THE STATIONARY DISC METHOD IN THE UNIQUE JET DETERMINATION OF CR AUTOMORPHISMS

FLORIAN BERTRAND

Introduction

The study of complex geometric invariants associated to a real submanifold $M \subset \mathbb{C}^N$ is essential in order to understand the group of automorphisms of M, and in particular, in the problem of distinguishing one automorphism from another. Poincaré initiated such a study of invariants associated to a real hypersurface by looking at relations between the Taylor series coefficients of a defining function and of the corresponding defining function obtained by a local biholomorphic change of variables. This approach of finding invariants from power series expansions was carried out much later in a significant way by Moser for Levi nondegenerate hypersurfaces [21]. For a real submanifold $M \subset \mathbb{C}^N$ and a point $p \in M$, we denote by $Aut(M, p)$ the set of germs of biholomorphic maps fixing p and such that $F(M) \subset M$. The following theorem is contained in the work of Chern and Moser [21] (see also [18, 19, 53, 54]):

Theorem 0.1. *If a real hypersurface $M \subset \mathbb{C}^N$ is real analytic Levi nondegenerate at a point $p \in M$, then elements of $Aut(M, p)$ are uniquely determined by their jet of order two at p.*

Theorem 0.1 is false without any assumption on the Levi form, as one can see by considering the hyperplane $M = \{(z, w) \in \mathbb{C}^2 \mid \Im m w = 0\}$ whose automorphism group at 0 is infinite dimensional (see e.g. [61]). Note that in $\mathbb{C}^2$, Levi-flat hypersurfaces are the only ones among real analytic hypersurfaces for which local biholomorphisms are not uniquely determined by their jet of any order [26].

Finite jet determination of holomorphic maps between *real analytic* real submanifolds has attracted considerable attention these past years. For a survey on this matter, we point out for instance to the articles of Zaitsev [61] or Baouendi, Ebenfelt and Rothschild [4]. Naturally, after Theorem 0.1, many other situations were investigated; finitely nondegenerate hypersurfaces [1, 60]; Ebenfelt, Lamel and Zaitsev [26] proved that 2-jet determination holds for hypersurfaces of finite type in $\mathbb{C}^2$ (see also [39]); finite (multi)type in $\mathbb{C}^N$ [5, 42, 40]. We note the paper of Juhlin [36] for holomorphically nondegenerate hypersurfaces which settles a conjecture due to Baouendi, Ebenfelt and Rotschild [2]. Important work on finite jet determination for holomorphic maps between real submanifolds of higher codimension has also been done; we refer for instance to the articles of Beloshapka [8], Zaitsev [60], Baouendi, Ebenfelt and Rotschild [2], Baouendi, Mir and Rotschild [6], Lamel and Mir [42], Juhlin [36], Juhlin and Lamel [37], Mir and Zaitsev [46]. Actually, in the real analytic setting one knows more; (formal) biholomorphisms between sufficiently nondegenerate real submanifolds can be reconstructed from their jets in an analytic way. See for instance the survey of Lamel [41] and the article of Juhlin and Lamel [37]).

In the *smooth* case, results on finite jet determination for holomorphic maps relies on the method of complete differential systems (which goes back to the works of Cartan [18, 19] and Chern and Moser [21], and developed further by Han [31, 32]; see also Peyron [48]) and have been restricted to the setting of finitely nondegenerate real submanifolds. We refer to the works of Ebenfelt [24], Ebenfelt and Lamel [25] for $\mathcal{C}^\infty$ real hypersurfaces and Kim and Zaitsev [38] for $\mathcal{C}^\infty$ real submanifolds of higher codimension.

In [9], the question of finite jet determination was studied from a more geometric perspective, using an important family of invariant objects attached to real submanifolds, namely the stationary discs. These discs are special analytic discs, attached to a given real submanifold $M \subset \mathbb{C}^N$, which admit a lift, with a pole of order at most one at the origin 0, to the conormal bundle of M. These particular discs were introduced by Lempert in [43] in his study of the complex geometry of bounded smooth strongly convex domains in $\mathbb{C}^N$. As applications, Lempert provided a precise form and a new proof of Fefferman theorem on boundary extension of biholomorphisms (see also [44]), introduced an analogue of the Riemann map which he used to construct solutions of Monge-Ampere type equations (see also [45]). Poletsky [49] studied extremal discs in pseudoconvex domains using a variational approach; in particular, he proved that the associated Euler-Lagrange equation corresponds to the stationary condition for a holomorphic disc. The existence and construction of stationary discs were later on developed in more general settings. In case of nonconvex bounded strictly pseudoconvex domains Pang [47] studied the relation between extremal discs for the Kobayashi metric and stationary discs, and obtained a smoothness result for the Kobayashi metric. We note the important work of Huang [35] on the construction of stationary discs for bounded strictly pseudoconvex domains and their application to the study of dynamical properties of self-maps (see also [34]). The case of strictly pseudodonvex real submanifolds of higher codimension was treated by Tumanov [56] who consequently obtained a regularity theorem for CR maps; see also the work of Sukhov and Tumanov [52] who construct stationary discs attached to small perturbations of $\mathbb{S}^3 \times \mathbb{S}^3 \subset \mathbb{C}^4$, where $\mathbb{S}^3$ denotes the unit sphere in $\mathbb{C}^2$. Finally, we refer to the works of Coupet, Gaussier and Sukhov [23], Spiro and Sukhov [51], and Gaussier and Joo [28] in the almost complex framework.

Recently, many results on finite jet determination for finitely smooth real submanifolds were obtained using stationary discs attached to such submanifolds [9, 11, 13, 10, 57, 14]. It is important to point out that, to the best of our knowledge, the method of stationary discs used in these articles is the only one which allows to treat finitely smooth real submanifolds. The main point in the approach developed in these papers is the construction of "enough" stationary discs with "good" geometric properties. The idea of attaching such a disc to a real submanifold is a boundary value problem, namely a nonlinear Riemann-Hilbert type problem. In case of a nondegenerate real subamanifold M - in which case its conormal bundle is totally real [59, 56] - inspired by the essential works of Forstnerič [27] and of Globevnik [29, 30] on analytic discs attached to totally real submanifolds, one can analyze the existence and the structure of solutions of such nonlinear Riemann-Hilbert problems. In the setting of Levi degenerate hypersurfaces, the corresponding conormal bundle admits complex tangencies, and therefore the attachment of discs is much more complicated. This issue can be settled by allowing a higher winding of the conormal part of the disc. In that setting, the construction and study of these higher order stationary discs, called k_0-stationary discs (see Definition 1.4) and introduced in [11], relies on a nonlinear singular Riemann-Hilbert problem which can be treated with the techniques developed in [12].

1. Preliminaries

We denote by Δ the unit disc in $\mathbb{C}$, by $\partial\Delta$ its boundary, and by $\mathbb{B} \subset \mathbb{C}^N$ the unit ball in $\mathbb{C}^N$.

1.1. **Real submanifolds of $\mathbb{C}^N$.** Let $M \subset \mathbb{C}^N$ be a $\mathcal{C}^4$ generic real submanifold of real codimension $d \geq 1$ and let $p \in M$. After a local biholomorphic change of coordinates, we may assume that $p = 0$ and that $M \subset \mathbb{C}^N = \mathbb{C}^n_z \times \mathbb{C}^d_w$ is given locally by

$$(1.1)\qquad \begin{cases} r_1 = \Re e w_1 - {}^t\bar{z}A_1 z + O(3) = 0 \\ \quad\vdots \\ r_d = \Re e w_d - {}^t\bar{z}A_d z + O(3) = 0 \end{cases}$$

where $A_1, \ldots, A_d$ are Hermitian matrices of size n (see [3] and Section 7.2 [17] for more details). In the remainder O(3), z is of weight one and $\Im m w$ of weight two. We set $r := (r_1, \ldots, r_d)$.

We recall the following biholomorphic invariant notions of nondegeneracy that coincide in the hypersurface case and in case $N = 4$. Let M be $\mathcal{C}^4$ generic real submanifold M of $\mathbb{C}^N$ of codimension d given by (1.1).

Definition 1.1 ([56]). The submanifold M is *generating strictly pseudoconvex* at 0 if the following two conditions are satisfied

($\mathfrak{a}$) A_1,...,A_d are linearly independent (equivalently on $\mathbb{R}$ or $\mathbb{C}$)

($\mathfrak{t}^+$) there exists a real linear combination $\sum_{j=1}^d c_j A_j$ that is positive definite.

Definition 1.2 ([10]). The submanifold M is *fully nondegenerate* at 0 if the following two conditions are satisfied

($\mathfrak{f}$) there exists $V \in \mathbb{C}^n$ such that $\mathrm{span}_{\mathbb{C}}\{A_1V, \ldots, A_dV\}$ is of dimension d,

($\mathfrak{t}$) there exists a real linear combination $\sum_{j=1}^d c_j A_j$ that is invertible.

Definition 1.3 ([14]). The submanifold M is $\mathfrak{D}$*-nondegenerate* at 0 if the following two conditions are satisfied

($\mathfrak{d}$) there exists $V \in \mathbb{C}^n$ such that $\mathrm{span}_{\mathbb{R}}\{A_1V, \ldots, A_dV\}$ is of dimension d,

($\mathfrak{t}$) there exists a real linear combination $\sum_{j=1}^d c_j A_j$ that is invertible.

