Eigenvalue estimates of differential operators on manifolds with boundary

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We denote by

- (M^n, g) an n-dimensional compact Riemannian manifold with nonempty smooth boundary ∂M.
- ν the inward unit normal along the boundary.
- ι the embedding $\partial M \longrightarrow M$.

The Dirichlet eigenvalue problem

$$\begin{cases} \Delta\omega &= \lambda\omega \quad \text{on } M \\ \omega &= 0 \quad \text{on } \partial M. \end{cases} \tag{1}$$

For that problem, the first eigenvalue $\lambda_{1,p}^D$ is characterized by

$$\lambda_{1,p}^D = \inf_{\substack{\omega\neq 0 \\ \omega_{|\partial M}=0}} \left\{ \frac{\|d\omega\|^2_{L^2(M)} + \|\delta\omega\|^2_{L^2(M)}}{\|\omega\|^2_{L^2(M)}} \right\}.$$

The Neumann eigenvalue problem

$$\begin{cases} \Delta\omega & = \lambda\omega \quad \text{on } M \\ \iota^*(\nu \lrcorner d\omega) & = 0 \quad \text{on } \partial M \\ \iota^*(\nu \lrcorner \omega) & = 0 \quad \text{on } \partial M \end{cases} \tag{2}$$

The first eigenvalue $\lambda_{1,p}^N$ of (2) is characterized by

$$\lambda_{1,p}^N = \inf_{\substack{\omega \neq 0 \\ \nu \lrcorner \omega = 0}} \left\{ \frac{\|d\omega\|_{L^2(M)}^2 + \|\delta\omega\|_{L^2(M)}^2}{\|\omega\|_{L^2(M)}^2} \right\}.$$

The problem (2) might have a kernel, given by the absolute de Rham cohomology

$$H_A^p(M) = \{\phi \in \Omega^p(M) |\ d\phi = \delta\phi = 0 \text{ on } M \text{ and } \nu \lrcorner \phi = 0 \text{ on } \partial M\}.$$

The Robin eigenvalue problem

$$\begin{cases} \Delta\omega & = \lambda\omega \quad \text{on } M \\ \iota^*(\nu \lrcorner d\omega - \tau\omega) & = 0 \quad \text{on } \partial M \\ \iota^*(\nu \lrcorner \omega) & = 0 \quad \text{on } \partial M, \end{cases} \tag{3}$$

where $\tau > 0$.

The Robin eigenvalue problem is a natural generalization of the scalar problem to differential forms.

The Robin eigenvalue problem

Theorem 1 (El Chami, Ginoux, Habib - 2021)

1. *The boundary value problem* (3) *is elliptic and self-adjoint. Then, it admits an unbounded sequence of real eigenvalues with finite multiplicities*

$$0 < \lambda_{1,p}(\tau) \leq \lambda_{2,p}(\tau) \leq \cdots$$

2. $$\lambda_{1,p}(\tau) = \inf \left\{ \frac{\|d\omega\|^2_{L^2(M)} + \|\delta\omega\|^2_{L^2(M)} + \tau \|\iota^*\omega\|^2_{L^2(\partial M)}}{\|\omega\|^2_{L^2(M)}} \right\},$$

   where ω *runs over all non-identically vanishing p-forms on* M *such that* $\nu \lrcorner \omega = 0$.

The biharmonic Steklov eigenvalue problem

$$\begin{cases} \Delta^2 f = 0 & \text{on } M \\ f = 0 & \text{on } \partial M \\ \Delta f = q\partial_\nu f & \text{on } \partial M. \end{cases}$$

This problem was studied by many authors : Kuttler, Ferrero-Gazzola-Weth, Raulot-Savo...

The first eigenvalue q_1 of the BS problem is characterized by

$$q_1 = \inf_{\substack{f\neq 0 \\ f|_{\partial M}=0}} \left\{ \frac{\|\Delta f\|^2_{L^2(M)}}{\|\partial_\nu f\|^2_{L^2(\partial M)}} \right\} = \inf_{\substack{f\neq 0 \\ \Delta f=0}} \left\{ \frac{\|f\|^2_{L^2(\partial M)}}{\|f\|^2_{L^2(M)}} \right\}.$$

Motivation

Let us consider the Serrin problem in a domain of the Euclidean space :

$$\begin{cases} \Delta f &= 1 \quad \text{on} \quad M, \\ f &= 0 \quad \text{on} \quad \partial M, \\ \partial_\nu f &= c \quad \text{on} \; \partial M, \end{cases}$$

where $c \in \mathbb{R}$ is a suitable constant.
If there is a solution f of this problem, then M is isometric to a round ball in $\mathbb{R}^n$ and the constant c is given by

$$c = \frac{\mathrm{Vol}\, M}{\mathrm{Vol}\, \partial M}.$$

Moreover, f is an eigenfunction of the BS problem associated to the eigenvalue $\frac{1}{c}$.

