

ON THE ENVELOPE OF POISSON FUNCTIONAL ON ALMOST COMPLEX MANIFOLDS

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ABSTRACT. We establish the plurisubharmonicity of the envelope of the Poisson functional on almost complex manifolds. That is, we generalize the corresponding result for complex manifolds and almost complex manifolds of complex dimension two.

We denote by $\mathbb{D} = \{\zeta \in \mathbb{C}; |\zeta| < 1\}$ the open unit disc in the complex plane. Given a smooth almost complex manifold (M, J) , we denote by $\mathcal{O}_J(\overline{\mathbb{D}}, M)$ the set of J -holomorphic discs in M , i.e., the set of smooth maps $v: \overline{\mathbb{D}} \rightarrow M$ which are J -holomorphic in a neighborhood of $\overline{\mathbb{D}}$.

A *disc functional* on M is a function

$$H_M: \mathcal{O}_J(\overline{\mathbb{D}}, M) \rightarrow \overline{\mathbb{R}} = [-\infty, +\infty].$$

The *envelope* $EH_M: M \rightarrow \overline{\mathbb{R}}$ associated to H_M in $p \in M$ is defined by

$$EH_M(p) = \inf \{H_M(v); v \in \mathcal{O}_J(\overline{\mathbb{D}}, M), v(0) = p\}.$$

Given an upper semi-continuous function f defined on M , the corresponding *Poisson functional* is the disc functional defined by

$$P_f(v) = \frac{1}{2\pi} \int_0^{2\pi} f(v(e^{it})) dt,$$

where $v \in \mathcal{O}_J(\overline{\mathbb{D}}, M)$. Our main result is the following:

Theorem 1. *Let $f: M \rightarrow \mathbb{R} \cup \{-\infty\}$ be an upper semi-continuous function defined on a smooth almost complex manifold (M, J) . Then EP_f is J -plurisubharmonic on M or identically $-\infty$.*

The envelope EP_f of the Poisson functional is the largest plurisubharmonic minorant of the upper semi-continuous function f . In case $M = \mathbb{C}^n$, the corresponding result was proved by Poletsky [16, 17], and by Bu and Schachermayer [2]. The result was then extended to complex manifolds by Lárusson and Sigurdsson [13, 14], and Rosay [20, 21]. Later on, Drinovec Drnovšek and Forstnerič proved it for locally irreducible complex spaces [5]. We provide an analogue for almost complex manifolds of any dimension.

In case $\dim_{\mathbb{C}} M = 2$, Theorem 1 was already established by the second named author [12]. The only missing step towards its higher dimensional analogue was a method of attaching a J -holomorphic disc to a real torus (often referred as a solution to the Riemann-Hilbert problem). More precisely, given an embedded J -holomorphic disc v , we associate to v a real 2-dimensional torus formed by the boundary circles of discs centered at the boundary points $v(\zeta)$, $\zeta \in \partial\mathbb{D}$. We then seek a disc centered at $v(0)$ and approximately attached to this torus. In low dimension, such a construction was settled by Coupet, Sukhov and Tumanov based on methods for elliptic PDEs [3, 23]; however their approach does not apply when $\dim_{\mathbb{C}} M > 2$. In the present paper, we then provide, in arbitrary dimension, a solution of the Riemann-Hilbert problem (Theorem 5) based on gluing techniques from [1, 10]. We emphasize that such a result is of an independent interest since solutions of such problems appear in various constructions of proper and complete holomorphic maps.

The plan of the paper is as follows. In §1 we introduce the gluing techniques related to [1, 10]. Section §2 is devoted to the Riemann-Hilbert problem. In §3 we prove Theorem 1. Finally, we present three applications in §4; more precisely, we describe some regularization results for J -plurisubharmonic functions, we give a characterization of compact Psh_J -hulls by J -holomorphic discs and we establish the J -plurisubharmonicity of the envelope of the Lelong functional.

1. GLUING TECHNIQUES FOR J -HOLOMORPHIC MAPS

An *almost complex structure* J on a real even dimensional smooth manifold M is a $(1, 1)$ tensor field which satisfies $J^2 = -Id$. Throughout the paper we assume that J is smooth. The pair (M, J) is called an *almost complex manifold*. We denote by J_{st} the standard integrable structure on \mathbb{R}^{2n} for every $n \in \mathbb{N}$. A differentiable map $v : (M', J') \rightarrow (M, J)$ between two almost complex manifolds is (J', J) -*holomorphic* if

$$(1) \quad J(v(p)) \circ d_p v = d_p v \circ J'(p),$$

for every $p \in M'$. When M' is a smooth bounded domain in \mathbb{C} the map v is called a *J -holomorphic disc*. We denote by \mathbb{D} the unit disc and by Δ any other smooth bounded domain in \mathbb{C} .

Let $V \subset M$ and $U \subset \mathbb{R}^{2n}$ be two open sets and let $\phi : V \rightarrow U$ be a coordinate diffeomorphism. The local structure $J_{loc} = d\phi_*(J)$ can then be represented as a $2n \times 2n$ real matrix map that satisfies $J_{loc}(p)^2 = -I$ for

all $p \in U$. We denote by $\mathcal{J}(U)$ the set of all such matrix maps satisfying $\det(J_{loc} + J_{st}) \neq 0$ on U . For such local structures and a map $u = \phi \circ v$, the holomorphicity condition (1) can be rewritten as

$$(2) \quad \mathcal{F}(u) = u_{\bar{\zeta}} + A(u)\overline{u_{\zeta}} = 0,$$

where $\zeta \in \Delta$ and $A(z)(w) = (J_{st} + J_{loc}(z))^{-1}(J_{loc}(z) - J_{st})(\bar{w})$ is a complex linear endomorphism for every $z \in \mathbb{C}^n$. Hence A can be considered as a $n \times n$ smooth complex matrix map acting on $w \in \mathbb{C}^n$ [22]. We call A the *complex matrix* associated to J by ϕ . Throughout the paper, we will use letter v for discs mapping into M and letter u for their local analogues.

Whenever we will work only in $U \subset \mathbb{C}^n$ we will denote the almost complex structure by $J \in \mathcal{J}(U)$ instead of J_{loc} . Also we will refer to (2) as the $\bar{\partial}_J$ -derivative of u and consider \mathcal{F} as a non-linear operator from the Sobolev space $W^{1,p}(\Delta, U)$ to the Lebesgue space $L^p(\Delta, U)$ for some $p > 2$. We will denote by $\mathcal{O}_J(\Delta, U)$ the set of maps $u \in W^{1,p}(\Delta, U)$ satisfying $\mathcal{F}(u) = 0$. Since the boundary of Δ is assumed to be smooth, the Sobolev embedding theorem implies that $\mathcal{O}_J(\Delta, U) \subset W^{1,p}(\Delta, U) \subset \mathcal{C}^{1-\frac{2}{p}}(\bar{\Delta}, U)$.

The crucial tool in the proof of Theorem 1 is a solution for the Cousin problem which was developed in [10]. In the following two subsections we present a brief overview for convenience of the reader.

