- two simply connected domains in $\mathbb{C}$ are always biholomorphic
- instead, the ball $\{|z_1|^2 + |z_2|^2 < 1\} \subset \mathbb{C}^2$ and the bidisc $\{|z_1| < 1, |z_2| < 1\} \subset \mathbb{C}^2$ are *not* biholomorphic

or real submanifolds:

- two real-analytic curves in $\mathbb{C}$ are locally biholomorphic
- the paraboloid $\{\operatorname{Im} z_2 = |z_2|^2\} \subset \mathbb{C}^2$ and the hyperplane $\{\operatorname{Im} z_2 = 0\} \subset \mathbb{C}^2$ are *not* biholomorphic.

The equivalence problem

One of the basic problems in Complex Analysis is to understand whether two objects can be identified via an invertible holomorphic map (*biholomorphism*).

Typical objects of interest are open domains:

- two simply connected domains in $\mathbb{C}$ are always biholomorphic
- instead, the ball $\{|z_1|^2 + |z_2|^2 < 1\} \subset \mathbb{C}^2$ and the bidisc $\{|z_1| < 1, |z_2| < 1\} \subset \mathbb{C}^2$ are *not* biholomorphic

or real submanifolds:

- two real-analytic curves in $\mathbb{C}$ are locally biholomorphic
- the paraboloid $\{\operatorname{Im} z_2 = |z_2|^2\} \subset \mathbb{C}^2$ and the hyperplane $\{\operatorname{Im} z_2 = 0\} \subset \mathbb{C}^2$ are *not* biholomorphic.

In fact, a common approach to understand the equivalence of domains is to study the equivalence of their boundaries.

The equivalence problem

The solution of this type of problems must necessarily pass through the identification and study of suitable invariants.

The equivalence problem

The solution of this type of problems must necessarily pass through the identification and study of suitable invariants.

In the case of the ball and the bidisc, one can for example show that the respective automorphism groups are not (algebraically) isomorphic.

The equivalence problem

The solution of this type of problems must necessarily pass through the identification and study of suitable invariants.

In the case of the ball and the bidisc, one can for example show that the respective automorphism groups are not (algebraically) isomorphic.

The comparison of submanifolds is often carried out by means of invariant tensors. The most important example is the *Levi form*, an Hermitian form whose signature is preserved by biholomorphic maps. The Levi form is positive definite on the paraboloid, and vanishes identically on the hyperplane.

Invariant metrics

A wealth of different metrics naturally associated to domains $\Omega \subset \mathbb{C}^n$ provide other important, well studied invariants. Here we will just just recall two of them.

Invariant metrics

A wealth of different metrics naturally associated to domains $\Omega \subset \mathbb{C}^n$ provide other important, well studied invariants. Here we will just just recall two of them.

Bergman metric

The space of L^2 holomorphic functions defined on Ω has the strong property that the evaluation functional $f \to f(z)$ is continuous in the L^2 metric, and can thus be represented as an integral

$$f(z) = \int_\Omega K(z,\zeta) f(\zeta) d\zeta \wedge d\bar{\zeta}.$$

The reproducing kernel $K(z,\zeta)$ is called the *Bergman kernel*. Its values on the diagonal can be used to define a Hermitian metric on Ω as $\partial\bar{\partial} \log K$, called the *Bergman metric*.

Invariant metrics

A wealth of different metrics naturally associated to domains $\Omega \subset \mathbb{C}^n$ provide other important, well studied invariants. Here we will just just recall two of them.

Kobayashi metric
For any $p \in \Omega$, the (infinitesimal) Kobayashi metric assigns to a vector $\xi \in T_p(\mathbb{C}^n)$ the number

$$K_\Omega(p, \xi) := \inf \left\{ \frac{1}{\lambda} > 0 \mid f : \Delta \to \Omega \text{ holomorphic}, f(0) = p, f'(0) = \lambda \xi \right\}$$

which measures "how large" a disc passing through p and tangent to ξ can be made to fit in Ω.
The integrated version of of this metric is the *Kobayashi (pseudo)distance.*

The Kobayashi metric

The Kobayashi metric and distance can degenerate, notably in case Ω contains (the holomorphic image of) a whole complex line. For instance the metric vanishes for $\Omega = \mathbb{C}^n$.