Motivation

Let M be a ball in $\mathbb{R}^n$, f a solution to the scalar Serrin value problem on M and ω_0 a parallel form on M. Then $\omega = f\omega_0$ is a solution (actually the unique solution) of the following Serrin problem

$$\begin{cases} \Delta\omega & = \omega_0 \quad \text{on } M \\ \omega_{|\partial M} & = 0 \quad \text{on } \partial M \\ \nu \lrcorner d\omega & = c\iota^*\omega_0 \quad \text{on } \partial M, \end{cases}$$

We easily check that the solution $\omega = f\omega_0$ is biharmonic and satisfies

$$\nu \lrcorner \Delta\omega = -\frac{1}{c}\iota^*\delta\omega, \quad \iota^*\Delta\omega = \frac{1}{c}\nu \lrcorner d\omega$$

on the boundary.

The BS boundary value problem

We consider the Biharmonic Steklov problem on differential forms

$$\begin{cases} \Delta^2\omega & = 0 \quad \text{on } M \\ \omega & = 0 \quad \text{on } \partial M \\ \nu \lrcorner \Delta\omega + q\iota^*\delta\omega & = 0 \quad \text{on } \partial M \\ \iota^*\Delta\omega - q\nu \lrcorner d\omega & = 0 \quad \text{on } \partial M. \end{cases} \tag{4}$$

Theorem 2

The boundary value problem (4) *admits a discrete spectrum consisting of an unbounded monotonously nondecreasing sequence of positive eigenvalues of finite multiplicities*

$$0 < q_{1,p} \leq q_{2,p} \leq \cdots$$

Proposition 3

We have the following characterizations of the first eigenvalue $q_{1,p}$ of the boundary value problem (4)

1.
$$q_{1,p} = \inf \left\{ \frac{\|\Delta\omega\|^2_{L^2(M)}}{\|\nu \lrcorner d\omega\|^2_{L^2(\partial M)} + \|\iota^*\delta\omega\|^2_{L^2(\partial M)}} \right\}, \tag{5}$$
where ω runs over all non-identically vanishing p-forms on M such that $\omega_{|_{\partial M}} = 0$.

2.
$$q_{1,p} = \inf \left\{ \frac{\|\omega\|^2_{L^2(\partial M)}}{\|\omega\|^2_{L^2(M)}} \right\}, \tag{6}$$
where ω runs over all non-identically vanishing p-forms on M such that $\Delta\omega = 0$.

Remark

If ω is an eigenform of the biharmonic Steklov problem (4) associated to an eigenvalue q, then $*\omega$ is also an eigenform of this problem associated to the same eigenvalue.

As a consequence, we have the identity

$$q_{i,p} = q_{i,n-p},$$

for all positive integer i and $p \in \{0, \dots, n\}$.

Recall that the shape operator of ∂M is defined by $S(X) := -\nabla_X \nu$, for any vector field X tangent to the boundary. The mean curvature of ∂M is $H := \dfrac{1}{n-1}\mathrm{tr}\, S$.

We assume that $\mathrm{Ric}_M \geq (n-1)K \cdot \mathrm{Id}$ on M and $H \geq H_0$ along ∂M for some real constants K, H_0.

We consider the function Θ defined for all r by

$$\Theta(r) := (s_K'(r) - H_0 s_K(r))^{n-1},$$

the function s_K being given by

$$s_K(r) := \begin{cases} \frac{1}{\sqrt{K}} \sin(r\sqrt{K}) & \text{if } K > 0, \\ r & \text{if } K = 0, \\ \frac{1}{\sqrt{|K|}} \sinh(r\sqrt{|K|}) & \text{if } K < 0. \end{cases}$$

Theorem 4 (Raulot & Savo - 2015)

Assume that (M^n, g) *satisfies* $\mathrm{Ric}_M \geq (n-1)K \cdot \mathrm{Id}$ *on* M *and* $H \geq H_0$ *along* ∂M *for some real constants* K, H_0*. For any nontrivial, nonnegative and subharmonic function* f*, we have*

$$\frac{\int_{\partial M} f}{\int_M f} \geq \frac{1}{\int_0^R \Theta(r)dr}.$$

Here $R := \max\{d(x, \partial M) \,|\, x \in M\}$ *is the so-called inner radius of* M*.*

Theorem 5

Assume that the Bochner operator $W_M^{[p]}$ is nonnegative, $\mathrm{Ric}_M \geq (n-1)K \cdot \mathrm{Id}$ on M and $H \geq H_0$ along ∂M for some real constants K, H_0. Then we obtain

$$q_{1,p} \geq \frac{1}{\int_0^R \Theta(r)\, dr}. \tag{7}$$

Proof

Let ω be an eigenform of problem (4) associated to $q_{1,p}$, and set $\alpha = \Delta\omega$.