1.1. The local Newton-type iteration. Note that the generalized Cauchy-Riemann system (2) is non-linear. Therefore, it is natural to use iterative methods when seeking solutions. Suppose that $\varphi: \Delta \rightarrow U$ is a map such that the L^p -norm of $\mathcal{F}(\varphi)$ is small. Using Newton-type iterations, we seek a disc $u \in \mathcal{O}_J(\Delta, U)$ which is $W^{1,p}$ -close to φ . For that purpose, we linearize the operator \mathcal{F} along the map $\varphi \in W^{1,p}(\Delta, U)$ and consider the operator $D_\varphi: W^{1,p}(\Delta, \mathbb{R}^{2n}) \rightarrow L^p(\Delta, \mathbb{R}^{2n})$ given by

$$D_\varphi(V) = V_{\bar{\zeta}} + A(\varphi)\overline{V_{\zeta}} + \sum_{j=1}^n \left(\frac{\partial A}{\partial z_j}(\varphi)V_j + \frac{\partial A}{\partial \bar{z}_j}(\varphi)\overline{V_j} \right) \overline{\varphi_{\zeta}}.$$

We point out two important properties of this operator. First, D_φ admits a bounded right inverse Q_φ for every $\varphi \in W^{1,p}(\Delta, U)$. Moreover, D_φ is *locally Lipschitz*, that is, there exists $L > 0$ depending on the $W^{1,p}$ -norm of φ , such that for any $\tilde{\varphi} \in W^{1,p}(\Delta, U)$ satisfying $\|\varphi - \tilde{\varphi}\|_{W^{1,p}(\Delta, U)} < 1$ we have

$$(3) \quad \|D_\varphi - D_{\tilde{\varphi}}\|_{op} < L \|\varphi - \tilde{\varphi}\|_{W^{1,p}(\Delta, U)},$$

where $\|\cdot\|_{op}$ denotes the operator norm. Both properties were established in [1] (see Theorem 2 and p. 7).

This enables us to apply the implicit function theorem [10, Theorem 4]. In particular, we have the following statement. If

$$\|\mathcal{F}(\varphi)\|_{L^p(\Delta, U)} < \min \left\{ \frac{1}{4 \|Q_\varphi\|_{op}}, \frac{1}{8 \|Q_\varphi\|_{op}^2 L} \right\},$$

where $L > 0$ is given by (3), then there exists a disc $u \in W^{1,p}(\Delta, U)$ with $\mathcal{F}(u) = 0$ and such that

$$\|u - \varphi\|_{W^{1,p}(\Delta, U)} \leq 2 \|Q_\varphi\|_{op} \|\mathcal{F}(\varphi)\|_{L^p(\Delta, U)}.$$

More precisely, provided that the starting map $u_0 = \varphi$ is 'good enough', u arises as the limit map of the iteration

$$(4) \quad u_{N+1} = u_N - Q_\varphi \mathcal{F}(u_N).$$

For details, see [10, Section 2]. Note also that, by the Sobolev embedding theorem, for $p > 2$ the $W^{1,p}$ -proximity implies that the limit u is \mathcal{C}^0 -close to φ up to the boundary of Δ .

Our strategy is then to use the above iteration method on a family of non-holomorphic maps for which the corresponding bounds are uniform. Therefore, we introduce the following definition.

Definition 2. *Let $J \in \mathcal{J}(U)$. We say that a family $\mathcal{W} \subset W^{1,p}(\Delta, U)$ satisfies the LQJ-condition if*

- i) *for every $\varphi \in \mathcal{W}$ the operator D_φ is locally Lipschitz for a unique constant $L > 0$,*
- ii) *there exists $Q > 0$ such that every D_φ , $\varphi \in \mathcal{W}$, admits a right inverse Q_φ satisfying $\|Q_\varphi\|_{op} < Q$.*

It is proved in [10, Section 2] that every precompact family $\mathcal{W} \subset W^{1,p}(\Delta, U)$ satisfies the LQJ-condition. However, in the present paper we will deal with families that are $W^{1,p}$ -unbounded. Therefore, we will have to seek these uniform bounds directly. In particular, we will use the following result which is a direct consequence of Theorem 4 [1].

Corollary 3. *Let $J \in \mathcal{J}(U)$ and let $\mathcal{W} \subset W^{1,p}(\Delta, U)$ be a family satisfying the LQJ-condition. Then there exist $\rho > 0$ and $C > 0$ depending on \mathcal{W} such that for every $\varphi \in \mathcal{W}$ with $\|\mathcal{F}(\varphi)\|_{L^p(\Delta, U)} < \rho$ there is a J -holomorphic map $u \in \mathcal{O}_J(\Delta, U)$ satisfying*

$$\|u - \varphi\|_{W^{1,p}(\Delta, U)} < C \|\mathcal{F}(\varphi)\|_{L^p(\Delta, U)}.$$

This corollary illustrates how these non-linear approximation techniques should be used in practice. In particular, finding a sequence of maps $(\varphi_N)_{N \in \mathbb{N}}$ such that $\|\mathcal{F}(\varphi_N)\|_{L^p}$ is decreasing is not enough. In order to find suitable J -holomorphic approximations one has to find uniform constants L and Q from Definition 2; otherwise the size of the neighborhood of φ_N on which the implicit function theorem is applied may shrink as $N \rightarrow \infty$. That is, $\rho > 0$ may tend to zero and $C > 0$ may tend to infinity.

Finally, for any fixed $a \in \Delta$, the right inverses of the corresponding linearized operators D_φ can be chosen so that $Q_\varphi(a) = 0$. This implies that the limit map of the iteration also satisfies $u(a) = \varphi(a)$. Indeed, such a pointwise restriction can be obtained by choosing an appropriate variation of the classical Cauchy-Green operator when defining the inverse Q_φ and is standard in the local approximation theory (see for instance [22]).

1.2. The non-linear Cousin problem. The above approximation techniques enable us to solve the following local version of the Cousin problem.

Let $\Delta_1, \Delta_2 \subset \mathbb{C}$ be two smoothly bounded and simply connected domains admitting a non-empty simply connected intersection $\Delta_1 \cap \Delta_2$ and a smooth cut-off function $\chi: \mathbb{C} \rightarrow [0, 1]$ that is equal to 1 on a neighborhood of $\overline{\Delta_1} \setminus \overline{\Delta_2}$ and that vanishes on a neighborhood of $\overline{\Delta_2} \setminus \overline{\Delta_1}$. In particular, such a function exists if $\overline{\Delta_1} \setminus \overline{\Delta_2} \cap \overline{\Delta_2} \setminus \overline{\Delta_1} = \emptyset$. We call (Δ_1, Δ_2) a *good pair*. Let $U \subset \mathbb{C}^n$ be an open set and let $J \in \mathcal{J}(U)$. We consider two J -holomorphic discs $u_1 \in \mathcal{O}_J(\Delta_1, U)$ and $u_2 \in \mathcal{O}_J(\Delta_2, U)$ whose images are $W^{1,p}$ -close on $\Delta_1 \cap \Delta_2$. The local Cousin problem consists in finding a J -holomorphic map that is \mathcal{C}^0 -close to the connected union of $u_1(\Delta_1)$ and $u_2(\Delta_2)$ (see [10, Theorem 1] for a precise statement). Its solution can be obtained by defining a pregluing map

$$\varphi = \chi u_1 + (1 - \chi) u_2$$

on $\Delta_1 \cup \Delta_2$. It turns out that the L^p -norm of its $\bar{\partial}_J$ -derivative is small. Thus this map can be used as the starting map $u_0 = \varphi$ in the Newton-type iteration (4) in order to obtain a map 'gluing together' u_1 and u_2 .