The Kobayashi metric

The Kobayashi metric and distance can degenerate, notably in case Ω contains (the holomorphic image of) a whole complex line. For instance the metric vanishes for $\Omega = \mathbb{C}^n$.

Also, the Kobayashi metric is generally not Hermitian (it is instead a *Finsler metric*), which means that it cannot be analyzed by means of tools like the curvature tensor.

The Kobayashi metric

The Kobayashi metric and distance can degenerate, notably in case Ω contains (the holomorphic image of) a whole complex line. For instance the metric vanishes for $\Omega = \mathbb{C}^n$.

Also, the Kobayashi metric is generally not Hermitian (it is instead a *Finsler metric*), which means that it cannot be analyzed by means of tools like the curvature tensor.

Instead, one usually tries to study the metric by means of *extremal discs*, that is, discs which actually realize the infimum in the definition, so that their derivative at p is as large as possible. The existence of extremal discs is guaranteed at least if Ω is bounded, but the study of their boundary behavior is subtle.

Invariants families of curves

Let $M \subset \mathbb{C}^n$ be a real-analytic submanifold; more in particular our focus will be on the case when M is a real hypersurface of $\mathbb{C}^2$. Some biholomorphic invariants associated to M can be described in terms of special families of curves on the manifold. Notable examples are

Invariants families of curves

Let $M \subset \mathbb{C}^n$ be a real-analytic submanifold; more in particular our focus will be on the case when M is a real hypersurface of $\mathbb{C}^2$. Some biholomorphic invariants associated to M can be described in terms of special families of curves on the manifold. Notable examples are

- (boundaries of) stationary discs attached to M
- (traces of) Segre varieties
- chains (if M is Levi-nondegenerate)

Invariants families of curves

Let $M \subset \mathbb{C}^n$ be a real-analytic submanifold; more in particular our focus will be on the case when M is a real hypersurface of $\mathbb{C}^2$. Some biholomorphic invariants associated to M can be described in terms of special families of curves on the manifold. Notable examples are

- (boundaries of) stationary discs attached to M
- (traces of) Segre varieties
- chains (if M is Levi-nondegenerate)

In the pursuit of understanding the relative behavior of these invariants, it is natural to ask in which cases they give rise to the same families of curves on M.

Stationary discs

Let $M \subset \mathbb{C}^n$ be a real hypersurface, and denote by Δ the unit disc in $\mathbb{C}$. Let $f : \overline{\Delta} \to \mathbb{C}^n$ be a map holomorphic on Δ and continuous up to the boundary. Then f is called a *stationary disc* for M if

Stationary discs

Let $M \subset \mathbb{C}^n$ be a real hypersurface, and denote by Δ the unit disc in $\mathbb{C}$. Let $f : \overline{\Delta} \to \mathbb{C}^n$ be a map holomorphic on Δ and continuous up to the boundary. Then f is called a *stationary disc* for M if

- it is *attached* to M, that is $f(\zeta) \in M$ for all $\zeta \in b\Delta$
- there exists a continuous function $c : b\Delta \to \mathbb{R}^*$ such that the map

$$b\Delta \ni \zeta \to \zeta c(\zeta)\partial\rho(f(\zeta)) \in \mathbb{C}^n$$

extends holomorphically to Δ.

Stationary discs

Let $M \subset \mathbb{C}^n$ be a real hypersurface, and denote by Δ the unit disc in $\mathbb{C}$. Let $f : \overline{\Delta} \to \mathbb{C}^n$ be a map holomorphic on Δ and continuous up to the boundary. Then f is called a *stationary disc* for M if

- it is *attached* to M, that is $f(\zeta) \in M$ for all $\zeta \in b\Delta$
- there exists a continuous function $c : b\Delta \to \mathbb{R}^*$ such that the map

$$b\Delta \ni \zeta \to \zeta c(\zeta)\partial\rho(f(\zeta)) \in \mathbb{C}^n$$

  extends holomorphically to Δ.