We point out that conditions ($\mathfrak{a}$), ($\mathfrak{f}$) and ($\mathfrak{d}$) impose a restriction on the codimension of M; namely ($\mathfrak{a}$) implies $d \leq n^2$, whiles ($\mathfrak{f}$) implies $d \leq n$, and ($\mathfrak{d}$) implies $d \leq 2n$. Note that ($\mathfrak{f}$) implies ($\mathfrak{d}$), and that ($\mathfrak{d}$) implies ($\mathfrak{a}$). We point out that in case $d = 2$, conditions ($\mathfrak{f}$), ($\mathfrak{d}$) and ($\mathfrak{a}$) are equivalent, but are not in general equivalent as illustrated by the following example quadric in $\mathbb{C}^8$ given by

$$\begin{cases} \Re e w_1 = |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 \\ \Re e w_2 = |z_1|^2 \\ \Re e w_3 = |z_2|^2 \\ \Re e w_4 = |z_1|^2 + \Re e(\overline{z_1} z_2) \end{cases}$$

Condition ($\mathfrak{f}$) is tightly related to the existence of analytic discs whose centers fill an open set and determined by their 1-jet (see Proposition 2.13). Recall that condition ($\mathfrak{t}$) was introduced by Tumanov [56] and is essential for the construction of stationary discs (see Theorem 2.1). Finally, we note that the quadric

$$\begin{cases} \Re e w_1 = |z_1|^2 \\ \Re e w_2 = |z_2|^2 \\ \Re e w_3 = z_1\overline{z_2} + \overline{z_1} z_2 \end{cases}$$

is $\mathfrak{D}$-nondegenerate but not fully nondegenerate; the quadric

$$\begin{cases} \Re e w_1 = |z_1|^2 - |z_2|^2 \\ \Re e w_2 = |z_3|^2 \end{cases}$$

is $\mathfrak{D}$-nondegenerate but not generating strictly pseudoconvex; and the quadric

$$\begin{cases} \Re e w_1 = |z_1|^2 \\ \Re e w_2 = |z_2|^2 \\ \Re e w_3 = z_1\overline{z_2} + \overline{z_1}z_2 \\ \Re e w_4 = iz_1\overline{z_2} - i\overline{z_1}z_2 \end{cases}$$

is generating strictly pseudoconvex but not $\mathfrak{D}$-nondegenerate.

1.2. **Stationary discs.** We first introduce relevant spaces of functions. Let $k \geq 0$ be an integer and let $0 < \alpha < 1$. We denote by $\mathcal{C}^{k,\alpha} = \mathcal{C}^{k,\alpha}(\partial\Delta, \mathbb{R})$ the space of real-valued functions defined on $\partial\Delta$ of class $\mathcal{C}^{k,\alpha}$. The space $\mathcal{C}^{k,\alpha}$ is endowed with its usual norm

$$\|f\|_{\mathcal{C}^{k,\alpha}} = \sum_{j=0}^{k} \|f^{(j)}\|_\infty + \sup_{\zeta\neq\eta\in\partial\Delta} \frac{\|f^{(k)}(\zeta) - f^{(k)}(\eta)\|}{|\zeta-\eta|^\alpha},$$

where $\|f^{(j)}\|_\infty = \max_{\partial\Delta}\|f^{(j)}\|$. We set $\mathcal{C}^{k,\alpha}_{\mathbb{C}} = \mathcal{C}^{k,\alpha} + i\mathcal{C}^{k,\alpha}$ and equip this space with the norm

$$\|f\|_{\mathcal{C}^{k,\alpha}_{\mathbb{C}}} = \|\Re e f\|_{\mathcal{C}^{k,\alpha}} + \|\Im m f\|_{\mathcal{C}^{k,\alpha}}.$$

We denote by $\mathcal{A}^{k,\alpha}$ the subspace of *analytic discs* in $\mathcal{C}^{k,\alpha}_{\mathbb{C}}$ consisting of functions $f : \overline{\Delta} \to \mathbb{C}$, holomorphic on Δ with trace on $\partial\Delta$ belonging to $\mathcal{C}^{k,\alpha}_{\mathbb{C}}$.

Let M be a $\mathcal{C}^4$ generic real submanifold of $\mathbb{C}^N$ of codimension d given by (1.1). An analytic disc $f \in (\mathcal{A}^{k,\alpha})^N$ is *attached to* M if $f(\partial\Delta) \subset M$. Following Lempert [43] and Tumanov [56] we define

Definition 1.4 ([11])**.** Let $k_0 \geq 1$ be an integer. A holomorphic disc $f : \Delta \to \mathbb{C}^N$ continuous up to $\partial\Delta$ and attached to M is *k_0-stationary for M* if there exists a holomorphic lift $\boldsymbol{f} = (f, \tilde{f})$ of f to the cotangent bundle $T^*\mathbb{C}^N$, continuous up to $\partial\Delta$ and such that for all $\zeta \in \partial\Delta$, $\boldsymbol{f}(\zeta) \in \mathcal{N}^{k_0}M(\zeta)$ where

$$\mathcal{N}^{k_0}M(\zeta) := \{(z, w, \tilde{z}, \tilde{w}) \in T^*\mathbb{C}^N \mid (z, w) \in M, (\tilde{z}, \tilde{w}) \in \zeta^{k_0} N^*_{(z,w)}M \setminus \{0\}\},$$

and where $N^*_{(z,w)}M$ is the conormal fiber at (z, w) of M. The set of these lifts $\boldsymbol{f} = (f, \tilde{f})$, with f nonconstant, is denoted by $\mathcal{S}^{k_0}(M)$.

Remark 1.5. If $k_0 = 1$, such discs correspond to usual stationary discs introduced by Lempert in [43]; we will drop the index 1 in all notations related to 1-stationary discs.

Note that, equivalently, an analytic disc $f \in (\mathcal{A}^{k,\alpha})^N$ attached to M is stationary for M if there exists d real valued functions $c_1, \ldots, c_d : \partial\Delta \to \mathbb{R}$ such that $\sum_{j=1}^d c_j(\zeta)\partial r_j(0) \neq 0$ for all $\zeta \in \partial\Delta$ and such that the map

$$\zeta \mapsto \zeta^{k_0} \sum_{j=1}^{d} c_j(\zeta)\partial r_j \left(f(\zeta), \overline{f(\zeta)}\right)$$

defined on $\partial\Delta$ extends holomorphically on Δ.

The set of such small discs is invariant under CR automorphisms; recall that if F is a CR automorphism of M and f an analytic disc attached to M then the map $F \circ f$ defined on $\partial\Delta$ extends holomorphically to Δ (see Proposition 6.2.2 in [3] or Theorem 1 p. 200 in [17]). Moreover recall the following essential result due to Webster [59] in the hypersurface case and to Tumanov [56] for higher codimension submanifolds.

Proposition 1.6 ([56]). *Let M be a $\mathcal{C}^4$ generic real submanifold of $\mathbb{C}^N$ of codimension d given by (1.1). Then M satisfies* $(\mathfrak{t})$ *if and only the conormal bundle N^*M is totally real at $\left(0, \sum_{j=1}^d c_j \partial r_j(0)\right)$, where the constant $c_1, \dots, c_d$ are such that $\sum_{j=1}^d c_j A_j$ is invertible.*

We end this section with an important remark on the smoothness of stationary discs. Let $k \geq 2$ be an integer and let M be a $\mathcal{C}^{k+2}$ generic real submanifold of $\mathbb{C}^N$ of codimension d given by (1.1). Assume that M is satisfies $(\mathfrak{t})$. Consider a lift of stationary disc $\boldsymbol{f} = (f, \tilde{f})$ for M satisfying $\boldsymbol{f}(1) = (0, \sum_{j=1}^d c_j \partial r_j(0))$ where $\sum_{j=1}^d c_j A_j$ is invertible. It follows from Proposition 1.6 and from Chirka (Theorem 33 in [22]) that such discs are of class $\mathcal{C}^{k,\alpha}$ for any $0 < \alpha < 1$ near $\zeta = 1$.

1.3. **Partial indices and Maslov index.** The construction of stationary discs attached to a generic real submanifold of $\mathbb{C}^N$ relies on a nonlinear Riemann-Hilbert problem whose study is related to certain geometric integers, namely the partial indices and the Maslov index, associated to the linearized problem. We recall the definition of these integers. We denote by $Gl_N(\mathbb{C})$ the general linear group on $\mathbb{C}^N$. Let $G : \partial\Delta \to Gl_N(\mathbb{C})$ be a smooth map. We consider a Birkhoff factorization (see Section 3 [29] or [58]) of $-\overline{G^{-1}}G$ on $\partial\Delta$:

$$-\overline{G(\zeta)}^{-1} G(\zeta) = B^+(\zeta) \begin{pmatrix} \zeta^{\kappa_1} & & & (0) \\ & \zeta^{\kappa_2} & & \\ & & \ddots & \\ (0) & & & \zeta^{\kappa_N} \end{pmatrix} B^-(\zeta),$$

where $\zeta \in \partial\Delta$, $B^+ : \bar{\Delta} \to Gl_N(\mathbb{C})$ and $B^- : (\mathbb{C} \cup \infty) \setminus \Delta \to Gl_N(\mathbb{C})$ are smooth maps, holomorphic on Δ and $\mathbb{C} \setminus \overline{\Delta}$ respectively. The integers $\kappa_1, \dots, \kappa_N$ are called the *partial indices* of $-\overline{G^{-1}}G$ and their sum $\kappa := \sum_{j=1}^N \kappa_j$ the *Maslov index* of $-\overline{G^{-1}}G$. Recall that the Maslov index κ is equal to the winding number of the function

$$\zeta \mapsto \det\left(-\overline{G(\zeta)^{-1}}G(\zeta)\right)$$

at the origin ([30], see also Lemma B.1 [15] for a proof); here $\det\left(-\overline{G(\zeta)^{-1}}G(\zeta)\right)$ denotes the determinant of $-\overline{G(\zeta)^{-1}}G(\zeta)$.