However, this construction is purely local and cannot be applied in our case since we will need to glue together discs whose images lie in different charts. Such a non-linear version of the Cousin problem was settled in [10, Theorem 2]. We briefly explain the method. Let (M, J) be an almost complex manifold and let $v_1: \Delta_1 \rightarrow M$ and $v_2: \Delta_2 \rightarrow M$ be two J -holomorphic discs defined on a good pair (Δ_1, Δ_2) and with images lying in different chart neighborhoods $\phi^1: V_1 \rightarrow U_1 \subset \mathbb{C}^n$ and $\phi^2: V_2 \rightarrow U_2 \subset \mathbb{C}^n$, respectively. We

assume that the intersection $V_1 \cap V_2$ is non-empty and contains the sets $v_1(\Delta_1 \cap \Delta_2)$ and $v_2(\Delta_1 \cap \Delta_2)$. We also assume that v_1 and v_2 are $W^{1,p}$ -close on $\Delta_1 \cap \Delta_2$. Then, as above, we can define a pregluing map φ that connects the images of v_1 and v_2 . The restrictions $\varphi_1 = \phi^1 \circ \varphi|_{\Delta_1}$ and $\varphi_2 = \phi^2 \circ \varphi|_{\Delta_2}$ are then taken as the starting maps of two Newton-type iterations that are now performed simultaneously, each in its own local chart equipped with the respective structures $J_1 = d\phi_*^1(J)$ and $J_2 = d\phi_*^2(J)$. In general, their limits $u_1 \in \mathcal{O}_{J_1}(\Delta_1, U_1)$ and $u_2 \in \mathcal{O}_{J_2}(\Delta_2, U_2)$ do not provide a global map $\widehat{v}: \Delta_1 \cup \Delta_2 \rightarrow M$ since $(\phi^1)^{-1} \circ u_1 \neq (\phi^2)^{-1} \circ u_2$ on $\Delta_1 \cap \Delta_2$. However, this can be settled with a subtle construction of appropriate right inverses Q_{φ_1} and Q_{φ_2} (see p. 4267 and p. 4270 in [10]). Such a map \widehat{v} solves the non-linear Cousin problem.

In this paper, we apply this construction on a collection of m good pairs. We do this as follows. Let $\Delta_0 \subset \mathbb{C}$ and $\Delta_j \subset \mathbb{C}$, $1 \leq j \leq m$, be a collection of non-empty, smoothly bounded and pairwise disjoint domains such that for each $j, k \in \{1, \dots, m\}$ we have:

- i) $\Delta_0 \cap \Delta_j \neq \emptyset$ and is smoothly bounded and simply connected,
- ii) $\overline{\Delta_0} \setminus \overline{\Delta_j} \cap \overline{\Delta_j} \setminus \overline{\Delta_0} = \emptyset$,
- iii) $\Delta_j \cap \Delta_k = \emptyset$ for $k \neq j$.

Moreover, we assume that $0 \in \Delta_0$ and that

$$\mathbb{D} = \Delta_0 \cup \left(\bigcup_{j=1}^m \Delta_j \right).$$

Since such sets Δ_j form m good pairs with Δ_0 , there is a smooth function $\chi: \mathbb{C} \rightarrow [0, 1]$ that is equal to 1 on some neighborhood of $\overline{\Delta_0} \setminus \bigcup_{j=1}^m \overline{\Delta_j}$ and that vanishes on some neighborhood of $\bigcup_{j=1}^m \overline{\Delta_j} \setminus \overline{\Delta_0}$. Therefore, given a collection of maps defined on Δ_0 and Δ_j 's, one can connect them with a map defined on \mathbb{D} provided that their images over $\Delta_0 \cap \Delta_j$ lie in the same chart neighborhood.

In this context, let $v_0: \Delta_0 \rightarrow M$ and $v_j: \Delta_j \rightarrow M$, $1 \leq j \leq m$, be J -holomorphic discs. We assume that the image $v_0(\Delta_0)$ lies in a fixed coordinate chart $\phi^0: V_0 \rightarrow U_0$ that intersects m distinct coordinate neighborhoods $\phi^j: V_j \rightarrow U_j$ of the images $v_j(\Delta_j)$. Then, provided that the images of v_0 and v_j are $W^{1,p}$ -close on $\Delta_0 \cap \Delta_j$, we seek a J -holomorphic map which is \mathcal{C}^0 -close to v_0 on Δ_0 and \mathcal{C}^0 -close to v_j 's on Δ_j 's. The following theorem solves this question. The statement is written in terms of local coordinates. That is, the maps u_0 and u_j correspond to $\phi^0 \circ v_0$ and $\phi^j \circ v_j$ respectively, while

the diffeomorphisms $\Psi^j: U_j^0 \rightarrow U_0^j$ represent the transition maps $\phi^0 \circ (\phi^j)^{-1}$ between the sets $U_j^0 = \phi_j(V_j \cap V_0) \subset U_j$ and $U_0^j = \phi_0(V_j \cap V_0) \subset U_0$.

Theorem 4. *For $1 \leq j \leq m$ let $\Delta_0, \Delta_j, U_0, U_j, U_0^j, U_j^0$ and Ψ_j be defined as above and let $J_0 \in \mathcal{J}(U_0)$ and $J_j \in \mathcal{J}(U_j)$ be such that $J_0 = d\Psi_*^j(J_j)$ on U_0^j . Moreover, let $u_0 \in \mathcal{O}_{J_0}(\Delta_0, U_0)$ be an embedding satisfying $u_0(\overline{\Delta_0 \cap \Delta_j}) \subset U_0^j$ and let $\mathcal{W}_j \subset \mathcal{O}_{J_j}(\Delta_j, U_j)$ $1 \leq j \leq m$, be families of J_j -holomorphic maps satisfying the LQJ_j -condition and such that every $u_j \in \mathcal{W}_j$ satisfies $u_j(\overline{\Delta_0 \cap \Delta_j}) \subset U_j^0$. Then the following conclusion holds: for every $\epsilon > 0$ there is $\delta > 0$ such that for any m -tuple of maps $u_j \in \mathcal{W}_j$, $1 \leq j \leq m$, satisfying $\|u_0 - \Psi^j(u_j)\|_{W^{1,p}(\Delta_0 \cap \Delta_j, U_0^j)} < \delta$, there exist $\hat{u}_0 \in \mathcal{O}_{J_0}(\Delta_0, U_0)$ and $\hat{u}_j \in \mathcal{O}_{J_j}(\Delta_j, U_j)$, $1 \leq j \leq m$, with the following properties:*

- i) $\Psi_j(\hat{u}_j) = \hat{u}_0$ on $\Delta_0 \cap \Delta_j$,
- ii) $\|\hat{u}_0 - u_0\|_{W^{1,p}(\Delta_0, U_0)} < \epsilon$ and $\|\hat{u}_j - u_j\|_{W^{1,p}(\Delta_j, U_j)} < \epsilon$,
- iii) $\hat{u}_0(0) = u_0(0)$.

The corresponding solution of the non-linear Cousin problem provided by this theorem is $\hat{v} \in \mathcal{O}_J(\mathbb{D}, M)$ given by $(\phi^0)^{-1} \circ \hat{u}_0$ on Δ_0 and by $(\phi^j)^{-1} \circ \hat{u}_j$ on Δ_j 's. Due to the properties of the local holomorphic maps which are of class $W^{1,p}$, $p > 2$, it is continuous up to the boundary.

This theorem is a corollary of [10, Theorem 2], hence we omit the proof. However, we point out three refinements that we have made. First, we work on $m \geq 2$ good pairs. This can be obtained with $m + 1$ simultaneous local Newton-type iterations or by induction. Secondly, in the original reference, the families $\mathcal{W}_j \subset \mathcal{O}_{J_j}(\Delta_j, U_j)$ are assumed to be $W^{1,p}$ -precompact. As explained after the Corollary 3 such an assumption implies the LQJ_j -conditions which are needed for the convergence of the local Newton-type iterations. However, in the present paper, the sets $\mathcal{W}_j \subset \mathcal{O}_{J_j}(\Delta_j, U_j)$ we will consider are $W^{1,p}$ -unbounded. Therefore, we will check the LQJ_j -conditions directly. Finally, we have added the pointwise condition $\hat{u}_0(0) = u_0(0)$. Once again, adding such a constraint is standard by [22].