Here $\rho : \mathbb{C}^n \to \mathbb{R}$ is a defining function for M and $\partial\rho$ is its complex gradient, i.e. $\partial\rho = \left(\frac{\partial\rho}{\partial z_1}, \ldots, \frac{\partial\rho}{\partial z_n}\right)$.

Stationary discs

Stationary discs were introduced by Lempert ('81) in the context of the study of the Kobayashi metric in strictly convex domains. Extremal discs for the Kobayashi metric are proper and geodesic in this setting, and satisfy the stationarity condition.

Stationary discs

Stationary discs were introduced by Lempert ('81) in the context of the study of the Kobayashi metric in strictly convex domains. Extremal discs for the Kobayashi metric are proper and geodesic in this setting, and satisfy the stationarity condition.

In the setting of strictly pseudoconvex domains, things are more subtle, and a general extremal disc may fail to be proper. However, works of Huang ('94) show that extremal discs which are sufficiently close to the boundary are again proper and geodesics.

Stationary discs

Stationary discs were introduced by Lempert ('81) in the context of the study of the Kobayashi metric in strictly convex domains. Extremal discs for the Kobayashi metric are proper and geodesic in this setting, and satisfy the stationarity condition.

In the setting of strictly pseudoconvex domains, things are more subtle, and a general extremal disc may fail to be proper. However, works of Huang ('94) show that extremal discs which are sufficiently close to the boundary are again proper and geodesics.

If M is a sphere, then the extremal discs are given by (intersections with the ball of) complex lines.

Segre varieties

Assume that $M \subset \mathbb{C}^n$ is real-analytic and is written as $M = \{z \in \mathbb{C}^n : \rho(z, \overline{z}) = 0\}$, where $\rho : \mathbb{C}^n \to \mathbb{R}$ is of class C^ω. For any $p \in \mathbb{C}^n$ one defines the corresponding *Segre variety* as $S_p = \{z \in \mathbb{C}^n : \rho(z, \overline{p}) = 0\}$.

Segre varieties

Assume that $M \subset \mathbb{C}^n$ is real-analytic and is written as $M = \{z \in \mathbb{C}^n : \rho(z, \overline{z}) = 0\}$, where $\rho : \mathbb{C}^n \to \mathbb{R}$ is of class C^ω. For any $p \in \mathbb{C}^n$ one defines the corresponding *Segre variety* as $S_p = \{z \in \mathbb{C}^n : \rho(z, \overline{p}) = 0\}$.

Some facts about Segre varieties

- if p is close enough to M then S_p is not empty
- $p \in M$ iff $p \in S_p$, and $q \in S_p$ iff $p \in S_q$
- if M is strongly pseudoconvex and $p \notin M$ is close enough to M, then $S_p \cap M$ is either empty or a $2n - 3$ dimensional manifold.

Segre varieties

Assume that $M \subset \mathbb{C}^n$ is real-analytic and is written as $M = \{z \in \mathbb{C}^n : \rho(z, \overline{z}) = 0\}$, where $\rho : \mathbb{C}^n \to \mathbb{R}$ is of class C^ω. For any $p \in \mathbb{C}^n$ one defines the corresponding *Segre variety* as $S_p = \{z \in \mathbb{C}^n : \rho(z, \overline{p}) = 0\}$.

Some facts about Segre varieties

- if p is close enough to M then S_p is not empty
- $p \in M$ iff $p \in S_p$, and $q \in S_p$ iff $p \in S_q$
- if M is strongly pseudoconvex and $p \notin M$ is close enough to M, then $S_p \cap M$ is either empty or a $2n-3$ dimensional manifold.

If $n = 2$, the traces of S_p on M thus form a family of 1-dimensional curves. If M is a sphere, then the Segre varieties are given by complex lines.