2. Construction of stationary discs

The idea of attaching an analytic disc to a real submanifold is a boundary value problem, namely a nonlinear Riemann-Hilbert type problem. Given an initial discs attached to a model submanifold, our approach consists in studying the existence and the structure of solutions of such Riemann-Hilbert problems by perturbing both the initial disc and the model submanifold. This is achieved via the implicit function theorem and, in particular, a careful choice of the Banach spaces of functions involved. In the nondegenerate settings that study is based on the important works of Forstnerič [27] and of Globevnik [29, 30] on analytic discs attached to totally real submanifolds. In the framework of Levi degenerate hypersurfaces, the attachment of discs is much more complicated and relies on the study of a singular Riemann-Hilbert problem; the article [12] provides the relevant techniques adapted to this problem.

2.1. **The case of nondegenerate real submanifolds.** In this section, we discuss the construction of lifts of stationary discs attached to small perturbations of a quadric submanifold, and satisfying condition $(\mathfrak{t})$.

2.1.1. *The model case.* Consider a quadric submanifold $Q \subset \mathbb{C}^N = \mathbb{C}^n_z \times \mathbb{C}^d_w$ of real codimension d

$$\begin{cases} \rho_1 = \Re e w_1 - {}^t\bar{z} A_1 z = 0 \\ \quad \vdots \\ \rho_d = \Re e w_d - {}^t\bar{z} A_d z = 0 \end{cases} \tag{2.1}$$

where $A_1, \ldots, A_d$ are hermitian matrices of size n; we set $\rho := (\rho_1, \ldots, \rho_d)$. By a straightforward computation one obtains a special family of stationary discs $\boldsymbol{f} = (h, g, \tilde{h}, \tilde{g})$ for Q of the form

$$\boldsymbol{f} = \left((1-\zeta)V, 2(1-\zeta)\,{}^t\overline{V} A_1 V, \ldots, 2(1-\zeta)\,{}^t\overline{V} A_d V, (1-\zeta)\,{}^t\overline{V} A, \frac{\zeta}{2} c \right), \tag{2.2}$$

where $V \in \mathbb{C}^n$, $c_1, \ldots, c_d \in \mathbb{R}$, and $A := \sum_{j=1}^d c_j A_j$. We emphasize that the quadric Q is only supposed to be of the form (2.1).

2.1.2. *The Riemann-Hilbert problem.* Let $M = \{r = 0\}$, with $r = (r_1, \ldots, r_d)$, be a generic real submanifold of $\mathbb{C}^N$ of codimension d given by (1.1). The fibration $\mathcal{N}Q(\zeta)$, $\zeta \in \partial\Delta$, can be defined by $2n + 2d$ real defining functions $\tilde{r} := (\tilde{r}_1, \ldots, \tilde{r}_{2n+2d})$. This allows to consider lifts of stationary discs as solutions of a nonlinear Riemann-Hilbert type problem. Indeed, an analytic disc $\boldsymbol{f} = (f, \tilde{f}) : \overline{\Delta} \to T^*\mathbb{C}^N$ is the lift of a stationary disc for $M = \{r = 0\}$ if and only if

$$\tilde{r}(\boldsymbol{f}) = 0 \quad \text{on } \partial\Delta. \tag{2.3}$$

The next section is devoted to the study of this problem.

2.1.3. *Construction of stationary discs.* The main result is

Theorem 2.1 ([10]). *Let $Q = \{\rho = 0\} \subset \mathbb{C}^{n+d}$ be a quadric submanifold of real codimension d of the form (2.1). Assume that Q satisfies $(\mathfrak{t})$ and consider an initial lift of a stationary disc, $\boldsymbol{f_0} = (h_0, g_0, \tilde{h}_0, \tilde{g}_0)$ of the form (2.2) where $c_1, \ldots, c_d$ are chosen such that the matrix $\sum_{j=1}^d c_j A_j$ is invertible. Then there exist open neighborhoods U of ρ in $(\mathcal{C}^4(\mathbb{B}))^d$ and V of 0 in $\mathbb{R}^{4n+4d}$, a real number $\varepsilon > 0$ and a map*

$$\mathcal{F} : U \times V \to \left(\mathcal{A}^{1,\alpha}\right)^{2n+2d}$$

of class $\mathcal{C}^1$ such that:

i. $\mathcal{F}(\rho, 0) = \boldsymbol{f_0}$,
ii. for all $r \in U$, the map

$$\mathcal{F}(r, \cdot) : V \to \{\boldsymbol{f} \in \mathcal{S}(\{r = 0\}) \mid \|\boldsymbol{f} - \boldsymbol{f_0}\|_{1,\alpha} < \varepsilon\}$$

is one-to-one and onto.

In particular,

$$\{\boldsymbol{f} \in \mathcal{S}(\{r = 0\}) \mid \|\boldsymbol{f} - \boldsymbol{f_0}\|_{1,\alpha} < \varepsilon\}$$

forms a $\mathcal{C}^1$ real submanifold of dimension $4n + 4d$ of $\left(\mathcal{A}^{1,\alpha}\right)^{2n+2d}$.

Note that the dimension of the above submanifold depends only on the dimension of the ambient space and not on the codimension; this point is related with the fact that the dimension of the conormal bundle of M does not depend on the codimension of M. Theorem 2.1 is proved in [9] in the case of codimension $d = 1$.

Remark 2.2. Working with the Banach spaces $\mathcal{C}^4(\mathbb{B})$ and $\mathcal{A}^{1,\alpha}$ is crucial for our approach which is based on the implicit function theorem. The required smoothness is indeed necessary for the below map F (see (2.4)) to be of class $\mathcal{C}^1$.

Remark 2.3. In [52], Sukhov and Tumanov proved Theorem 2.1 in case the model quadric is $\mathbb{S}^3 \times \mathbb{S}^3 \subset \mathbb{C}^4$, where $\mathbb{S}^3$ denotes the unit sphere in $\mathbb{C}^2$ (see Corollary 3.2 and Theorem 3.4 [52]). Their approach also relies on the study of the corresponding Riemann-Hilbert problem using the methods developed by [27, 29, 30].

Remark 2.4. In the context of generating strictly pseudoconvex submanifolds, the analogous of Theorem 2.1 was proved by Tumanov [56], based on a rather different approach, namely the Bishop equation (see Theorem 5.1 [56]); it is important for his approach that the submanifold is generating strictly pseudoconvex submanifolds, that is, satsifies ($\mathfrak{a}$) and ($\mathfrak{t}^+$). We emphasize that Theorem 2.1 only requires the model to satisfy ($\mathfrak{t}$); in particular, note that there is no codimension restriction in Theorem 2.1.

Proof of Theorem 2.1. The proof of Theorem 2.1 relies on the implicit function theorem and is inspired by the works of Forstnerič [27] and of Globevnik [29, 30]. In a neighborhood of $(\rho, \boldsymbol{f_0})$ in $(\mathcal{C}^4(\mathbb{B}))^d \times \left(\mathcal{A}^{1,\alpha}\right)^{2n+2d}$, Consider the following map between Banach spaces

$$F : (\mathcal{C}^4(\mathbb{B}))^d \times \left(\mathcal{A}^{1,\alpha}\right)^{2n+2d} \to (\mathcal{C}^{1,\alpha})^{2n+2d}$$

by

$$F(r, \boldsymbol{f}) := \tilde{r}(\boldsymbol{f}). \tag{2.4}$$

Here we use the notation

$$\tilde{r}(\boldsymbol{f})(\zeta) = \tilde{r}(\zeta)(\boldsymbol{f}(\zeta)),\ \zeta \in \partial\Delta.$$