2. APPROXIMATELY ATTACHING A DISC TO A TORUS

We present in this section a construction similar to the ones obtained in [3, 23]. Since we deal with manifolds of arbitrary dimension, our approach is based on gluing techniques. The construction is inspired by the work of Rosay [21] in the integrable case.

Theorem 5. *Let (M, J) be a smooth almost complex manifold. We fix $\epsilon > 0$, $m \in \mathbb{N}$ and an embedding $v \in \mathcal{O}_J(\overline{\mathbb{D}}, M)$. Let $G: \partial\mathbb{D} \times \overline{\mathbb{D}} \rightarrow M$ be a smooth map given by $G(z, \zeta) = v_z(\zeta)$ where for every $z \in \partial\mathbb{D}$ the fiber $v_z \in \mathcal{O}_J(\overline{\mathbb{D}}, M)$ is centered at $v_z(0) = v(z)$. For $1 \leq j \leq m$ let $I_j \subset \partial\mathbb{D}$ be disjoint closed arcs and let $V_j \subset M$ be open sets such that $|\partial\mathbb{D} \setminus \cup_{j=1}^m I_j| < \epsilon$ and $G(I_j, \overline{\mathbb{D}}) \subset V_j$. Moreover, for some open sets $U_j \subset \mathbb{C}^n$, we consider local charts $\phi^j: V_j \rightarrow U_j$ in which we have the following properties:*

- i) *the structure $J_j = d\phi_*^j(J) \in \mathcal{J}(U_j)$;*
- ii) *the map $\zeta \mapsto v(\zeta)$ coincides with $w_0^j(\zeta) = (\zeta, 0, \dots, 0)$;*
- iii) *the map $\zeta \mapsto G(z, \zeta)$ coincides with $\zeta \mapsto (z, 0, \dots, \zeta)$ for $z \in I_j$;*
- iv) *the fibers $\zeta \mapsto (z_1, z_2, \dots, z_{n-1}, \zeta)$ are J_j -holomorphic, that is, the complex matrix A_j associated to J by ϕ^j has a vanishing last column.*

Then there exist a set $E \subset \partial\mathbb{D}$ of measure $|E| < \epsilon$ and a map $h \in \mathcal{O}_J(\overline{\mathbb{D}}, M)$ such that $h(0) = v(0)$ and $\text{dist}(h, G(\partial\mathbb{D} \times \partial\mathbb{D})) < \epsilon$ on $\partial\mathbb{D} \setminus E$.

As opposed to the above mentioned results [3, 23], our theorem requires the existence of special coordinate neighborhoods ϕ^j . This is due to the fact that we apply Theorem 4 which requires that the corresponding families satisfy the LQJ_j -conditions. These conditions can not be established without property iv) in the above theorem. Moreover, in contrast with [3, 23], the boundary $h(\partial\mathbb{D})$ of the disc we obtain is not entirely approximately attached to the given torus. Indeed, the values $h(E)$ may be far from $G(\partial\mathbb{D} \times \partial\mathbb{D})$. Nevertheless, since $|E|$ is arbitrarily small such a solution of the Riemann-Hilbert problem is enough to establish Theorem 1.

Proof. Fix $\epsilon > 0$. For $\alpha > 0$ we define the sets

$$\Delta_0 = \{z \in \mathbb{D}; \text{dist}(z, \cup_{j=1}^m I_j) > \alpha\}, \Delta_j = \{z \in \mathbb{D}; \text{dist}(z, I_j) < 2\alpha\}.$$

We set α small enough to ensure that the sets Δ_j , $1 \leq j \leq m$, are pairwise disjoint and that $v(\Delta_j) \subset V_j$. Moreover, we perturb their boundary slightly at the edges, so that we obtain m smoothly bounded good pairs (Δ_0, Δ_j) with the properties required in Theorem 4.

Next, we make a transition into the local coordinates provided by ϕ^j . In such coordinates, we have the local structure $J_j = d\phi_*^j(J)$, the corresponding complex matrix A_j and the generalized Cauchy-Riemann operator \mathcal{F}_j defined in (2). The image of the maps that we are working with are assumed to lie in $\phi^j(V_j)$. Therefore we simplify our notation concerning the spaces of functions and, e.g. write $W^{1,p}(\Delta_j) = W^{1,p}(\Delta_j, \phi^j(V_j))$. Moreover, since the

last column of A_j vanishes, we introduce the notation $A_j = [A'_j, 0] \in \mathbb{C}^{n \times n}$ and $u = [u', u_n]^T \in \mathbb{C}^n$, where we have $A'_j \in \mathbb{C}^{n \times (n-1)}$, $u' \in \mathbb{C}^{n-1}$ and $u_n \in \mathbb{C}$. Note that we have

$$\mathcal{F}_j(u) = u_{\bar{\zeta}} + A'_j(u) \overline{u'_\zeta} = 0.$$

Moreover, the linearization of \mathcal{F}_j along $\varphi \in W^{1,p}(\Delta_j)$ is given by

$$D_\varphi(V) = V_{\bar{\zeta}} + A'_j(\varphi) \overline{V'_\zeta} + \sum_{k=1}^n \left(\frac{\partial A'_j}{\partial z_k}(\varphi) V'_k + \frac{\partial A'_j}{\partial \bar{z}_k}(\varphi) \overline{V'_k} \right) \overline{\varphi'_\zeta}.$$

The following lemma which holds only in these particular coordinates is crucial.

Lemma 6. *Suppose that we are given a family of maps $\varphi_j \in W^{1,p}(\Delta_j)$ mapping into a compact subset of \mathbb{C}^n and satisfying*

$$\left\| \varphi_j - u_0^j \right\|_{L^p(\Delta_j)} < c_j \quad \text{and} \quad \left\| \varphi'_j - (u_0^j)' \right\|_{W^{1,p}(\Delta_j)} < c_j.$$

Provided that $c_j > 0$ is small enough, this family satisfies the LQJ $_j$ -condition.

Proof of Lemma 6. Let $V \in W^{1,p}(\Delta_j)$. We have

$$D_{\varphi_j}(V) - D_{u_0^j}(V) = I(V) + II(V) + III(V),$$

where

$$\begin{aligned} I(V) &= \left(A'_j(\varphi_j) - A'_j(u_0^j) \right) \overline{V'_\zeta}, \\ II(V) &= \sum_{k=1}^n \left(\frac{\partial A'_j}{\partial z_k}(\varphi_j) \overline{(\varphi'_j)_\zeta} - \frac{\partial A'_j}{\partial z_k}(u_0^j) \overline{(u_0^j)'_\zeta} \right) V_k, \\ III(V) &= \sum_{k=1}^n \left(\frac{\partial A'_j}{\partial \bar{z}_k}(\varphi_j) \overline{(\varphi'_j)_\zeta} - \frac{\partial A'_j}{\partial \bar{z}_k}(u_0^j) \overline{(u_0^j)'_\zeta} \right) \overline{V_k}. \end{aligned}$$

Applying the Hölder inequality we get

$$\|I(V)\|_{L^p(\Delta_j)} \leq \left\| A'_j(\varphi_j) - A'_j(u_0^j) \right\|_{L^{2p}(\Delta_j)} \left\| \overline{V'_\zeta} \right\|_{L^{2p}(\Delta_j)}.$$

Moreover, Δ_j is relatively compact hence the $L^{2p}(\Delta_j)$ -norm of a given function may be bounded from above by its $L^p(\Delta_j)$ -norm multiplied by $\text{area}(\Delta_j)^{\frac{1}{2p}}$. If we apply this fact twice, we obtain

$$\|I(V)\|_{L^p(\Delta_j)} \leq \text{area}(\Delta_j)^{\frac{1}{p}} \left\| A'_j(\varphi_j) - A'_j(u_0^j) \right\|_{L^p(\Delta_j)} \|V\|_{W^{1,p}(\Delta_j)}.$$

Since φ_j is assumed to be L^p -close to u_0^j , the operator norm of $I(V)$ can be made as small as desired. Indeed, the images of all such φ_j are contained in a compact subset of \mathbb{C}^n on which A'_j is a smooth matrix map.