Chains

Suppose that $M \subset \mathbb{C}^n$ is Levi non-degenerate. In this case, a distinguished invariant family of curves, called *chains* was identified in Chern-Moser ('75) in the context of the solution to the equivalence problem in that class.

Chains

Suppose that $M \subset \mathbb{C}^n$ is Levi non-degenerate. In this case, a distinguished invariant family of curves, called *chains* was identified in Chern-Moser ('75) in the context of the solution to the equivalence problem in that class.

Chains are transversal to the complex tangent distribution, and are characterized intrinsically by the vanishing of a certain connection defined on a principal bundle naturally associated to M. As such, they solve a system of differential equations of order 2.

Chains

Suppose that $M \subset \mathbb{C}^n$ is Levi non-degenerate. In this case, a distinguished invariant family of curves, called *chains* was identified in Chern-Moser ('75) in the context of the solution to the equivalence problem in that class.

Chains are transversal to the complex tangent distribution, and are characterized intrinsically by the vanishing of a certain connection defined on a principal bundle naturally associated to M. As such, they solve a system of differential equations of order 2.

If M is a hyperquadric, then the chains are given by intersections with complex lines.

Comparison of the invariant families

As transpired from the discussion above, these invariants do give rise to the same family of curves if $M \subset \mathbb{C}^2$ is a sphere. This immediately suggest the following

Comparison of the invariant families

As transpired from the discussion above, these invariants do give rise to the same family of curves if $M \subset \mathbb{C}^2$ is a sphere. This immediately suggest the following

Basic question: is the sphere the only case for which these notions coincide?

Segre varieties and chains

An affirmative answer was given in Faran ('81) for the case of Segre varieties vs. chains:

Segre varieties and chains

An affirmative answer was given in Faran ('81) for the case of Segre varieties vs. chains:

Theorem (Faran)

Let $M \subset \mathbb{C}^2$ be a strongly pseudoconvex real-analytic hypersurface. If the intersections of all Segre varieties with M are chains, then M is locally spherical (i.e. locally biholomorphically equivalent to the sphere).

Segre varieties and chains

An affirmative answer was given in Faran ('81) for the case of Segre varieties vs. chains:

Theorem (Faran)

Let $M \subset \mathbb{C}^2$ be a strongly pseudoconvex real-analytic hypersurface. If the intersections of all Segre varieties with M are chains, then M is locally spherical (i.e. locally biholomorphically equivalent to the sphere).

A version of this result is also obtained in higher dimension, where the assumption, however, involves the intersection of M with sufficiently many Segre varieties rather than only one.

Segre varieties and extremal discs

Our first aim was to study the analogous question of the relationship between Segre varieties and extremal discs.

Segre varieties and extremal discs

Our first aim was to study the analogous question of the relationship between Segre varieties and extremal discs.

Theorem (Bertrand, D., Lamel '20)

Let $M \subset \mathbb{C}^2$ be a connected real-analytic hypersurface, $p \in M$. Assume that there exist open neighbourhoods $U, V \subset \mathbb{C}^2$ of p such that the Segre varieties $S_q \subset U$, $q \in V$ are defined and such that $S_q \cap U_+$ is an extremal disc for U_+ for every $q \in V_-$. Then M is umbilical at every strictly pseudoconvex point of $V \cap M$, and hence generically locally spherical.

Segre varieties and extremal discs

Remark: If M is also the (connected) boundary of a simply connected domain, then it is in fact globally biholomorphic to a sphere:

Corollary

Let $\Omega \subset \mathbb{C}^2$ be a bounded, simply connected strictly pseudoconvex domain with connected real-analytic boundary, and assume that U is a neighbourhood of $b\Omega$ such that $S_q \cap \Omega$ is defined for all $q \in U \cap \overline{\Omega}^c$. If $S_q \cap \Omega$ is an extremal disc (for Ω) for every $q \in U \cap \overline{\Omega}^c$, then Ω is biholomorphic to the the unit ball B.