The map F is of class $\mathcal{C}^1$ (see Lemma 5.1 in [33] and Lemma 6.1 and Lemma 11.2 in [29] which generalizes to $\mathcal{C}^{1,\alpha}$). Recall that an analytic disc $\boldsymbol{f} \in \left(\mathcal{A}^{1,\alpha}\right)^{2n+2d}$ is the lift of a stationary disc for $\{r = 0\}$ if and only if it solves the nonlinear Riemann-Hilbert problem (2.3). In other words, for any fixed $r \in (\mathcal{C}^4(\mathbb{B}))^d$, the zero set of $F(r, \cdot)$ coincides with $\mathcal{S}(\{r = 0\})$. We wish to apply the implicit function theorem to the map F. We need to consider the partial derivative of F with respect to $\left(\mathcal{A}^{1,\alpha}\right)^{2n+2d}$ at $(\rho, \boldsymbol{f_0})$, namely

$$\partial_2 F(\rho, \boldsymbol{f_0})\boldsymbol{f} = 2\Re e\left[\overline{G(\zeta)}\boldsymbol{f}\right]$$

where $G(\zeta)$ is the following complex valued square matrix of size $2n + 2d$

$$G(\zeta) := \left(\tilde{\rho}_{\overline{z}}(\boldsymbol{f_0}), \tilde{\rho}_{\overline{w}}(\boldsymbol{f_0}), \tilde{\rho}_{\overline{\tilde{z}}}(\boldsymbol{f_0}), \tilde{\rho}_{\overline{\tilde{w}}}(\boldsymbol{f_0})\right). \tag{2.5}$$

Recall that condition ($\mathfrak{t}$) is equivalent to the fact that the conormal bundle is totally real (see Proposition 1.6). Due to the choice of the initial disc, this ensures that the matrix $G(\zeta)$ is invertible for all $\zeta \in \partial\Delta$; this point is proven below. We need to show that (see p. 39 [30])

i. the map $\partial_2 F(\rho, \boldsymbol{f_0})$ is onto, and
ii. the real dimension of the kernel of $\partial_2 F(\rho, \boldsymbol{f_0})$ is $4n + 4d$.

Surjectivity of $\partial_2 F(\rho, \mathbf{f_0})$. It is more convenient to reorder coordinates and consider $(w, z, \tilde{z}, \tilde{w})$ instead of $(z, w, \tilde{z}, \tilde{w})$. Accordingly, discs $\boldsymbol{f}$ are of the form $(g, h, \tilde{h}, \tilde{g})$. We still denote by $G(\zeta)$ the corresponding reordered matrix, namely

$$G(\zeta) := \left(\tilde{\rho}_{\overline{w}}(\boldsymbol{f_0}), \tilde{\rho}_{\overline{z}}(\boldsymbol{f_0}), \tilde{\rho}_{\overline{\tilde{z}}}(\boldsymbol{f_0}), \tilde{\rho}_{\overline{\tilde{w}}}(\boldsymbol{f_0})\right).$$

The matrix $G(\zeta)$ is square of size $2n+2d$, upper block triangular and a direct computation gives

$$G(\zeta) = \begin{pmatrix} \frac{1}{2}I_d & & (*) \\ & G_2(\zeta) & \\ (0) & & -i\zeta I_d \end{pmatrix},$$

where I_d denotes the identity matrix of size d and $G_2(\zeta)$ is the following square matrix of size $2n$

$$G_2 = \begin{pmatrix} 2\sum_{j=1}^{d}\tilde{g}_j(A_j)_{11} & \dots & 2\sum_{j=1}^{d}\tilde{g}_j(A_j)_{n1} & 1 & & 0 \\ \vdots & \vdots & \vdots & & \ddots & \\ 2\sum_{j=1}^{d}\tilde{g}_j(A_j)_{1n} & \dots & 2\sum_{j=1}^{d}\tilde{g}_j(A_j)_{nn} & 0 & & 1 \\ 2i\sum_{j=1}^{d}\tilde{g}_j(A_j)_{11} & \dots & 2i\sum_{j=1}^{d}\tilde{g}_j(A_j)_{n1} & -i & & 0 \\ \vdots & \vdots & \vdots & & \ddots & \\ 2i\sum_{j=1}^{d}\tilde{g}_j(A_j)_{1n} & \dots & 2\sum_{j=1}^{d}\tilde{g}_j(A_j)_{nn} & 0 & & -i \end{pmatrix}.$$

where $\tilde{g}_j = c_j/2\zeta$ and $(A_j)_{kl}$, $k=1,\ldots,n$, $l=1,\ldots,n$, denotes the kl coefficient of A_j. Recall that $c_1,\ldots,c_d \in \mathbb{R}$ are chosen such that $A := \sum_{j=1}^{d} c_j A_j$ is invertible. It follows that

$$G_2(\zeta) = \begin{pmatrix} \zeta\, {}^tA & I_n \\ i\zeta\, {}^tA & -iI_n \end{pmatrix}$$

is invertible on $\partial\Delta$, and that, accordingly, so is $G(\zeta)$. Due to the expression of $G(\zeta)$, in order to show its surjectivity, it is enough to show that the map

$$L_1 : \left(\mathcal{A}^{1,\alpha}\right)^{2n} \to (\mathcal{C}^{1,\alpha})^{2n}$$

defined by $L_1 = 2\Re e\left[\overline{G_2(\zeta)}\,\cdot\right]$ is surjective. For this purpose, we will show that the partial indices $k_1,\ldots,k_{2n}$ of $-\overline{G_2^{-1}}G_2$ are nonnegative (see [29] or Section 4 in [30]). Right multiplication by the constant matrix $\begin{pmatrix} {}^tA^{-1} & 0 \\ 0 & I_n \end{pmatrix}$ does not change the partial indices, and gives us the matrix

$$\begin{pmatrix} \zeta I_n & I_n \\ i\zeta I_n & -iI_n \end{pmatrix}$$

After permuting rows and columns, which also does not change the partial indices, we obtain

$$G_2^\flat = \begin{pmatrix} R & & 0 \\ & \ddots & \\ 0 & & R \end{pmatrix}, \text{ with } R(\zeta) = \begin{pmatrix} \zeta & 1 \\ i\zeta & -i \end{pmatrix}.$$

By a direct computation we have

$$-\overline{(G_2^\flat)^{-1}}G_2^\flat = \begin{pmatrix} P & & 0 \\ & \ddots & \\ 0 & & P \end{pmatrix} \text{ with } P(\zeta) = -\begin{pmatrix} 0 & \zeta \\ \zeta & 0 \end{pmatrix},$$

which, for instance, decomposes as

$$P(\zeta) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that the partial indices of $-\overline{(G_2^\flat)^{-1}}G_2^\flat$ are all equal to one and that, therefore, the map $\partial_2 F(\rho, \boldsymbol{f_0})$ is onto.

Kernel of $\partial_2 F(\rho, \mathbf{f_0})$. Recall (see [29] or Section 5 in [30]) that the real dimension of the kernel of $\partial_2 F(\rho, \boldsymbol{f_0})$ is given by $\kappa + 2n + 2d$, where κ is the Maslov index of $-\overline{G(\zeta)^{-1}}G(\zeta)$ and is equal to the winding number of the function $\zeta \mapsto \det\left(-\overline{G(\zeta)^{-1}}G(\zeta)\right)$ at the origin. Using the form of $G(\zeta)$ and in particular $G_2(\zeta)$, a direct computation shows that $\kappa = 2n + 2d$.

□

2.2. **The case of degenerate real hypersurfaces.** In this section, we discuss the construction of stationary discs attached to Levi degenerate real hypersurfaces of $\mathbb{C}^{n+1} = \mathbb{C}^n_z \times \mathbb{C}_w$. However, in this context, it is not clear whether or not one can ensure the existence of smooth stationary discs, and not much seems to be known in that direction. Note that in such case the corresponding conormal bundle is no longer totally real and the method described in Section 2.1.3 falls apart. Recall that stationary discs are holomorphic discs attached to a given hypersurface, admitting a meromorphic lift to the cotangent bundle with at most one pole of order one at the origin and attached to the conormal bundle; surprisingly, when one allows the pole to be of greater order, there might exist many such discs, which still form a biholomorphically invariant family. This point led to the introduction in [11] of the notion of k_0-stationary discs (see Definition 1.4). A first observation is that one can attach to any polynomial rigid model hypersurface $\{-\Re e w + P(z, \overline{z}) = 0\}$ many k_0-stationary discs; with k_0 carefully chosen, depending on the highest power of $\overline{z}$ in the polynomial $P(z, \overline{z})$. The important question is to construct k_0-stationary discs by perturbing both a model hypersurface and an initial disc in a similar manner as in Section 2.1.3; unfortunately not any model hypersurface will work for our approach. The next section is devoted to understand which model hypersurface to consider.