Furthermore, since $\left\| (u_0^j)' \right\|_{L^\infty(\overline{\Delta_j})} = 1$ we have

$$\begin{aligned} \|II(V)\|_{L^p(\Delta_j)} &\leq \sum_{k=1}^n \left(\left\| \frac{\partial A'_j}{\partial z_k}(\varphi_j) - \frac{\partial A'_j}{\partial z_k}(u_0^j) \right\|_{L^p(\Delta_j)} \|V\|_{L^\infty(\overline{\Delta_j})} \right) + \\ &+ \sum_{k=1}^n \left(\left\| \frac{\partial A'_j}{\partial z_k}(\varphi_j) \right\|_{L^\infty(\overline{\Delta_j})} \left\| (\varphi'_j)_\zeta - (u_0^j)' \right\|_{L^p(\Delta_j)} \|V\|_{L^\infty(\overline{\Delta_j})} \right). \end{aligned}$$

The L^p -norms in the first sum can be assumed to be small by a similar argument as above. The same is true for the second sum since φ'_j is $W^{1,p}$ -close to $(u_0^j)'$. Hence, by the Sobolev embedding theorem stating that $W^{1,p}(\Delta_j) \subset L^\infty(\overline{\Delta_j})$, the operator norm of $II(V)$ can be made as small as desired. The same conclusion can be obtained for $III(V)$.

Provided that $c_j > 0$ is small we have proved that D_{φ_j} and $D_{u_0^j}$ are arbitrarily close in the operator norm. Let $Q_{u_0^j}$ be the right inverse for $D_{u_0^j}$. Assume that $c_j > 0$ is small enough to ensure

$$\left\| Id - D_{\varphi_j} Q_{u_0^j} \right\|_{op} \leq \frac{1}{2}.$$

Then the following operator

$$Q_{\varphi_j} = Q_{u_0^j} \left(D_{\varphi_j} Q_{u_0^j} \right)^{-1}$$

is a right inverse for D_{φ_j} and satisfies $\|Q_{\varphi_j}\|_{op} \leq 2 \|Q_{u_0^j}\|_{op}$. We have provided uniformly bounded right inverses Q_{φ_j} .

Assume now that $\|\varphi_j - \tilde{\varphi}_j\|_{W^{1,p}(\Delta)} < 1$. We seek a bound $L_j > 0$ depending only on $c_j > 0$ and satisfying

$$\|D_{\varphi_j} - D_{\tilde{\varphi}_j}\|_{op} \leq L_j \|\varphi_j - \tilde{\varphi}_j\|_{W^{1,p}(\Delta)}.$$

Such a constant can be provided along the same above computations. The only difference is that the L^p -difference between $A'_j(\varphi_j)$ and $A'_j(\tilde{\varphi}_j)$ is not small; it can, however, be uniformly bounded. The same applies to the L^p -norm of $\frac{\partial A'_j}{\partial z_k}(\varphi_j) - \frac{\partial A'_j}{\partial z_k}(\tilde{\varphi}_j)$. \square

This lemma provides the necessary bounds for local J_j -holomorphic approximation and the solution of the non-linear Cousin problem. In particular, our strategy is to define a set of almost holomorphic maps $\varphi_j^{c,N}$, $c \in \partial\mathbb{D}$, $N \in \mathbb{N}$, that are L^p -close to u_0^j on Δ_j and have the desired properties. After that, we will provide their J_j -holomorphic approximations and then glue these maps with v using Theorem 4. We point out again that it is essential

that we require only $(\varphi_j^{c,N})'$ to be $W^{1,p}$ -close to $(u_0^j)'$. Indeed, derivatives of the n -th component in $\varphi_j^{c,N}$ will grow linearly with N .

Let Ω_j , $1 \leq j \leq m$, be a range of smooth Jordan domains in \mathbb{C} such that $\Delta_j \subset \Omega_j$, $\partial\Omega_j \cap \partial\Delta_j = I_j$, $v(\Omega_j) \subset V_j$ and $\overline{\Omega_j} \cap \overline{\Delta_k} = \emptyset$ for $k \neq j$. Denote by h_j the Riemann map between $\overline{\Omega_j}$ and $\overline{\mathbb{D}}$, and define for $\zeta \in \overline{\Delta_j}$ and $N \in \mathbb{N}$

$$(5) \quad h_j^N(\zeta) = (h_j(\zeta))^N.$$

These functions are holomorphic in the usual sense and map into $\overline{\mathbb{D}}$. Moreover, note that $|h_j^N| = 1$ on I_j and $|h_j^N| \rightarrow 0$ uniformly on every compact subset of $\overline{\Delta_j} \setminus I_j$ when $N \rightarrow \infty$.

Fix $c \in \partial\mathbb{D}$. For $\zeta \in \overline{\Delta_j}$, we now define

$$\varphi_j^{c,N}(\zeta) = (\zeta, 0, \dots, 0, c \cdot h_j^N(\zeta)).$$

Due to the properties of h_j^N , we have the following limit as $N \rightarrow \infty$

$$\left\| \varphi_j^{c,N} - u_0^j \right\|_{L^p(\Delta_j)} \rightarrow 0.$$

Since $(\varphi_j^{c,N})' = (u_0^j)'$, we have

$$\mathcal{F}_j(\varphi_j^{c,N}) = A_j(\varphi_j^{c,N}) \overline{(\varphi_j^{c,N})_\zeta} = A_j'(\varphi_j^{c,N}) \overline{(u_0^j)_\zeta}.$$

Recall that the map u_0^j is J_j -holomorphic which implies $A_j'(u_0^j) \overline{(u_0^j)_\zeta} = 0$. The images of $\varphi_j^{c,N}$ lie in a compact set on which A is smooth. Hence, the L^p -norm of $\mathcal{F}_j(\varphi_j^{c,N})$ tends to zero when $N \rightarrow \infty$.

We have now reached the situation in which we can perform the local Newton-type iteration. Indeed, let $c_j > 0$ be the constant from Lemma 6. We can fix N large enough so that for every $c \in \partial\mathbb{D}$ we have

$$\left\| \varphi_j^{c,N} - u_0^j \right\|_{L^p(\Delta_j)} < \frac{c_j}{2}.$$

Since such a family of maps satisfies the LQJ_j -condition, by Corollary 3, there exist J_j -holomorphic maps $u_j^{c,N} \in W^{1,p}(\Delta_j)$ for which the $W^{1,p}$ -norm of $u_j^{c,N} - \varphi_j^{c,N}$ is as small as desired. In particular, we need

$$\left\| u_j^{c,N} - \varphi_j^{c,N} \right\|_{W^{1,p}(\Delta_j)} < \epsilon',$$

where $\epsilon' > 0$ is associated to $\epsilon > 0$ and will be determined below. Moreover, we can assume that

$$\left\| u_j^{c,N} - u_0^j \right\|_{L^p(\Delta_j)} < c_j \quad \text{and} \quad \left\| (u_j^{c,N})' - (u_0^j)' \right\|_{W^{1,p}(\Delta_j)} < c_j.$$

This ensures that for large N and every $c \in \mathbb{D}$ the family \mathcal{W}_j of maps $u_j^{c,N}$ also satisfies the LQJ_j -condition. Moreover, due to the properties of the maps h_j^N , the $W^{1,p}(\Delta_0 \cap \Delta_j)$ -difference between $u_j^{c,N}$ and u_0^j tends to zero as $N \rightarrow \infty$. Indeed, $h_j^N \rightarrow 0$ on compact subsets of $\overline{\Delta_j} \setminus I_j$ while the $W^{1,p}$ -norm of $u_j^{c,N} - \varphi_j^{c,N}$ tends to zero by Corollary 3. At this stage, everything is set to apply Theorem 4 and solve the non-linear Cousin problem.