On the one hand, the form α is harmonic and $q_{1,p} = \frac{\|\alpha\|^2_{L^2(\partial M)}}{\|\alpha\|^2_{L^2(M)}}$.

On the other hand, the function $|\alpha|^2$ is nonnegative and subharmonic. Indeed, $\Delta(|\alpha|^2) \leq 0$ by the Bochner formula :

$$\langle \Delta\alpha, \alpha\rangle = |\nabla\alpha|^2 + \frac{1}{2}\Delta(|\alpha|^2) + \langle W_M^{[p]}(\alpha), \alpha\rangle.$$

By Theorem 4, we get

$$\frac{\int_{\partial M} |\alpha|^2}{\int_M |\alpha|^2} \geq \frac{1}{\int_0^R \Theta(r)dr},$$

from which the inequality (7) follows. □

Examples

Assume that the Bochner operator $W_M^{[p]}$ is nonnegative, $\mathrm{Ric}_M \geq 0$ on M and $H \geq H_0$ along ∂M for some real constant H_0. Then

$$\Theta(r) = (1 - H_0 r)^{n-1}.$$

We can compute $\int_0^R \Theta(r)dr$ explicitly in this case. We get

1. $q_{1,p} \geq \dfrac{nH_0}{1-(1-RH_0)^n}$ if $H_0 > 0$;
2. $q_{1,p} \geq \dfrac{1}{R}$ if $H_0 \geq 0$;
3. $q_{1,p} \geq \dfrac{-nH_0}{(1-RH_0)^n - 1}$ if $H_0 < 0$.

Theorem 6

Let M be a compact domain in $\mathbb{R}^n$ with smooth boundary. Then for any $p \geq 1$,

$$q_{1,p} \geq q_{1,p-1}.$$

Proof 1/3

Let ω be any eigenform of the Biharmonic Steklov problem associated to $q_{1,p}$.

For each i, the $(p-1)$-form $\partial_{x_i} \lrcorner \omega$ satisfies

$$(\partial_{x_i} \lrcorner \omega)_{|\partial M} = 0.$$

Then, we can apply to it the variational characterization (5). We get

$$q_{1,p-1}\left(\int_{\partial M} |\nu \lrcorner d(\partial_{x_i} \lrcorner \omega)|^2 + \int_{\partial M} |\iota^* \delta(\partial_{x_i} \lrcorner \omega)|^2\right) \leq \int_M |\Delta(\partial_{x_i} \lrcorner \omega)|^2. \tag{8}$$

Proof 2/3

We compute each term of (8), then we sum up over i. We obtain

$$\begin{aligned}
\sum_{i=1}^{n} |\nu \lrcorner d(\partial_{x_i} \lrcorner \omega)|^2 &= |\nu \lrcorner \nabla_\nu \omega|^2 + p|\nu \lrcorner d\omega|^2 \\
\sum_{i=1}^{n} |\iota^* \delta(\partial_{x_i} \lrcorner \omega)|^2 &= (p-1)|\iota^* \delta\omega|^2 \\
\sum_{i=1}^{n} |\Delta(\partial_{x_i} \lrcorner \omega)|^2 &= p|\Delta\omega|^2.
\end{aligned}$$

Proof 3/3

Since $\omega_{|_{\partial M}} = 0$, we know that $\iota^*\delta\omega = -\nu\lrcorner\nabla_\nu\omega$. Then we get

$$q_{1,p-1}\int_{\partial M}\left(|\nu\lrcorner d\omega|^2 + |\iota^*\delta\omega|^2\right) \leq \int_M |\Delta\omega|^2.$$

Hence

$$q_{1,p-1} \leq q_{1,p}$$

since ω is an eigenform associated to $q_{1,p}$. $\square$

Corollary 7

Let M be a domain of $\mathbb{R}^n$, then for any $p \in \{0, \ldots, n\}$,

$$q_{1,p} = q_{1,0}.$$

In particular, if M is the unit ball, then

$$q_{1,p} = n.$$

Proposition 8

Let (M^n, g) be an n-dimensional compact Riemannian manifold with nonempty smooth boundary.
We have the following estimate

$$\frac{1}{\lambda_{1,p}(\tau)} \leq \frac{1}{\lambda^D_{1,p}} + \frac{1}{\tau q_{1,p}},$$

where $q_{1,p}$ (resp. $\lambda_{1,p}(\tau)$, $\lambda^D_{1,p}$) is the first eigenvalue of the biharmonic Steklov (resp. Robin, Dirichlet) boundary problem on p-forms.

