EXTREMAL DISCS AND SEGRE VARIETIES FOR REAL-ANALYTIC HYPERSURFACES IN $\mathbb{C}^2$

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ABSTRACT. We show that if the Segre varieties of a strictly pseudoconvex hypersurface in $\mathbb{C}^2$ are extremal discs for the Kobayashi metric, then that hypersurface has to be locally spherical. In particular, this gives yet another characterization of the unit sphere in terms of two important invariant families of objects coinciding.

1. INTRODUCTION

There is a deep link between complex analysis in a smoothly bounded domain $\Omega \subset \mathbb{C}^N$ and the biholomorphically invariant (or short, CR) geometry of its boundary $b\Omega$. This link is known for strictly pseudoconvex smoothly bounded domains, where the relationship between one of the most important analytical objects associated to the domain, its Bergman Kernel function $K(z, \tilde{z})$, and the CR invariants of its boundary (computable through the Chern-Moser normal form) have been well investigated: The asymptotic expansion of the Bergman kernel can be recovered from boundary invariants (and vice versa), a line of research instigated by Fefferman's work on the biholomorphically invariant geometry of strictly pseudoconvex boundaries, for which we refer the reader to [1] and also to Hirachi's work [10].

In this paper, we will mostly deal with strictly pseudoconvex domains $\Omega \subset \mathbb{C}^2$, whose boundary $b\Omega =: M$ we also assume to be real-analytic. In that case, there are two important families of boundary invariants: First, the Chern-Moser normal form which gives rise to *chains*, i.e. families of biholomorphically invariant curves in M intrinsically defined (we recall the basics of this in subsection 2.1); and the Segre families of invariant complex curves defined near M. It is a theorem of Faran [9] that if the intersections of the Segre family with M and the chains agree, then M is locally biholomorphically equivalent to the unit sphere (Faran's result is valid in higher dimensions as well, but we will concentrate on $\mathbb{C}^2$ in this paper).

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There is yet another important family of invariant curves in a strictly pseudoconvex domain $\mathbb{C}^2$ which are associated to the Kobayashi pseudometric,

$$k_\Omega(z,v) := \inf\left\{a > 0\colon f \in \mathcal{H}(\Delta,\Omega), f(0) = z, af'(0) = v\right\},$$

where Δ denotes the unit disc in $\mathbb{C}$. The corresponding integrated pseudodistance, the *Kobayashi pseudodistance*, gives us, for settings in which it actually is a distance, a *biholomorphically invariant* distance notion, and with it, a natural hyperbolic geometry. A holomorphic disc $f\colon \Delta \to \Omega$ with $f(0) = z$ is said to be *extremal* for $(z,v) \in \Omega \times T_z\Omega$ if $f'(0)k_\Omega(z,v) = v$; extremal discs are natural biholomorphic invariants of a bounded domain just like the chains and Segre families discussed above. In the setting of the Kobayashi metric, things are a bit subtle: Even though extremal discs are proper and geodesics for the Kobayashi distance in strictly convex domains by the work of Lempert [16], the hyperbolic geometry of strictly pseudoconvex domains is more complicated; in particular, a general extremal disc may fail to be proper. However, work of Huang [11, 12] shows that for z sufficiently close to $p \in b\Omega$ and for v sufficiently close to the complex tangent space $T_p^c b\Omega$, extremal discs are again proper and complex geodesics. We will consider such extremals as yet another biholomorphically invariant family.

The relationship of the geometry of the boundary with the biholomorphically invariant hyperbolic geometry has been studied less, and in particular, the question answered by Faran about the Segre family and chains is open when asked about the Segre family and extremal discs. Our purpose in this paper is to settle this question (in $\mathbb{C}^2$). In order to state our theorem, let us write for a neighbourhood U of a point $p \in M$, where M is a strictly pseudoconvex real hypersurface, the decomposition $U = U_+ \cup (U \cap M) \cup U_-$ where U_+ lies on the pseudoconvex side of M.