Segre varieties and extremal discs

Remark: If M is also the (connected) boundary of a simply connected domain, then it is in fact globally biholomorphic to a sphere:

Corollary

Let $\Omega \subset \mathbb{C}^2$ be a bounded, simply connected strictly pseudoconvex domain with connected real-analytic boundary, and assume that U is a neighbourhood of $b\Omega$ such that $S_q \cap \Omega$ is defined for all $q \in U \cap \overline{\Omega}^c$. If $S_q \cap \Omega$ is an extremal disc (for Ω) for every $q \in U \cap \overline{\Omega}^c$, then Ω is biholomorphic to the the unit ball B.

The corollary follows from the previous theorem by applying the results in Chern-Ji ('96). Without topological assumptions, a locally spherical hypersurface might indeed fail to be globally CR equivalent to the sphere, cf. Burns-Shnider ('76).

Segre varieties and extremal discs

The assumption that M is strongly pseudoconvex is needed:

Example

Consider the ellipsoids $M = \{|z|^{2\alpha} + |w|^{2\beta} = 1\} \subset \mathbb{C}^2$. The stationary discs for such a hypersurface were computed in Jarnicki-Pflug ('95) and coincide with the Segre varieties. Note however that M is locally spherical where it is strongly pseudoconvex.

Segre varieties and extremal discs

The assumption that M is strongly pseudoconvex is needed:

Example

Consider the ellipsoids $M = \{|z|^{2\alpha} + |w|^{2\beta} = 1\} \subset \mathbb{C}^2$. The stationary discs for such a hypersurface were computed in Jarnicki-Pflug ('95) and coincide with the Segre varieties. Note however that M is locally spherical where it is strongly pseudoconvex.

It is in general not enough to just have a one-parameter family of Segre varieties which are stationary:

Example

Let $M = \{\operatorname{Im} w = |z|^2 + |z|^8\} \subset \mathbb{C}^2$. Then the Segre varieties $w = c$ with $\operatorname{Im} c > 0$ correspond to stationary discs.

Normal form

Since the result is local we fix a point $p \cong 0 \in M$ and choose coordinates in which M is in Chern-Moser normal form, i.e. $M = \{\rho = 0\}$ with

$$\rho(z, w, \overline{z}, \overline{w}) = \mathrm{Re} w - |z|^2 + \alpha z^2 \overline{z}^4 + \overline{\alpha} z^4 \overline{z}^2 + \mathrm{Im} w \, h(z, \overline{z}, \mathrm{Im} w) + g(z, \overline{z})$$

where $\alpha \in \mathbb{C}$, $g(z, \overline{z}) = O(|z|^7)$ and $h(z, \overline{z}, \mathrm{Im} w) = O(|z|^6)$.

Normal form

Since the result is local we fix a point $p \cong 0 \in M$ and choose coordinates in which M is in Chern-Moser normal form, i.e. $M = \{\rho = 0\}$ with

$$\rho(z, w, \overline{z}, \overline{w}) = \mathrm{Re} w - |z|^2 + \alpha z^2 \overline{z}^4 + \overline{\alpha} z^4 \overline{z}^2 + \mathrm{Im} w \, h(z, \overline{z}, \mathrm{Im} w) + g(z, \overline{z})$$

where $\alpha \in \mathbb{C}$, $g(z, \overline{z}) = O(|z|^7)$ and $h(z, \overline{z}, \mathrm{Im} w) = O(|z|^6)$.

In these coordinates, a subfamily of the Segre varieties can be conveniently written as $\{w = c\}$, $c \in \mathbb{C}$. In particular the varieties of the form $\{w = t^2\}$, $t \in \mathbb{R}$ intersect M in a closed curve.

Normal form

Since the result is local we fix a point $p \cong 0 \in M$ and choose coordinates in which M is in Chern-Moser normal form, i.e. $M = \{\rho = 0\}$ with

$$\rho(z, w, \overline{z}, \overline{w}) = \mathrm{Re} w - |z|^2 + \alpha z^2 \overline{z}^4 + \overline{\alpha} z^4 \overline{z}^2 + \mathrm{Im} w\, h(z, \overline{z}, \mathrm{Im} w) + g(z, \overline{z})$$

where $\alpha \in \mathbb{C}$, $g(z, \overline{z}) = O(|z|^7)$ and $h(z, \overline{z}, \mathrm{Im} w) = O(|z|^6)$.