2.2.1. *Admissible hypersurfaces.* A polynomial $P : \mathbb{C}^n \to \mathbb{R}$ is *weighted homogeneous of weight* $M = (m_1, \cdots, m_n) \in \mathbb{N}^n$ *and degree* $d \in \mathbb{N}$ if for any $t \in \mathbb{R}$ and $z \in \mathbb{C}^n$ we have

$$P(t^{m_1} z_1, \cdots, t^{m_n} z_n, t^{m_1} \bar{z}_1, \cdots, t^{m_n} \bar{z}_n) = t^d P(z, \bar{z}).$$

Using the notation $t^M z = (t^{m_1} z_1, \cdots, t^{m_n} z_n)$, the condition can be written as $P(t^M z, t^M \bar{z}) = t^d P(z, \bar{z})$. For our approach it is more convenient to assume that $m_1, \cdots, m_n$ are all even; this is not a problem since the size of the weights $(m_1, \ldots, m_n)$ is often not important. Consider a real-valued, weighted homogeneous of weight $M = (m_1, \cdots, m_n)$ polynomial P of degree d. We write

$$P(z, \bar{z}) = \sum_{\substack{M \cdot J + M \cdot K = d \\ d - k_0 \leq M \cdot J \leq k_0}} \alpha_{JK} z^J \overline{z}^K \tag{2.6}$$

where k_0 is the largest k with $\dfrac{d}{2} \leq k \leq d - 1$ for which there exists two multi-indices $\tilde{J}, \tilde{K}$ with $M \cdot \tilde{K} = k$ satisfying $\alpha_{\tilde{J}\tilde{K}} \neq 0$. Here, for two multi-indices $M = (m_1, \cdots, m_n)$ and $J = (j_1, \cdots, j_n)$, we set $M \cdot J = \sum_{i=1}^n m_i j_i$. Moreover, since P is real, $\alpha_{JK} = \overline{\alpha}_{KJ}$ for all multi-indices J, K. We define the *model hypersurface* $S_P = \{\rho = 0\} \subset \mathbb{C}^{n+1}$ where

$$\rho(z, w) = -\Re e w + P(z, \overline{z}). \tag{2.7}$$

Define for $v = (v_1, \cdots, v_n) \in \mathbb{C}^n$ the analytic disc $h^v : \Delta \to \mathbb{C}^n$

$$h^v(\zeta) = (1 - \zeta)^M v = ((1 - \zeta)^{m_1} v_1, (1 - \zeta)^{m_2} v_2, \ldots, (1 - \zeta)^{m_n} v_n).$$

We will need to control the Levi form of S_P along the boundary of h^v,

$$P_{z\overline{z}}(h^v(\zeta),\overline{h^v(\zeta)}) = \begin{pmatrix} P_{z_1\overline{z}_1}(h^v(\zeta),\overline{h^v(\zeta)}) & \cdots & P_{z_1\overline{z}_n}(h^v(\zeta),\overline{h^v(\zeta)}) \\ \vdots & \ddots & \vdots \\ P_{z_n\overline{z}_1}(h^v(\zeta),\overline{h^v(\zeta)}) & \cdots & P_{z_n\overline{z}_n}(h^v(\zeta),\overline{h^v(\zeta)}) \end{pmatrix}.$$

For $\zeta \in \partial\Delta$ we can write

$$\begin{cases} \zeta^{k_0} P_{z_i\overline{z}_j}(h^v(\zeta),\overline{h^v(\zeta)}) = (1-\zeta)^{d-m_i-m_j} Q^v_{i\overline{j}}(\zeta) \\ \\ \zeta^{k_0} P_{z_i z_j}(h^v(\zeta),\overline{h^v(\zeta)}) = (1-\zeta)^{d-m_i-m_j} S^v_{ij}(\zeta) \end{cases}$$

where $Q^v_{i\overline{j}}$ and S^v_{ij} are holomorphic polynomials, and where each $Q^v_{i\overline{j}}$ has degree at most $2k_0 - d + m_j$ and each S^v_{ij} has degree at most $2k_0 - d$. Our assumption is now that not only does h^v only pass through Levi nondegenerate points for $\zeta \neq 1$, but also, that the Levi form of S_P along h^v has the generic order of vanishing at 1 (so that the order of vanishing of the Levi form stays constant under small perturbations of both P and v). More precisely

Definition 2.5 ([13])**.** We say that v is *admissible for* P if there exists an analytic disc $g^v : \overline{\Delta} \to \mathbb{C}$ such that for $f^v = (h^v, g^v)$ we have that $f^v(\partial\Delta) \subset S_P$, $f^v(\Delta) \not\subset S_P$, and for $\zeta \in \partial\Delta$

$$Q^v(\zeta) = \det \begin{pmatrix} Q_{1\overline{1}}(\zeta) & \dots & Q_{1\overline{n}}(\zeta) \\ \vdots & \ddots & \vdots \\ Q_{n\overline{1}}(\zeta) & \dots & Q_{n\overline{n}}(\zeta) \end{pmatrix} \neq 0.$$

Remark 2.6. Note that under generic conditions, we do find admissible vectors. Indeed if S_P is generically Levi nondegenerate and if the set of Levi-degenerate points $\Sigma_P = \{(z,w) \in S_P \colon \det P_{z_i\overline{z}_j}(z,\overline{z}) = 0\}$ does not have any branches of dimension $2n-1$ near 0, then one can prove that there exists an admissible vector v for P.

Finally, we define

Definition 2.7 ([13])**.** A model hypersurface $S_P = \{\rho = 0\} \subset \mathbb{C}^{n+1}$ with ρ given by (2.7) is *admissible* if P has an admissible vector.

Remark 2.8. In complex dimension two, a model hypersurface $\{-\Re e w + P(z,\overline{z}) = 0\}$ is admissible if the zero locus of the laplacian of the homogeneous polynomial P is trivial, namely

$$\{z \in \mathbb{C} \mid P_{z\overline{z}}(z,\overline{z}) = 0\} = \{0\}.$$

Example 2.9. *For $t \geq 0$, the model $\{-\Re e w + |z|^4 + t\Re e(z^3\overline{z})\}$ is admissible if and only if $0 \leq t < \frac{2}{3}$. The model $\{-\Re e w + \Re e(z^3\overline{z})\}$ is not admissible.*

2.2.2. *Construction of stationary discs.* In order to construct stationary discs by perturbing both an admissible model hypersurface and an initial disc it is important to focus on discs passing through the degeneracy locus of S_P. Indeed, one cannot expect the family of lifts of stationary discs passing through the degeneracy locus of S_P or avoiding it (for which the method developed in Section 2.1.3 is well adapted) to be of the same dimension. To this end we introduce spaces of discs with prescribed pointwise constraints. Let $k \geq 0$ be an integer and let $0 < \alpha < 1$. For a positive integer m, we denote by $\mathcal{A}^{k,\alpha}_{0^m}$ the subspace of $\mathcal{C}^{k,\alpha}_{\mathbb{C}}$ of functions of the form $(1-\zeta)^m f$, with $f \in \mathcal{A}^{k,\alpha}$. We equip $\mathcal{A}^{k,\alpha}_{0^m}$ with the norm

$$\|(1-\zeta)^m f\|_{\mathcal{A}^{1,\alpha}_{0^m}} = \|f\|_{\mathcal{C}^{k,\alpha}_{\mathbb{C}}} \tag{2.8}$$

which makes it a Banach space, isomorphic to $\mathcal{A}^{k,\alpha}$. We also denote by $\mathcal{C}^{k,\alpha}_{0^m}$ the subspace of $\mathcal{C}^{k,\alpha}$ of functions of the form $(1-\zeta)^m v$ with $v \in \mathcal{C}^{k,\alpha}_{\mathbb{C}}$. The space $\mathcal{C}^{k,\alpha}_{0^m}$ is equipped with the norm

$$\|(1-\zeta)^m f\|_{\mathcal{C}^{k,\alpha}_{0^m}} = \|f\|_{\mathcal{C}^{k,\alpha}_{\mathbb{C}}}$$

and is also a Banach space.

Consider now an admissible model hypersurface $S_P = \{\rho = 0\} \subset \mathbb{C}^{n+1}$ with ρ given by (2.7); that is, with P of the form (2.6) with weight $M = (m_1, \cdots, m_n)$ and degree d. Let $v = (v_1, \cdots, v_n)$ be an admissible vector for P and let f^0 be an initial k_0-stationary disc attached to S_P of the form

$$f^0 = (h^v, g^0) = ((1-\zeta)^{m_1} v_1, \ldots, (1-\zeta)^{m_n} v_n), g^0)$$

where g^0 is determined by $\Re e g^0 = P(h^v, \overline{h^v})$ on $\partial\Delta$, with lift

$$\boldsymbol{f^0} = (h^v, g^0, \tilde{h}^0, \tilde{g}^0) = (h^v, g^0, \zeta^{k_0} P_z(h^v, \overline{h^v}), -\zeta^{k_0}/2).$$

We introduce the product space

$$Y^{M,d} := \prod_{i=1}^{n} \left(\mathcal{A}^{k,\alpha}_{0^{m_i}}\right) \times \mathcal{A}^{k,\alpha}_0 \times \prod_{i=1}^{n} \left(\mathcal{A}^{k,\alpha}_{0^{d-m_i}}\right) \times \mathcal{A}^{k,\alpha}$$

endowed with the product norm (2.8). We denote by $\mathcal{S}^{k_0,r}$ the set of lifts $\boldsymbol{f} \in Y^{M,d}$ of k_0-stationary discs for a given real hypersurface $S = \{r = 0\}$; note that $\boldsymbol{f^0} \in \mathcal{S}^{k_0,\rho}$. The main theorem regarding the construction of k_0-stationary discs is

Theorem 2.10 ([13]). *Under the above assumptions, there exist an integer N, open neighborhoods U of ρ in X (see the remark below) and V of 0 in $\mathbb{R}^N$, a real number $\varepsilon > 0$ and a map*

$$\mathcal{F} : U \times V \to Y^{M,d}$$

of class $\mathcal{C}^1$ such that:

i. $\mathcal{F}(\rho, 0) = \boldsymbol{f^0}$,

ii. for all $r \in U$ the map

$$\mathcal{F}(r, \cdot) : V \to \{\boldsymbol{f} \in \mathcal{S}^{k_0,r} \mid \|\boldsymbol{f} - \boldsymbol{f^0}\|_{Y^{M,d}} < \varepsilon\}$$

is one-to-one and onto.