Theorem 1.1. *Let $M \subset \mathbb{C}^2$ be a connected real-analytic hypersurface, $p \in M$. Assume that there exist open neighbourhoods $U, V \subset \mathbb{C}^2$ of p such that the Segre varieties $S_q \subset U$, $q \in V_-$ are defined and such that $S_q \cap U_+$ is an extremal disc for U_+ for every $q \in V_-$. Then M is umbilical at every strictly pseudoconvex point of $V \cap M$, and hence generically locally spherical.*

In particular, we also have the following characterization of the unit ball:

Corollary 1.2. *Let $\Omega \subset \mathbb{C}^2$ be a bounded, simply connected strictly pseudoconvex domain with connected real-analytic boundary, and assume that U is a neighbourhood of $b\Omega$ such that $S_q \cap \Omega$ is defined for all $q \in U \cap \bar{\Omega}^c$. If $S_q \cap \Omega$ is an extremal disc (for Ω) for every $q \in U \cap \bar{\Omega}^c$, then Ω is biholomorphic to the the unit ball $\mathbb{B}$.*

We point out that the strict pseudoconvexity of the domain is crucial as can be seen by considering the domain $\Omega = \{|z|^2 + |w|^4 < 1\} \subset \mathbb{C}^2$, where it can been shown using [13] that any Segre variety $S_q \cap \Omega$ for $q \in \overline{\Omega}^c$ near $b\Omega$ is an extremal disc. Finally, we note that the local version Theorem 1.1

is really stronger than Corollary 1.2, which follows from Theorem 1.1 after applying [8, Theorem C]. Indeed, one does not expect a locally spherical hypersurface to be globally CR equivalent to the sphere, by examples due to Burns and Shnider [4]; for these, by the localization theorem of Huang [12], (small) stationary discs are exactly the intersections of (small) Segre varieties with the hypersurface.

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2. Preliminaries

In this section, we recall some preliminaries which are needed later in the proof.

2.1. **The Chern-Moser normal form.** Here we introduce the basics of the Chern-Moser normal form for real-analytic (or formal) hypersurfaces in $\mathbb{C}^2$ which we need for our main argument; the normal form was introduced and its relation with the equivalence problem studied in [7]. Recall that a germ of a real-analytic hypersurface $(M,p)\subset(\mathbb{C}^2,p)$ is defined by the vanishing locus of a germ of a real-valued real analytic function $\varrho(\tilde z,\bar{\tilde z},\tilde w,\bar{\tilde w})\in\mathbb{C}\{\tilde z,\bar{\tilde z},\tilde w,\bar{\tilde w}\}$. Strict pseudoconvexity of M at p means that the bordered complex Hessian of ϱ satisfies

$$\begin{vmatrix} 0 & \varrho_{\tilde z}(p) & \varrho_{\tilde w}(p) \\ \varrho_{\bar{\tilde z}}(p) & \varrho_{\tilde z\bar{\tilde z}}(p) & \varrho_{\tilde w\bar{\tilde z}}(p) \\ \varrho_{\bar{\tilde w}}(p) & \varrho_{\tilde z\bar{\tilde w}}(p) & \varrho_{\tilde w\bar{\tilde w}}(p) \end{vmatrix} \neq 0.$$

The *model hypersurface* for strictly pseudoconvex hypersurfaces in $\mathbb{C}^2$ is the Heisenberg hypersurface $\mathbb{H}$: $\operatorname{Re} w=|z|^2$. The automorphisms of the Heisenberg hypersurface are given by linear fractional maps of the form

$$(z,w)\mapsto\left(\lambda\frac{z+aw}{1+2\bar a z+(|a|^2+it)w},|\lambda|^2\frac{w}{1+2\bar a z+(|a|^2+it)w}\right),$$

for $(\lambda,a,t)\in\mathbb{C}^*\times\mathbb{C}\times\mathbb{R}$.