In these coordinates, a subfamily of the Segre varieties can be conveniently written as $\{w = c\}$, $c \in \mathbb{C}$. In particular the varieties of the form $\{w = t^2\}$, $t \in \mathbb{R}$ intersect M in a closed curve.

The idea is to use the assumption that the discs $\{w = t^2\}$ are stationary to conclude that $\alpha = 0$, showing that p is an umbilical point.

Stationarity condition

Let $\Omega_t = \{w = t^2\} \cap \{\rho > 0\}$; the discs we are interested in are thus the $f_t : \Delta \to \mathbb{C}^2$ defined as $f_t(\zeta) = (R_t(\zeta), t^2)$, where $R_t : \Delta \to \Omega_t$ is the Riemann map such that $R_t(0) = 0$, $R_t'(0) > 0$.

Stationarity condition

Let $\Omega_t = \{w = t^2\} \cap \{\rho > 0\}$; the discs we are interested in are thus the $f_t : \Delta \to \mathbb{C}^2$ defined as $f_t(\zeta) = (R_t(\zeta), t^2)$, where $R_t : \Delta \to \Omega_t$ is the Riemann map such that $R_t(0) = 0$, $R_t'(0) > 0$.

The stationarity condition for f_t can be written explicitly as follows: there exists a continuous function $a_t : b\Delta \to \mathbb{R}^+$ and holomorphic functions $\widetilde{z}_t, \widetilde{w}_t \in \mathcal{O}(\Delta) \cap C(\overline{\Delta})$ satisfying

$$\begin{cases} \widetilde{z}_t(\zeta) = \zeta a_t(\zeta) \frac{\partial \rho}{\partial z}(R_t(\zeta), t^2, \overline{R_t(\zeta)}, t^2) \\ \widetilde{w}_t(\zeta) = \zeta a_t(\zeta) \frac{\partial \rho}{\partial w}(R_t(\zeta), t^2, \overline{R_t(\zeta)}, t^2) \end{cases}$$

for all $\zeta \in b\Delta$.

Stationarity condition

Let $\Omega_t = \{w = t^2\} \cap \{\rho > 0\}$; the discs we are interested in are thus the $f_t : \Delta \to \mathbb{C}^2$ defined as $f_t(\zeta) = (R_t(\zeta), t^2)$, where $R_t : \Delta \to \Omega_t$ is the Riemann map such that $R_t(0) = 0$, $R_t'(0) > 0$.

The stationarity condition for f_t can be written explicitly as follows: there exists a continuous function $a_t : b\Delta \to \mathbb{R}^+$ and holomorphic functions $\widetilde{z}_t, \widetilde{w}_t \in \mathcal{O}(\Delta) \cap C(\overline{\Delta})$ satisfying

$$\begin{cases} \widetilde{z}_t(\zeta) = \zeta a_t(\zeta) \frac{\partial \rho}{\partial z}(R_t(\zeta), t^2, \overline{R_t(\zeta)}, t^2) \\ \widetilde{w}_t(\zeta) = \zeta a_t(\zeta) \frac{\partial \rho}{\partial w}(R_t(\zeta), t^2, \overline{R_t(\zeta)}, t^2) \end{cases}$$

for all $\zeta \in b\Delta$.

In order to obtain the desired conclusion one needs to explore the consequences of this condition as $t \to 0$.

Scaling

In order to make the method work, one needs to understand the asymptotic behavior of the functions a_t, R_t^{-1} as $t \to 0$. This is rather singular, though, since Ω_t degenerate to a point at $t = 0$.

Scaling

In order to make the method work, one needs to understand the asymptotic behavior of the functions a_t, R_t^{-1} as $t \to 0$. This is rather singular, though, since Ω_t degenerate to a point at $t = 0$.