Remark 2.11. The Banach space X should be thought as the set of allowable perturbation M of S_P. Its definition and specially its norm is rather technical. More importantly, being an allowable perturbation is an independent condition with respect to smooth (enough) CR diffeomorphisms whose linear parts preserve weights. Hence the definition of allowable deformation actually gives rise to a well-defined class of real hypersurfaces, independent of the coordinates used. Details can be found in [13]. For instance, in complex dimension two, we consider small perturbations of $\{-\Re e w + P(z, \overline{z}) = 0\}$, where P is homogeneous of degree d, of the form

$$\{-\Re e w + P(z, \overline{z}) + O\left(|z|^{d+1}\right) + \Im m w \; O\left(|z, \operatorname{Im} w|^{d-1}\right) = 0\}.$$

The proof of Theorem 2.10 is also based on the implicit function theorem and follows the strategy of the proof of Theorem 2.1. However, in the present case, the conormal bundle is no longer totally real and thus, the corresponding Riemann-Hilbert problem is singular. Indeed, the initial stationary disc f^0 passes through the degeneracy locus of the initial model S_P at

$\zeta = 1$. It follows that the matrix map $\zeta \mapsto G(\zeta)$ (see (2.5)) is no longer invertible-valued; more precisely, $G(\zeta)$ is invertible for all $\zeta \in \partial\Delta \setminus \{1\}$ but $G(1)$ is not invertible. Our approach is, roughly speaking, to "factorize the non-singular part" $\widetilde{G}$ of G and to study a non-singluar Riemann-Hilbert problem – with $\widetilde{G}$ in place of G – which is, however, defined between different spaces. In such case, standard techniques developed in [27, 29, 30] cannot be applied directly and versions of Riemann-Hilbert problems with pointwise constraints are needed. To the best of our knowledge, relevant results are not covered in the vast literature on Riemann-Hilbert problems. The paper [12] provides the tools, such as index formulas (see Theorem 2.1 and Theorem 2.4 [12]) required for the construction of stationary discs with pointwise constraints.

Remark 2.12. In complex dimension two, Theorem 2.10 was proved in [11] with a rather different approach, based on functional analysis and involves Toeplitz, Hankel and Fredholm operators. As opposed to figuring out which holomorphic discs are attached to a given real submanifold (Riemann-Hilbert problem), the paper [11] deals with circles on a given real submanifold which extends holomorphically to the unit disc. The properties of the associated Fredholm indices such as their invariance under homotopy, ensure the existence of nearby small k_0-stationary discs attached to perturbed hypersuface, and the number of real variables parametrizing the perturbed discs is completely determined by those indices. The approach used in [11] does not seem to be well suited for higher dimension and, in fact, the method based on the study of a singular Riemann-Hilbert problem developed later on in [13] is also much more satisfactory from a geometric viewpoint.

2.3. **Geometric properties of stationary discs.** It is important for applications to mapping problems to study geometric properties of the manifold of stationary discs constructed in Theorem 2.1 and Theorem 2.10.

We notice first that since this manifold is of finite dimension, such discs are determined by a finite jet at $\zeta = 1$, in the sense that there is a positive integer k such that the jet map $\mathfrak{j}_k$

$$\boldsymbol{f} \mapsto \mathfrak{j}_k(\boldsymbol{f}) = \left(\boldsymbol{f}(1), \frac{\partial \boldsymbol{f}}{\partial \zeta}(1), \ldots, \frac{\partial^{(k)} \boldsymbol{f}}{\partial^{(k)} \zeta}(1)\right)$$

is injective (see Lemma 5.3 [13] or the Appendix in [14]). Naturally, one seeks the smallest such integer k and in order to achieve it, it is important to restrict the jet map to a smaller family of discs. It is also important to establish some filling properties of such discs. In the context of the papers [9, 11, 13, 10], centers of stationary discs fill an open set. This is achieved by showing that the map $\boldsymbol{f} \mapsto f(0)$ restricted to a smaller family of discs is a diffeomorphism onto its image; for dimensional reasons, it is also essential to impose pointwise constraints on stationary discs. For instance, in case of a fully nondegenerate real submanifold

Proposition 2.13 ([10]). *Let $Q \subset \mathbb{C}^{n+d}$ be a quadric submanifold of real codimension d given by (2.1), fully nondegenerate at* 0. *Consider an initial disc $\boldsymbol{f_0}$ of the form (2.2) where V is given by* (f) *and $c = (c_1, \ldots, c_d)$ is such that the matrix $\sum_{j=1}^d c_j A_j$ is invertible. Then there exist an open neighborhood U of ρ in $(\mathcal{C}^4(\mathbb{B}))^d$ and $\varepsilon > 0$ such that for all $r \in U$:*

i. The map $\mathfrak{j}_1$ is injective on the $2n+2d$ submanifold $\{\boldsymbol{f} \in \mathcal{S}_0(\{r=0\}) \mid \|\boldsymbol{f} - \boldsymbol{f_0}\|_{\mathcal{A}_0^{1,\alpha}} < \varepsilon\}$.

ii. The set $\{f(0) \mid \boldsymbol{f} \in \mathcal{S}_0(\{r=0\}),\ \|\boldsymbol{f} - \boldsymbol{f_0}\|_{\mathcal{A}_0^{1,\alpha}} < \varepsilon\}$ contains an open set O of $\mathbb{C}^{n+d}$. Moreover, for any $q \in O$ there exists an unique lift of stationary disc $\boldsymbol{f} = (f, \tilde{f})$ such that $f(0) = q$.

Remark 2.14. The set $\mathcal{S}_0(\{r=0\})$ corresponds to lifts of stationary discs tied to the origin. More precisely, we define the affine space $\mathcal{A}$ to be the subset of $(\mathcal{A}^{1,\alpha})^N$ of discs of the form

$$\left((1-\zeta)h,(1-\zeta)g,(1-\zeta)\tilde{h},(1-\zeta)\tilde{g}+\frac{\zeta}{2}c\right),$$

and we set

$$\mathcal{S}_0(\{r=0\}):=\mathcal{S}(\{r=0\})\cap\mathcal{A}.$$

Notice that $\boldsymbol{f_0}\in\mathcal{S}_0(\{r=0\})$. Recall also that the norm on $\mathcal{A}_0^{1,\alpha}$ has been defined in (2.8).

In [9], Proposition 2.13 is obtained for small deformations of a given Levi nondegenerate hyperquadric; it is important for that approach to have an explicit description of all stationary discs attached to the model hypersurface. However, in the higher codimensional setting, it is unclear at the moment how to obtain an explicit description of all stationary discs attached to the model quadric in general, and therefore, it is more subtle to prove that lifts of stationary discs are determined by their 1-jet at 1 and that the center of stationary discs fill an open set. We emphasize that the full nondegeneracy condition is natural in the context of Proposition 2.13 since part *ii.* is proved by showing that the evaluation map $\boldsymbol{f}\mapsto f(0)$ from $T_{\boldsymbol{f_0}}\mathcal{S}_0(Q)$ to $\mathbb{C}^N$ is injective; in case Q satisfies $(\mathfrak{t})$, this occurs if and only if Q is fully nondegenerate.

In the framework of the recent papers [57, 14], it seems rather unclear how to prove that centers of stationary discs fill an open set. Instead authors focus on boundaries of discs filling an open set in either the submanifold M (consequence of Proposition 2.2 in [50]) or its conormal bundle N^*M (see Proposition 3.4 in [14]). Surprisingly, authors have independently noticed that such filling property was tightly connected to the notion of defect of a stationary disc. This notion, which we recall now, was introduced by Tumanov [55], and Baouendi, Rothschild and Trépreau [7]. Following [7], we say that a stationary disc f is *defective* if it admits a lift $\boldsymbol{f}=(f,\tilde{f}):\Delta\to T^*\mathbb{C}^N$ such that $1/\zeta.\boldsymbol{f}=(f,\tilde{f}/\zeta)$ is holomorphic on Δ. The disc is called *nondefective* if it is not defective. In the model case (2.1), a stationary disc f for Q is defective if there exists $c=(c_1,\ldots,c_d)\in\mathbb{R}^d\setminus\{0\}$ such that the map $\zeta\mapsto c\partial_z\rho(f(\zeta))=\sum_{j=1}^d c_j\partial_z\rho_j(f(\zeta))$ defined on $\partial\Delta$ extends holomorphically on Δ; this relates with Tumanov's equivalent definition of the defect in [55]. A key observation in [14] is that condition $(\mathfrak{d})$ in Definition 1.3 is in fact equivalent to the existence of a nondefective stationary disc with lift of the form (2.2). The authors then show that, in that context, part *i.* of Proposition 2.13 on the injectivity of the map $\mathfrak{j}_1$ remains true whenever the initial disc $\boldsymbol{f_0}$ projects to a nondefective stationary disc; the filling property on the conormal bundle then follows from the injectivity of the map $\mathfrak{j}_1$. The paper [14] shows the clear limitations of the family of lifts of stationary discs of the form (2.2); roughly speaking, results in [14] are the strongest one can obtain in higher codimension if one wants to work exclusively with the family (2.2). Actually, this provides also a guideline to treat more general submanifolds (satisfying $(\mathfrak{t})$); one needs to find more general explicit stationary discs in the model case and find one nondefective disc among that family. We emphasize that, incidentally, earlier papers [11, 13, 10] also rely on this special family of discs (2.2). In the setting of generating strictly pseudoconvex submanifolds, Tumanov [56] gave an explicit description of the entire family of all stationary discs attached to a model quadric. Tumanov has long conjectured the existence of a nondefective stationary disc among that family; and he was able to settle the conjecture, which relies essentially on linear algebra, very recently in [57]. Geometric properties of discs for generating strictly pseudconvex submanifolds such as their 1-jet determination and their filling property were in fact obtained in [56, 50].