Since v is embedded its image admits a chart neighborhood $\phi^0: V_0 \rightarrow U_0$ such that $d\phi_*^0(J) = J_{st}$ along $\phi^0 \circ v(\overline{\mathbb{D}})$ and that $\det(d\phi_*^0(J) + J_{st}) \neq 0$ on U_0 (see the appendix of [9]). Next, we set $u_0 = \phi^0 \circ v$, $J_0 = d\phi_*^0(J)$, $U_0^j = \phi^0(V_0 \cap V_j)$, $U_j^0 = \phi^j(V_0 \cap V_j)$ and $\Psi^j = \phi^0 \circ (\phi^j)^{-1}: U_j^0 \rightarrow U_0^j$. Note that $u_0^j = \Psi_j^{-1} \circ u_0 = \phi^j \circ v$ on Δ_j . Therefore, provided that N is large, we can assume that

$$\left\| u_0 - \Psi_j(u_j^{c,N}) \right\|_{W^{1,p}(\Delta_0 \cap \Delta_j)} < \delta,$$

where $\delta > 0$ is the constant associated to $\epsilon' > 0$ in Theorem 4. Indeed, the maps $u_j^{c,N}$ are arbitrarily $W^{1,p}$ -close to the maps $\varphi_j^{c,N}$, while the last are arbitrarily $W^{1,p}$ -close to u_0^j on $\Delta_0 \cap \Delta_j$. Note also that the diffeomorphisms Ψ_j are fixed and therefore we have the control over the size of their derivatives. Applying Theorem 4, we can now provide maps $\widehat{u}_0 \in \mathcal{O}_{J_0}(\Delta_0, U_0)$ and $\widehat{u}_j \in \mathcal{O}_{J_j}(\Delta_j, U_j)$, $1 \leq j \leq m$, such that

$$\|\widehat{u}_0 - u_0\|_{W^{1,p}(\Delta_0)} < \epsilon' \text{ and } \left\| \widehat{u}_j^{c,N} - u_j^{c,N} \right\|_{W^{1,p}(\Delta_j)} < \epsilon'.$$

Moreover, we have $\widehat{u}_0(0) = u_0(0)$ and $\Psi_j(\widehat{u}_j^{c,N}) = \widehat{u}_0$ on $\Delta_0 \cap \Delta_j$. Thus, the J -holomorphic map $h: \overline{\mathbb{D}} \rightarrow M$ given by

$$h = \begin{cases} \phi_0^{-1} \circ \widehat{u}_0 & \text{on } \Delta_0 \\ \phi_j^{-1} \circ \widehat{u}_j^{c,N} & \text{on } \Delta_j \end{cases}$$

is well defined and satisfies $h(0) = v(0)$. Finally, we chose $\epsilon' > 0$ such that the estimate

$$\left\| \widehat{u}_j^{c,N} - \varphi_j^{c,N} \right\|_{W^{1,p}(\Delta_j)} < 2\epsilon'$$

implies

$$\text{dist} \left(\phi_j^{-1} \circ \widehat{u}_j^{c,N}(\overline{\Delta_j}), \phi_j^{-1} \circ \varphi_j^{c,N}(\overline{\Delta_j}) \right) < \epsilon$$

for all $1 \leq j \leq m$. It follows that h is the map we seek. \square

3. PROOF OF THEOREM 1

An upper semi-continuous function f defined on an open set V in (M, J) is J -plurisubharmonic if $f \circ v$ is subharmonic for any $v \in \mathcal{O}_J(\overline{\mathbb{D}}, V)$. We denote by $\text{Psh}_J(V)$ the set of J -plurisubharmonic functions on V .

The proof of Theorem 1 follows the one given in [12] but uses the new solution of the Riemann-Hilbert problem presented in Section 2 in order to avoid dimensional restrictions. The reader should note that a large part of the original proof goes through in any dimension. Hence, we drop some details. In particular, we do not prove that EP_f is upper semi-continuous in higher dimensions. This is clear by reading p. 267 in [12]. Finally, since Theorem 1 was already proved in real dimension 4, we will assume that the real dimension of M is greater than 4.

Proof of Theorem 1. Let f be an upper semi-continuous function defined on M . Since the envelope EP_f is upper semi-continuous it can be approximated from above by continuous functions. Hence we can assume the function f to be continuous (see [5, p. 8]). We need to show that for every $p \in M$ with $EP_f(p) > -\infty$ and every smooth J -holomorphic embedding $v_p \in \mathcal{O}_J(\overline{\mathbb{D}}, V)$ centered at p , we have

$$(6) \quad EP_f(p) = EP_f(v_p(0)) \leq \int_0^{2\pi} EP_f \circ v_p(e^{i\theta}) \frac{d\theta}{2\pi}.$$

Fix such a disc v_p and a number $\varepsilon > 0$. Recall that v_p is defined and J -holomorphic on some neighborhood \mathbb{D}_{v_p} of $\overline{\mathbb{D}}$. Let $z' \in \partial\mathbb{D}$. There exists a J -holomorphic disc $v_{z'}$ centered at $v_p(z')$ and transverse to v_p which satisfies the following extremality condition

$$(7) \quad \int_0^{2\pi} f \circ v_{z'}(e^{it}) \frac{dt}{2\pi} < EP_f(v_p(z')) + \frac{\varepsilon}{4\pi}.$$

Since $\dim_{\mathbb{R}} M \geq 6$ we may assume that $v_{z'}$ is embedded [24]. Moreover, applying the deformation theory from [22] one can perturb $v_{z'}$ and obtain a J -holomorphic disc centered at any $q \in M$ that is sufficiently close to p . Our strategy is to perform these perturbations in such a way that we will obtain the coordinate charts ϕ^j needed to apply Theorem 5.

Let $V \subset M$ be a small neighborhood of the image $v_{z'}(\overline{\mathbb{D}})$. Since $v_{z'}$ and v_p are both embedded one can set a local chart $\phi: V \rightarrow \mathbb{C}^n$, in which the disc $\phi \circ v_{z'}$ agrees with $u_{z'}(\zeta) = (0, \dots, 0, \zeta)$. Moreover, for values close to z' the disc v_p is identified with the map $z \mapsto (z, 0, \dots, 0)$ in such a way that $v_p(z')$ corresponds to the origin. It is proved in [22, Theorem 1.3.] that the

manifold of $\phi_*(J)$ -holomorphic discs is a Banach manifold modeled on the space of usual J_{st} -holomorphic disc. In particular, in a $W^{1,p}$ -neighborhood of $u_{z'}$ there exists a one-to-one correspondence Φ based on singular integrals and such that $\Phi(u_{z'}) = u_{z'}$ and that the map $\Phi(u)$ is J_{st} -holomorphic if and only if u is $\phi_*(J)$ -holomorphic. Moreover, this correspondence can be constructed to ensure $\Phi(u)(0) = u(0)$. It follows that given a point $q \in \mathbb{C}^{n-1}$ with a small norm, one can define a J -holomorphic disc u_q centered at $(q, 0)$ by mapping the translation $\zeta \mapsto (q, \zeta)$ with Φ^{-1} . We then obtain a diffeomorphism $H(q, \zeta) \mapsto u_q(\zeta)$ with $\phi_*(J)$ -holomorphic ζ -fibers. This allows us to define the desired coordinate chart $\phi^{z'} = H \circ \phi$ on a possibly smaller open neighborhood of $v_{z'}(\overline{\mathbb{D}})$ contained in V .