If $H(z,w)=(f(z,w),g(z,w))$ is a linear fractional map of this form, H can be determined from $f_z(0)$, $f_w(0)$, and $\operatorname{Im} g_{ww}(0)$. Thus, the jet map $j_0^2\colon \operatorname{Aut}(\mathbb{H},0)\to G_0^2(\mathbb{C}^2)$, $H\mapsto j_0^2H$, is an injective homeomorphism onto its image Γ. The group $\operatorname{Aut}(\mathbb{H},0)$ is of maximal dimension amongst all automorphism groups of strictly pseudoconvex hypersurfaces in $\mathbb{C}^2$ and therefore gives the natural space for parameters of a normal form for the family of strictly pseudoconvex hypersurfaces under the action of the group of local biholomorphisms.

The *Chern-Moser normal form* gives, for any choice of a parameter $\gamma\in\Gamma$, a change of coordinates $(\tilde z,\tilde w)=H_M^p(z,w,\gamma)=(f(z,w,\gamma),g(z,w,\gamma))$, which is uniquely determined under the following conditions:

- in the new coordinates (z,w), p is the origin (i.e. $H_M^p(0,\gamma)=p$);
- the defining equation of M in the new coordinates has the form

$$\operatorname{Re} w=\varphi(z,\bar z,\operatorname{Im} w)=\sum_{j,k}\varphi_{j,k}(\operatorname{Im} w)z^j\bar z^k,$$

  where φ satisfies the *normalization conditions*

$$\begin{aligned} \varphi_{j,0}(t)&=0,\quad j\geq 0,\\ \varphi_{1,1}(t)&=1,\\ \varphi_{j,1}(t)&=0,\quad j\geq 2,\\ \varphi_{2,2}(t)=\varphi_{2,3}(t)=\varphi_{3,3}(t)&=0. \end{aligned}$$

One can in addition require that $j_0^2 H_M^p(\cdot,\gamma) = \gamma \cdot j_0^2 H_M^p(\cdot,\mathrm{id})$. The *chains* through p are the (parametrized) curves given by $t \mapsto H_M^p(0,t,\gamma)$.

The term $\varphi_{j,k}(t) z^j \bar{z}^k$ is said to be of *type* (j,k). The lowest order (non-trivial) invariant terms in the normal form are therefore the terms of type $(4,2)$ (and $(2,4)$), $\varphi_{4,2}$ and $\varphi_{2,4}$ respectively; they correspond to *Cartan's cubic tensor* from [5, 6]. The condition that $\varphi_{2,4}(0) = \varphi_{4,2}(0) = 0$ is invariant under different choices of γ, and if it is satisfied for one (and hence, all) $\gamma \in \Gamma$ we say that the point p is *umbilic*.

Umbilicity is quite different in higher dimensions, which is why we concentrate on the two-dimensional case here. Umbilicity at a point means that the order of approximation of a given strictly pseudoconvex with the model hypersurface is higher than generically expected, and the model hypersurface is the only one which is everywhere umbilical: If M is a strictly pseudoconvex real-analytic hypersurface, and if $p \in M$ has the property that it possesses a neighbourhood consisting of umbilical points, then there exists a neighbourhood $U \subset M$ of p which is biholomorphically equivalent to a piece of the model hypersurface. The same holds for smooth M if one replaces "biholomorphically equivalent" by "$\mathcal{C}^\infty$-CR equivalent", and the fact can simply be stated by saying that every umbilical hypersurface is locally spherical.