3. Jet determination of CR automorphisms

Stationary discs are particularly adapted to mapping problems, and in particular to jet determination problems of CR automorphisms. This approach was pioneered in [9] in case of Levi nondegenerate hypersurfaces of class $\mathcal{C}^4$ and consequently developed for more general real submanifolds in a series of papers [11, 13, 10, 57, 14]. Not only the use of such invariants provides a new geometric insight on jet determination problems, it is the only approach in the literature which allows for the consideration of finitely smooth real submanifolds.

Let $M \subset \mathbb{C}^N$ be a finitely smooth generic real submanifold and let $p \in M$. For a positive integer k, we denote by $Aut^k(M,p)$ the set of germs at p of CR automorphisms F of M of class $\mathcal{C}^k$; in particular $F(p) = p$ and $F(M) \subset M$. The main results are

Theorem 3.1 ([14]). *Let $M \subset \mathbb{C}^N$ be a $\mathcal{C}^4$ generic real submanifold. Assume that M is $\mathfrak{D}$-nondegenerate at $p \in M$. Then elements of $Aut^3(M,p)$ are uniquely determined by their 2-jet at p.*

Theorem 3.2 ([57]). *Let $M \subset \mathbb{C}^N$ be a $\mathcal{C}^4$ generic real submanifold. Assume that M is generating strictly pseudoconvex at $p \in M$. Then elements of $Aut^3(M,p)$ are uniquely determined by their 2-jet at p.*

Theorem 3.3 ([13]). *Let $S_P = \{\rho = 0\} \subset \mathbb{C}^{n+1}$ be an admissible model hypersurface with ρ given by (2.7). Then for any allowable perturbation M of S_P (see Remark 2.11) near $0 \in S_P$, there exists an integer ℓ such that elements of $Aut^\ell(M,0)$ are uniquely determined by their ℓ-jet at 0.*

Remark 3.4. Assume that a jet determination result of order k' holds in the formal setting, in the sense that every ℓ-jet of a formal biholomorphisms which preserves a formal hypersurface (up to the order ℓ) and is trivial up to order k' necessarily coincides with the ℓ-jet of the identity map. Then the conclusion of Theorem 3.3 holds for k'-jet determination as long as M is smooth enough. It follows for instance that, for the version of Theorem 3.3 in $\mathbb{C}^2$ (see Theorem 1.2 in [11]), one can always achieve 2-jet determination of CR diffeomorphisms as in the real analytic case (see [26, 39]). In higher dimension one can achieve the order of jet determination established in the formal setting, see for instance [37, 42] and for the model case [40].

Theorem 3.1 (and Theorem 3.2) were first proved in the hypersurface case in [9] and for fully nondegenerate submanifolds in [10]; Theorem 3.3 was first proved in complex dimension two in [11]. Theorem 3.2 is the strongest result to date for strictly pseudoconvex submanifolds, that is, satisfying ($\mathfrak{t}^+$), in higher codimension; on the other hand, the strict pseudoconvexity condition is quite restrictive. Although the $\mathfrak{D}$-nondegeneracy condition imposes a strong restriction on the codimension ($d \leq 2n$), Theorem 3.1 is, up to now, the strongest result for submanifolds satisfying ($\mathfrak{t}$) in higher codimension. The assumption made on the hypersurface in Theorem 3.3 is relatively technical and still too restrictive; for instance, Theorem 3.3 does not apply to (finitely smooth) perturbations of $\{-\Re e w + |z|^4 + \Re e(z^3\overline{z}) = 0\} \subset \mathbb{C}^2$; and the stationary disc method does not allow yet to treat such hypersurfaces.

The proofs of Theorem 3.1, Theorem 3.2, Theorem 3.3 and their analogue [9, 10, 11] are similar and relies only on the construction of stationary discs and their geometric properties mentioned in Section 2.3. We prove Theorem 3.1 when M is assumed to be fully nondegenerate.

Proof of Theorem 3.1. Let $M \subset \mathbb{C}^N$ be a $\mathcal{C}^4$ generic real submanifold, fully nondegenerate at $p \in M$. We assume that $p = 0$ and that M is locally given by $\{r = 0\}$ (1.1). We denote by Q the associated quadric part of M defined by $\{\rho = 0\}$ (2.1). Let $F \in Aut^3(M, 0)$ with a trivial 2-jet at 0. We wish to show that F is the identity.

Since M satisfies $(\mathfrak{f})$ at 0, there exists $V \in \mathbb{C}^n$ such that $\text{span}_{\mathbb{C}}\{A_1V, \ldots, A_dV\}$ is of dimension d. Consider an initial lift of stationary disc $\boldsymbol{f_0}$ of the form

$$\boldsymbol{f_0} = \left((1-\zeta)V, 2(1-\zeta)\,{}^t\overline{V}A_1V, \ldots, 2(1-\zeta)\,{}^t\overline{V}A_dV, (1-\zeta)\,{}^t\overline{V}A, \frac{\zeta}{2}c\right)$$

where $c_1, \ldots, c_d$ are chosen such that $\sum_{j=1}^d c_jA_j$ is invertible. Consider now the dilation $\Lambda_t : \mathbb{C}^{n+d} \to \mathbb{C}^{n+d}$ defined by

$$\Lambda_t(z, w) := (tz, t^2w).$$

We set $r_t := \frac{1}{t^2} r \circ \Lambda_t$ and $F_t := \Lambda_t^{-1} \circ F \circ \Lambda_t$. We also recall that for an analytic disc $\boldsymbol{f} = (f, \tilde{f}) \in \left(\mathcal{A}^{1,\alpha}\right)^{2n+2d}$ where $0 < \alpha < 1$, we have

$$(F_t)_*\boldsymbol{f}(\zeta) = \left(F_t \circ f(\zeta), \tilde{f}(\zeta)(d_{f(\zeta)}F_t)^{-1}\right)$$

for $\zeta \in \Delta$, and $F_t \circ f(\zeta)$ is well defined thanks to Proposition 6.2.2 [3].

Denote by U the neighborhood of ρ obtained in Proposition 2.13. For t small enough, one can show that the defining function $r_t = \dfrac{1}{t^2} r \circ \Lambda_t \in U$. Proposition 2.13 also provides an open set $O \subset \mathbb{C}^{n+d}$ such that

$$O \subset \{f(0) \mid \boldsymbol{f} \in \mathcal{S}_0(\{r_t = 0\}),\ \|\boldsymbol{f} - \boldsymbol{f_0}\|_{\mathcal{A}_0^{1,\alpha}} < \varepsilon/2\}.$$

We will show that F_t is equal to the identity on the open set O. Let $q \in O$ and let $\boldsymbol{f}$ be the unique lift of stationary disc in $\mathcal{S}_0(\{r_t = 0\})$ with $\|\boldsymbol{f} - \boldsymbol{f_0}\|_{\mathcal{A}_0^{1,\alpha}} < \varepsilon/2$ and such that $f(0) = q$. By invariance and since F_t has a trivial 2-jet, we have $(F_t)_*\boldsymbol{f} \in \mathcal{S}_0(\{r_t = 0\})$ and a technical computation shows that $\|(F_t)_*\boldsymbol{f} - \boldsymbol{f_0}\|_{\mathcal{A}_0^{1,\alpha}} < \varepsilon$ for t small enough. Moreover, the discs $(F_t)_*\boldsymbol{f}$ and $\boldsymbol{f}$ have the same 1-jet. By Proposition 2.13 we have $(F_t)_*\boldsymbol{f} = \boldsymbol{f}$ and therefore $F_t \circ f(0) = f(0)$, that is $F_t(q) = q$. This proves Theorem 3.1. □

References

[1] M.S. Baoudendi, P. Ebenfelt, L.P. Rothschild, *Parametrization of local biholomorphisms of real-analytic hypersurfaces*, Asian J. Math. **1** (1997), 1-16.

[2] M.S. Baoudendi, P. Ebenfelt, L.P. Rothschild, *CR automorphisms of real analytic CR manifolds in complex space*, Comm. Anal. Geom. **6** (1998), 291-315.

[3] M.S. Baouendi, P. Ebenfelt, L.P. Rothschild, *Real submanifolds in complex space and their mappings*, Princeton Mathematical Series, **47**. Princeton University Press, Princeton, NJ, 1999. xii+404 pp.