If $z \in \partial\mathbb{D}$ is close enough to z' the disc $v_z^{z'}(\zeta) = (\phi^{z'})^{-1}(z, 0, \dots, \zeta)$ is well defined, centered at $v_p(z)$, and still satisfies the extremality condition (7). By compactness, we cover $\partial\mathbb{D}$ with a finite number of open arcs

$$I_{z'} = \left\{ z \in \partial\mathbb{D}; v_z^{z'} \text{ is well defined} \right\}.$$

We shrink these sets into pairwise disjoint closed arcs $I_{z'_1}, I_{z'_2}, \dots, I_{z'_m}$ to ensure that the measure of $E = \partial\mathbb{D} \setminus \cup_{j=1}^m I_{z'_j}$ is arbitrarily small. These arcs are the one required in Theorem 5 while $\phi^{z'_j}$ are the the corresponding charts.

Finally, we define the map $G: \partial\mathbb{D} \times \overline{\mathbb{D}} \rightarrow M$. Let $J_{z'_1}, J_{z'_2}, \dots, J_{z'_m} \subset \partial\mathbb{D}$ be pairwise disjoint open arcs satisfying $I_{z'_j} \subset J_{z'_j}$. Moreover, we assume that $v_z^{z'_j}$ is defined for every $z \in J_{z'_j}$. Let $\chi: \partial\mathbb{D} \rightarrow [0, 1]$ be a smooth cut-off function that is equal to 1 on the intervals $I_{z'_j}$ and vanishes off the union of sets $J_{z'_j}$. We set $G(z, \zeta) = v_z^{z'_j}(\chi(z)\zeta)$ for $z \in J_{z'_j}$ and $G(z, \zeta) = v_p(z)$ elsewhere. That is, for $z \in I_{z'_j}$ we have $G(z, \zeta) = v_z^{z'_j}(\zeta)$, while off these closed arcs we shrink these discs into constant maps. Note that for all $z \in \partial\mathbb{D}$ the disc $\zeta \mapsto G(z, \zeta)$ is J -holomorphic. In addition, since the measure of E is small and almost every ζ -fiber of G satisfies (7), we can assume that the following crucial inequality is fulfilled:

$$(8) \quad \int_0^{2\pi} \int_0^{2\pi} f \circ G(e^{i\theta}, e^{it}) \frac{dt}{2\pi} \frac{d\theta}{2\pi} < \int_0^{2\pi} EP_f \circ v_p(e^{i\theta}) \frac{d\theta}{2\pi} + \frac{\varepsilon}{2}.$$

Let h be a J -holomorphic disc centered at p and with its boundary approximately attached to $G(\partial\mathbb{D} \times \partial\mathbb{D})$ given by Theorem 5. By construction the distance between $h(\zeta)$ and $G(\zeta, c \cdot h_j^N(\zeta))$ is small for every $\zeta \in I_{z'_j}$, $c = e^{it} \in \partial\mathbb{D}$ and $N \in \mathbb{N}$ large enough. Here h_j^N is the map defined in (5).

Note that we also have $|h_j^N(\zeta)| = 1$ for $\zeta \in I_{z'_j}$. Hence for values $e^{i\theta}$ outside the set E we can write $h_j^N(e^{i\theta}) = e^{i\beta_N(\theta)}$. Let us extend β_N continuously to a map defined on $[0, 2\pi]$ and satisfying $\beta_N(0) = \beta_N(2\pi)$. We claim that

$$\int_0^{2\pi} f \circ h(e^{i\theta}) \frac{d\theta}{2\pi} < \int_0^{2\pi} f \circ G(e^{i\theta}, e^{i(t+\beta_N(\theta))}) \frac{d\theta}{2\pi} + \frac{\varepsilon}{2}.$$

Indeed, for $\theta, t \in [0, 2\pi]$ the values of $G(e^{i\theta}, e^{i(t+\beta_N(\theta))})$ lie in the torus $G(\partial\mathbb{D} \times \partial\mathbb{D})$ while the values of $h(e^{i\theta})$ are by construction contained in a small neighborhood of the set $G(\overline{\mathbb{D}} \times \overline{\mathbb{D}})$. Therefore, for $e^{i\theta} \in E$ the difference between these two terms can be uniformly bounded. The rest follows by the fact that the measure of E is as small as desired.

We now apply the key trick which is due to Poletsky [17]. We define for $t \in [0, 2\pi]$

$$I(t) = \int_0^{2\pi} f \circ G(e^{i\theta}, e^{i(t+\beta_N(\theta))}) \frac{d\theta}{2\pi}.$$

According to the mean value theorem there exists $\nu \in [0, 2\pi)$ such that

$$I(\nu) = \int_0^{2\pi} I(t) \frac{dt}{2\pi} = \int_0^{2\pi} \int_0^{2\pi} f \circ G(e^{i\theta}, e^{i(t+\beta_N(\theta))}) \frac{dt}{2\pi} \frac{d\theta}{2\pi}$$

Furthermore, after a change of variables in the double integral, we get

$$I(\nu) = \int_0^{2\pi} \int_0^{2\pi} f \circ G(e^{i\theta}, e^{it}) \frac{dt}{2\pi} \frac{d\theta}{2\pi}.$$

Together with (8) we then obtain

$$EP_f(u_p(0)) \leq \int_0^{2\pi} f \circ h(e^{i\theta}) \frac{d\theta}{2\pi} \leq I(\nu) + \frac{\varepsilon}{2} \leq \int_0^{2\pi} EP_f \circ v_p(e^{i\theta}) \frac{d\theta}{2\pi} + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (6) holds and the theorem is proved. \square

4. APPLICATIONS

In this section we provide a few applications of Theorem 1. The proofs are omitted since they contain very few original ideas.

4.1. Regularization of J -plurisubharmonic functions. In real dimension four, Pliś proved in [19] that every J -plurisubharmonic function can be locally approximated by smooth J -plurisubharmonic functions. Along with his almost complex analogue of the Richberg theorem [18], the key ingredient was the low dimensional analogue of our main theorem given in [12]. In fact, as observed by Pliś himself, this was the only reason why the result was proved in real dimension four. Therefore, as a corollary of Theorem 1 and [18, 19] we obtain the following statement.

Corollary 7. *Let (M, J) be an almost complex manifold and let $p \in M$. There exists a neighborhood V of p such that for every J -plurisubharmonic function f on V there exists a decreasing sequence of J -plurisubharmonic C^∞ functions $(f_N)_{N \in \mathbb{N}}$ on V such that $f_N \rightarrow f$ when $N \rightarrow \infty$.*

It is worth pointing out that Harvey, Lawson and Pliś [8] proved the corresponding global approximation result in case when M is J -pseudoconvex (see Theorem 4.1 in [8]), that is, when M admits a smooth strictly J -plurisubharmonic exhaustion function. In particular, they also obtain the above Corollary 7 (see Corollary 4.2) since any almost complex manifold admits a system of (strictly) J -pseudoconvex neighborhoods. However, their techniques are very different from the ones presented here.

4.2. Psh $_J$ -hull and Poletsky discs. In addition to providing new J -plurisubharmonic functions by variational methods, Poletsky theory gives beautiful characterizations of hulls in term of J -holomorphic discs. We refer to [17] for the most classical one in (\mathbb{C}^n, J_{st}) . In our case, we can obtain a characterization of compact Psh $_J$ -hulls of compact sets in terms of J -holomorphic discs, similar to those obtained in [15, 13, 20, 5].