Proof 1/3

Let ω be an eigenform of the Robin boundary problem (3) associated to $\lambda_{1,p}(\tau)$. We denote by ω_1 the p-form satisfying

$$\begin{cases} \Delta\omega_1 & = 0 \quad \text{on } M \\ \iota^*\omega_1 & = \iota^*\omega \quad \text{on } \partial M \\ \nu\lrcorner\,\omega_1 & = 0 \quad \text{on } \partial M \end{cases}$$

and let $\omega_2 := \omega - \omega_1$. The p-form ω_2 satisfies

$$\begin{cases} \Delta\omega_2 & = \Delta\omega \quad \text{on } M \\ \omega_2 & = 0 \quad \text{on } \partial M. \end{cases}$$

Proof 2/3

Then we get

$$\begin{aligned} q_{1,p}||\omega_1||^2_{L^2(M)} &\leq ||\omega_1||^2_{L^2(\partial M)} \\ ||\omega_1||_{L^2(\partial M)} &= ||\omega||_{L^2(\partial M)}, \quad (\text{since } \omega_{2|_{\partial M}} = 0) \end{aligned}$$

$$\begin{aligned} \lambda^D_{1,p}||\omega_2||^2_{L^2(M)} &\leq ||d\omega_2||^2_{L^2(M)} + ||\delta\omega_2||^2_{L^2(M)} \\ ||d\omega_2||^2_{L^2(M)} + ||\delta\omega_2||^2_{L^2(M)} &\leq ||d\omega||^2_{L^2(M)} + ||\delta\omega||^2_{L^2(M)}, \end{aligned}$$

where the last inequality is obtained by a partial integration.

Proof 3/3

Therefore, by the triangle inequality and the property $2ab \leq \tau^{-1}a^2 + \tau b^2$ for any $a, b \in \mathbb{R}$, we get

$$
\begin{aligned}
&||\omega||^2_{L^2(M)} \\
\leq \;& \left(\sqrt{q^{-1}_{1,p}}||\omega||_{L^2(\partial M)} + \sqrt{(\lambda^D_{1,p})^{-1}}\left(||d\omega||^2_{L^2(M)} + ||\delta\omega||^2_{L^2(M)}\right)^{\frac{1}{2}}\right)^2 \\
\leq \;& (\tau^{-1}q^{-1}_{1,p} + (\lambda^D_{1,p})^{-1})\left(||d\omega||^2_{L^2(M)} + ||\delta\omega||^2_{L^2(M)} + \tau||\omega||^2_{L^2(\partial M)}\right)
\end{aligned}
$$

from which the result follows since ω is an eigenform associated to $\lambda_{1,p}(\tau)$. □

Proposition 9

Assume that the absolute de Rham cohomology $H_A^p(M)$ does not vanish. Let ω_D be an eigenform of the Dirichlet boundary problem associated to $\lambda_{1,p}^D$ and ω_0 be the orthogonal projection of ω_D on the space $H_A^p(M)$. If $\omega_0 \neq 0$, then

$$\frac{1}{\lambda_{1,p}(\tau)} \geq \frac{1}{\lambda_{1,p}^D} + \frac{||\omega_0||^4_{L^2(M)}}{\tau ||\omega_0||^2_{L^2(\partial M)}}.$$

Proposition 10

Assume that the first eigenvalue $\lambda_{1,p}^N$ of the Neumann boundary problem is positive, then

$$\frac{1}{\lambda_{1,p}(\tau)} \geq \frac{\lambda_{1,p}^D - \lambda_{1,p}^N + \tau\alpha_N}{\tau\alpha_N\lambda_{1,p}^D + \lambda_{1,p}^N(\lambda_{1,p}^D - \lambda_{1,p}^N)},$$

where $\alpha_N = \dfrac{||\omega_N||^2_{L^2(\partial M)}}{||\omega_N||^2_{L^2(M)}}$, ω_N *being an eigenform of the Neumann boundary problem associated to* $\lambda_{1,p}^N$.

Corollary 11

If $\lambda_{1,p}^N > 0$, then $\lambda_{1,p}(\tau) < \lambda_{1,p}^N + \tau \frac{||\omega_N||^2_{L^2(\partial M)}}{||\omega_N||^2_{L^2(M)}}$ for every eigenform ω_N associated to $\lambda_{1,p}^N$.

Proof. It is easy to check that if ω_N is an eigenform associated to $\lambda_{1,p}^N$, then

$$\frac{\lambda_{1,p}^D - \lambda_{1,p}^N + \tau\alpha_N}{\tau\alpha_N\lambda_{1,p}^D + \lambda_{1,p}^N(\lambda_{1,p}^D - \lambda_{1,p}^N)} > \frac{1}{\lambda_{1,p}^N + \tau\alpha_N},$$

where $\alpha_N = \frac{||\omega_N||^2_{L^2(\partial M)}}{||\omega_N||^2_{L^2(M)}}$. □

Thank you for your attention !