[4] M.S. Baouendi, P. Ebenfelt, L.P. Rothschild, *Local geometric properties of real submanifolds in complex space*, Bull. Amer. Math. Soc. (N.S.) **37** (2000), 309-336.

[5] M.S. Baoudendi, P. Ebenfelt, L.P. Rothschild, *Convergence and finite determination of formal CR mappings*, J. Amer. Math. Soc. **13** (2000), 697-723.

[6] M.S. Baouendi, N. Mir, L.P. Rothschild, *Reflection Ideals and mappings between generic submanifolds in complex space*, J. Geom. Anal. **12** (2002), 543-580.

[7] M.S. Baouendi, L.P. Rothschild, J.-M. Trépreau *On the geometry of analytic discs attached to real manifolds*, J. Differential Geom. **39** (1994), 379-405.

[8] V.K. Beloshapka, *Finite dimensionality of the group of automorphisms of a real-analytic surface*, Math. USSR Izvestiya **32** (1989), 239-242.

[9] F. Bertrand, L. Blanc-Centi, *Stationary holomorphic discs and finite jet determination problems*, Math. Ann. **358** (2014), 477-509.

[10] F. Bertrand, L. Blanc-Centi, F. Meylan *Stationary discs and finite jet determination for non-degenerate generic real submanifolds*, Adv. Math. **343** (2019), 910-934.

[11] F. Bertrand, G. Della Sala, *Stationary discs for smooth hypersurfaces of finite type and finite jet determination*, J. Geom. Anal. **25** (2015), 2516-2545.

[12] F. Bertrand, G. Della Sala, *Riemann-Hilbert problems with constraints*, Proc. Amer. Math. Soc. **147** (2019), 2123-2131.

[13] F. Bertrand, G. Della Sala, B. Lamel, *Jet determination of smooth CR automorphisms and generalized stationary discs*, preprint.

[14] F. Bertrand, F. Meylan, *Nondefective stationary discs and 2-jet determination in higher codimension*, arXiv:1912.10034.

[15] L. Blanc-Centi, *Stationary discs glued to a Levi non-degenerate hypersurface*, Trans. Amer. Math. Soc. **361** (2009), 3223-3239.

[16] L. Blanc-Centi, F. Meylan, *On nondegeneracy conditions for the Levi map in higher codimension: a survey*, preprint.

[17] A. Boggess, *CR Manifolds and the Tangential Cauchy-Riemann Complex.* CRC Press, 1991.

[18] E. Cartan, *Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes I*, Ann. Math. Pura Appl. **11** (1932), 17-90.

[19] E. Cartan, *Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes II*, Ann. Sc. Norm. Sup. Pisa **1** (1932), 333-354.

[20] H. Cartan, *Sur les groupes de transformations analytiques*, Act. Sc. et Int., Hermann, Paris (1935)

[21] S.S. Chern, J.K. Moser, *Real hypersurfaces in complex manifolds*, Acta math. **133** (1975), 219-271.

[22] E.M. Chirka, *Regularity of the boundaries of analytic sets*, Mat. Sb. **45** (1983), 291-336.

[23] B. Coupet, H. Gaussier, A. Sukhov, *Riemann maps in almost complex manifolds*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **2** (2003) 761-785.

[24] P. Ebenfelt, *Finite jet determination of holomorphic mappings at the boundary*, Asian J. Math. **5** (2001), 637-662.

[25] P. Ebenfelt, B. Lamel, *Finite jet determination of CR embeddings*, J. Geom. Anal. **14** (2004), 241-265.

[26] P. Ebenfelt, B. Lamel, D. Zaitsev, *Finite jet determination of local analytic CR automorphisms and their parametrization by 2-jets in the finite type case*, Geom. Funct. Anal. **13** (2003), 546-573.

[27] F. Forstnerič, *Analytic disks with boundaries in a maximal real submanifold of* $\mathbb{C}^2$, Ann. Inst. Fourier **37** (1987), 1-44.

[28] H. Gaussier, J.-C. Joo,*Extremal discs in almost complex spaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **9** (2010), 759-783.

[29] J. Globevnik, *Perturbation by analytic discs along maximal real submanifolds of* $\mathbb{C}^N$, Math. Z. **217** (1994), 287-316.

[30] J. Globevnik, *Perturbing analytic discs attached to maximal real submanifolds of* $\mathbb{C}^N$, Indag. Math. **7** (1996), 37-46.

[31] C.K. Han, *Analyticity of CR equivalences between real hypersurfaces in* $\mathbb{C}^n$ *with degenerate Levi form*, Invent. Math. **73** (1983), 51-69.

[32] C.K. Han, *Complete system for the mappings of CR manifolds of nondegenerate Levi forms*, Math. Ann. **309** (1997), 401-409.

[33] C.D. Hill, G. Taiani, *Families of analytic discs in* $\mathbb{C}^n$ *with boundaries on a prescribed CR submanifold*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **5** (1978), 327-380.

[34] X. Huang, *A preservation principle of extremal mappings near a strongly pseudoconvex point and its applications*, Illinois J. Math. **38** (1994), 283-302.

[35] X. Huang, *A non-degeneracy property of extremal mappings and iterates of holomorphic self-mappings*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **21** (1994), 399-419.

[36] R. Juhlin, *Determination of formal CR mappings by a finite jet*, Adv. Math. **222** (2009), 1611-1648.

[37] R. Juhlin, B. Lamel, *Automorphism groups of minimal real-analytic CR manifolds*, J. Eur. Math. Soc. (JEMS) **15** (2013), 509-537.

[38] S.-Y. Kim, D. Zaitsev, *Equivalence and embedding problems for CR-structures of any codimension*, Topology **44** (2005), 557-584.

[39] M. Kolář, F. Meylan, *Infinitesimal CR automorphisms of hypersurfaces of finite type in* $\mathbb{C}^2$, Arch. Math. (Brno) **47** (2011), 367-375.

[40] M. Kolář, F. Meylan, D. Zaitsev, *Chern-Moser operators and polynomial models in CR geometry*, Adv. Math. **263** (2014), 321-356.

[41] B. Lamel. *Jet embeddability of local automorphism groups of real-analytic CR manifolds*. Geometric analysis of several complex variables and related topics, 89-108, Contemp. Math., 550, Amer. Math. Soc., Providence, RI, 2011.
[42] B. Lamel, N. Mir, *Finite jet determination of CR mappings*, Adv. Math. **216** (2007), 153-177.
[43] L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France **109** (1981), 427-474.
[44] L. Lempert, *A precise result on the boundary regularity of biholomorphic mappings*, Math. Z. **193** (1986), 559-579. Erratum, Math. Z. **206** (1991), 501-504.
[45] L. Lempert, *Holomorphic invariants, normal forms, and the moduli space of convex domains*, Ann. of Math. (2) **128** (1988), 43-78.
[46] N. Mir, D. Zaitsev, *Unique jet determination and extension of germs of CR maps into spheres*, preprint.
[47] M.-Y. Pang, *Smoothness of the Kobayashi metric of nonconvex domains*, Internat. J. Math. **4** (1993), 953-987.
[48] M. Peyron, *Analyticité des applications CR dans des variétés presque complexes*, preprint, http://arxiv.org/abs/1205.3262
[49] E. Poletsky, *The Euler-Lagrange equations for extremal holomorphic mappings of the unit disk*, Michigan Math. J. **30** (1983), 317-333.
[50] A. Scalari, A. Tumanov, *Extremal discs and analytic continuation of product CR maps*, Michigan Math. J. **55** (2007), 25-33.
[51] A. Spiro, A. Sukhov, *An existence theorem for stationary discs in almost complex manifolds*, J. Math. Anal. Appl. **327** (2007), 269-286.
[52] A. Sukhov, A. Tumanov, *Stationary discs and geometry of CR manifolds of codimension two*, Internat. J. Math. **12** (2001), 877-890.
[53] N. Tanaka, *On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables*, J. Math. Soc. Japan **14** (1962), 397-429.
[54] N. Tanaka, *On generalized graded Lie algebras and geometric structures*, J. Math. Soc. Japan **19** (1967), 215-254.
[55] A. Tumanov, *Extension of CR-functions into a wedge from a manifold of finite type*, (Russian) Mat. Sb. (N.S.) **136 (178)** (1988), 128-139; translation in Math. USSR-Sb. **64** (1989), 129-140.
[56] A. Tumanov, *Extremal discs and the regularity of CR mappings in higher codimension*, Amer. J. Math. **123** (2001), 445-473.
[57] A. Tumanov, *Stationary discs and finite jet determination for CR mappings in higher codimension*, arXiv:1912.03782.
[58] N.P. Vekua, *Systems of singular integral equations*, Noordhoff, Groningen (1967) 216 pp.
[59] S. Webster, *On the reflection principle in several complex variables*, Proc. Amer. Math. Soc. **71** (1978), 26-28.
[60] D. Zaitsev, *Germs of local automorphisms of real analytic CR structures and analytic dependence on the k-jets*, Math. Res. Lett. **4** (1997), 1-20.
[61] D. Zaitsev, *Unique determination of local CR-maps by their jets: a survey*, Rend. Mat. Acc. Lincei **13** (2002), 135-145.

Florian Bertrand
Department of Mathematics,
American University of Beirut, Beirut, Lebanon
E-mail address: fb31@aub.edu.lb