Let $K \subset (M, J)$ be a compact set. Its Psh $_J$ -hull is defined by

$$\widehat{K} = \{p \in M; f(p) \leq \sup_K f, \forall f \in \text{Psh}_J(M)\}.$$

Note that, if M is J -pseudoconvex, it is enough to consider only continuous or smooth J -plurisubharmonic functions. This is due to the above mentioned result of Harvey, Lawson and Pliś (see Corollary 6.3 in [8]). The following statement is standard and we refer to e.g. [5] for its proof.

Corollary 8. *Let $K \subset (M, J)$ be a compact set whose Psh $_J$ -hull \widehat{K} is also compact. Let V be a relatively compact open set containing \widehat{K} . Then $p \in M$ belongs to \widehat{K} if and only if for any $\varepsilon > 0$ and for any neighborhood $U \subset V$ of K , there exists a J -holomorphic disc $v \in \mathcal{O}_J(\overline{\Delta}, M)$ centered at p and a set $E \subset \partial\Delta$ of measure $|E| < \varepsilon$ such that $v(\partial\Delta \setminus E) \subset U$.*

Note that if M admits a global strictly J -plurisubharmonic function, the compactness assumption made on \widehat{K} may be replaced by any of the equivalent conditions stated in Theorem 5.4 of Diederich and Sukhov [4]; e.g. one obtains a characterization of Psh $_J$ -hulls of compact sets contained in Stein domains.

In the case when M does not admit any non-constant bounded J -plurisubharmonic function (e.g. if M is compact), Theorem 1 provides J -holomorphic

discs with prescribed center and with most of their boundary in a given open set. This was pointed out by Lárússon and Sigurdsson in [13] (see also [21]).

Corollary 9. *Let (M, J) be an almost complex manifold admitting no non-constant bounded J -plurisubharmonic function. Let $V \subset M$ be a non-empty open set and let $p \in M$. For every $\varepsilon > 0$, there exists $v \in \mathcal{O}_J(\overline{\Delta}, M)$ centered at p and a set $E \subset \partial\Delta$ of measure $|E| < \varepsilon$ such that $v(\partial\Delta \setminus E) \subset V$.*

It is worth mentioning that the second named author gave a direct construction of such discs in a certain class of compact almost complex manifolds in which the Runge-type approximation results are possible [11].

4.3. Envelope of the Lelong functional. We consider the *Lelong functional* associated to a non-negative real function α on M , defined by

$$L_\alpha(v) = \sum_{\zeta \in \mathbb{D}} \alpha(v(\zeta)) \log |\zeta|,$$

where $v \in \mathcal{O}_J(\overline{\mathbb{D}}, M)$. Given a J -plurisubharmonic function $f \in \text{Psh}_J(M)$ and a point $p \in M$ we denote by $\nu_f(p) \in [0, +\infty]$ its *Lelong number* at p ; in any local coordinate system ϕ on M , with $\phi(p) = 0$, we have

$$\nu_f(p) = \lim_{r \rightarrow 0^+} \frac{\sup_{|\phi(q)| \leq r} f(q)}{\log r}.$$

Note that we consider the constant function $f = -\infty$ as J -plurisubharmonic and set $\nu_{-\infty} = +\infty$. Given a non-negative function $\alpha: M \rightarrow \mathbb{R}_+$, consider

$$\mathcal{F}_\alpha = \{f \in \text{Psh}_J(M); f \leq 0, \nu_f \geq \alpha\},$$

and the corresponding extremal function on M

$$f_\alpha = \sup \mathcal{F}_\alpha.$$

The following theorem establishes the correspondence between f_α and the envelope EL_α of the Lelong functional.

Theorem 10. *Let $\alpha: M \rightarrow \mathbb{R}_+$ be a non-negative function defined on a smooth almost complex manifold (M, J) . Then EL_α is J -plurisubharmonic and coincides with f_α .*

This disc formula was first obtained by Poletsky [17]. For manifolds this result was proved by Lárússon and Sigurdsson [13, 14] and by Drinovec Drnovšek and Forstnerič [6] for locally irreducible complex spaces. We explain its relation with the envelope of the Poisson functional.

If $f \in \mathcal{F}_\alpha$ and $v \in \mathcal{O}_J(\overline{\mathbb{D}}, M)$, then $f \circ v \leq 0$ is a subharmonic function defined in a neighborhood of $\overline{\mathbb{D}}$ whose Lelong number at any point $\zeta \in \mathbb{D}$ satisfies

$$\nu_{f \circ v}(\zeta) \geq \alpha(v(\zeta)).$$

Hence $f \circ v$ is bounded above by the largest subharmonic function $f_{\alpha \circ v} \leq 0$ on \mathbb{D} satisfying $\nu_{f_{\alpha \circ v}} \geq \alpha \circ v$. This maximal function $f_{\alpha \circ v}$ is the weighted sum of Green functions with coefficients $\alpha \circ v$

$$f_{\alpha \circ v}(z) = \sum_{\zeta \in \mathbb{D}} \alpha(v(\zeta)) \log \left| \frac{z - \zeta}{1 - \bar{\zeta}z} \right|, \quad z \in \mathbb{D}.$$

If the sum is divergent, we take $f_{\alpha \circ v} \equiv -\infty$.

Setting $z = 0$, we note that for every $f \in \mathcal{F}_\alpha$ and $v \in \mathcal{O}_J(\overline{\mathbb{D}}, M)$ we have

$$f(v(0)) \leq \sum_{\zeta \in \mathbb{D}} \alpha(v(\zeta)) \log |\zeta| \leq \inf_{\zeta \in \mathbb{D}} \alpha(v(\zeta)) \log |\zeta| =: K_\alpha(v).$$

By taking the infimum over all J -holomorphic discs v centered at $p \in M$ we obtain for any $f \in \mathcal{F}_\alpha$:

$$(9) \quad f(p) \leq EL_\alpha(p) \leq EK_\alpha(p) =: k_\alpha(p).$$

It turns out that f_α coincides with the envelope of the Poisson functional EP_{k_α} for the function k_α . Moreover, by taking the supremum over all $f \in \mathcal{F}_\alpha$ in (9) we get $EP_{k_\alpha} = f_\alpha \leq EL_\alpha$. Thus, in order to prove Theorem 10 it is enough to prove that $EL_\alpha \leq EP_{k_\alpha}$.

On almost complex manifolds all the above considerations were settled by the second named author and Drinovec Drnovšek [7]. However, similarly to [12], the paper [7] focuses on real dimension four due to the lack of solutions of the Riemann-Hilbert problem. In particular, proving $EL_\alpha \leq EP_{k_\alpha}$ is equivalent to showing that for every continuous function $\varphi: M \rightarrow \mathbb{R}$ with $\varphi \geq k_\alpha$, embedded J -holomorphic disc $v \in \mathcal{O}_J(\overline{\mathbb{D}}, M)$, and $\epsilon > 0$ there exists a J -holomorphic disc h with center $h(0) = v(0)$ and satisfying

$$L_\alpha(h) = \sum_{z \in \mathbb{D}} \alpha(h(z)) \log |z| < \frac{1}{2\pi} \int_0^{2\pi} \varphi \circ v(e^{it}) dt + \epsilon.$$

Such a map can now be obtained via the new methods developed in Section 2. Nevertheless, some extra care is needed. In particular, in the proof of Theorem 1 we require that the J -holomorphic disc $h(\zeta)$ approximating the map $G(\zeta, c \cdot h_j^N(\zeta))$ satisfies $h(0) = G(0, 0)$. In order to prove the above inequality, we must add finitely many such pointwise restrictions (see the

proof of Theorem 1 in [7]). Moreover, the functions h_j^N from (5) need to be slightly changed.

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