A suitable scaling method is thus needed to resolve the singularity at $t = 0$, and turn the (suitably rescaled) domains Ω_t into small perturbations of the unit disc.

Scaling

In order to make the method work, one needs to understand the asymptotic behavior of the functions a_t, R_t^{-1} as $t \to 0$. This is rather singular, though, since Ω_t degenerate to a point at $t = 0$.

A suitable scaling method is thus needed to resolve the singularity at $t = 0$, and turn the (suitably rescaled) domains Ω_t into small perturbations of the unit disc.

Another crucial issue is the smoothness of the functions involved with respect to t; this is established by an appropriate choice of the function a_t, as given in Pang ('93), and smoothness results for Riemann maps depending on parameters.

Chains as null geodesics

In the approach of Faran ('81), chains were treated within the original framework introduced by Chern and Moser; for our problem, however, this does not appear to lead to computations that are sufficiently tractable.

Chains as null geodesics

In the approach of Faran ('81), chains were treated within the original framework introduced by Chern and Moser; for our problem, however, this does not appear to lead to computations that are sufficiently tractable.

For our purpose, it seems that a more convenient treatment is the one provided by Fefferman ('76). In that paper, approximate solutions u are constructed for a suitable Monge-Ampère equation defined on a strictly pseudoconvex domain $D \subset \mathbb{C}^n$. This solution is also an asymptotic approximation of the Bergman kernel $K(z, z)$ near the boundary.

Chains as null geodesics

In the approach of Faran ('81), chains were treated within the original framework introduced by Chern and Moser; for our problem, however, this does not appear to lead to computations that are sufficiently tractable.

For our purpose, it seems that a more convenient treatment is the one provided by Fefferman ('76). In that paper, approximate solutions u are constructed for a suitable Monge-Ampère equation defined on a strictly pseudoconvex domain $D \subset \mathbb{C}^n$. This solution is also an asymptotic approximation of the Bergman kernel $K(z,z)$ near the boundary.

In turn, the function u is used to define a biholomorphically invariant, conformal Lorentz metric on $bD \times S^1$. As it turns out, the projections onto bD of null geodesics of this metric coincide with the Chern-Moser chains.

Chains and extremal discs

The construction in Fefferman ('76) is very concrete, and the coefficients of the metric can be computed explicitly from a defining function for bD.

Chains and extremal discs

The construction in Fefferman ('76) is very concrete, and the coefficients of the metric can be computed explicitly from a defining function for bD.

Nevertheless, computing exact solutions of the corresponding geodesic equation is not possible in general. Our approach is rather to study the asymptotic behavior of the family of geodesics, as their starting point approaches the origin (in the same normal-form coordinates considered above).

Chains and extremal discs

The construction in Fefferman ('76) is very concrete, and the coefficients of the metric can be computed explicitly from a defining function for bD.

Nevertheless, computing exact solutions of the corresponding geodesic equation is not possible in general. Our approach is rather to study the asymptotic behavior of the family of geodesics, as their starting point approaches the origin (in the same normal-form coordinates considered above).

At least in $\mathbb{C}^2$, an asymptotic expansion can be computed explicitly up to the smallest order affected by the presence of non-umbilical terms in the normal form. An analysis similar to the one employed for Segre varieties (modulo many technical complications) can then be carried out. The computational complexity involved is, however, considerably larger than in the previous case.

Chains and extremal discs

Theorem (Bertrand, D., Lamel)

Let $M \subset \mathbb{C}^2$ be a connected real-analytic hypersurface, $p \in M$. Assume that there exist open neighbourhoods $U \subset \mathbb{C}^2$ of p such that every chain contained in U is the boundary of an extremal disc contained in U_+. Then M is umbilical at every strictly pseudoconvex point of $V \cap M$, and hence generically locally spherical.

Under the same topological assumptions as in the previous case, the local result can be globalized, yielding the sphericity of the domain bounded by M.

Thank you!