2.2. **Segre varieties.** Let $M \subset \mathbb{C}^N$ be a real-analytic hypersurface, defined locally at $p \in M$ by a real-analytic equation $\varrho(Z,\bar{Z}) = 0$. To be more precise, here we assume that $\varrho(Z,W)$ is holomorphic on $U \times U^* \subset \mathbb{C}^{2N}$, where $U^* = \{Z \colon \bar{Z} \in U\}$, and $\varrho_W(Z,W) \neq 0$ for $(Z,W) \in U \times U^*$. then one can define the Segre variety associated to the point q (in a suitable neighbourhood V of p) by

$$S_q = \{z \in U \colon \varrho(z,\bar{q}) = 0\}, \quad q \in V.$$

For good choices of U and V, for every $q \in V$, the variety $S_q \subset U$ is a connected, smooth complex hypersurface in U. One can check that for $p \in M$, we have that $T_p S_p = T_p^c M$. Actually, a bit more is true: given $p \in M$, and a real-analytic curve $\Gamma \subset M$ through p transverse to $T_p^c M$, one can choose coordinates $Z = (z_1,\dots,z_{N-1},w)$ near p such that in these coordinates, $p = 0$, and $\Gamma = \{(0,\dots,0,it)\}$ and for small s, we have that $S_{(0,s)} = \{w = -s\}$ (see e.g. [15, Lemma 4.1]).

The importance of Segre varieties is that they transform very nicely with respect to holomorphic maps: If H is a germ of a holomorphic map taking a real-analytic submanifold $M \subset \mathbb{C}^N$ into a real-analytic submanifold $M' \subset \mathbb{C}^{N'}$, then $H(S_p) \subset S'_{H(p)}$, where $S'_{q'}$ denotes the Segre variety of M' (associated to q').

2.3. **Stationary discs.** Let $M = \{\varrho = 0\}$ be a smooth hypersurface in $\mathbb{C}^N$. A disc $f : \overline{\Delta} \to \mathbb{C}^N$ continuous up to $b\Delta$ and holomorphic in Δ is *attached* to M if $f(b\Delta) \subset M$. Following Lempert [16], such a map is called *stationary* if there exists a continuous function $c : b\Delta \to \mathbb{R}^*$ such that the map $\zeta c(\zeta)\partial\varrho(f(\zeta))$, defined on $b\Delta$, extends holomorphically into Δ. Here $\partial\varrho = \partial_Z\varrho$ denotes the complex gradient of ϱ. Equivalently, one can require that f allows for a meromorphic lift with a pole of order at most 1 to the conormal bundle $\mathcal{N}^*M$. For details on that, we refer the reader e.g. to [2]. We will only deal with small discs f.

2.4. **Spaces of functions with parameters.** Here, we recall spaces of functions with parameters defined in [17], suitable for the study of deformations of Riemann maps [3, Corollary 9.4] we are going to use. Let $I = [0,1] \subset \mathbb{R}$, let Ω be a bounded open set in a Euclidean space and let $k, j \geq 0$ be integers and let $0 \leq \alpha < 1$, We denote by $\mathcal{C}^{k+\alpha}(\overline{\Omega})$ the standard Hölder space with its usual norm $|\cdot|_{k+\alpha}$. Define $\hat{\mathcal{C}}^{k+\alpha,j}(\overline{\Omega}, I)$ to be the set of functions f defined on $\overline{\Omega} \times I$ such that for all integers $0 \leq l \leq j$, the map $t \mapsto \partial_t^l f(.,t)$ is continuous from I into $\mathcal{C}^k(\overline{\Omega})$ and such that

$$\|f\|_{k+\alpha,j} := \max_{0\leq l\leq j} \sup_{t\in I} |\partial_t^l f(\cdot,t)|_{k+\alpha} < \infty.$$

We now define

$$\mathcal{C}^{k+\alpha,j}(\overline{\Omega}, I) := \bigcap_{0\leq l\leq j} \hat{\mathcal{C}}^{k-l+\alpha,l}(\overline{\Omega}, I),$$

and

$$|f|_{k+\alpha,j} := \max_{0\leq l\leq j} \|f\|_{k-l+\alpha,l}.$$

As pointed out in [3], we have the following inclusion:

$$\mathcal{C}^{k+\alpha}(\overline{\Omega} \times I) \subset \mathcal{C}^{k+\alpha,k}(\overline{\Omega}, I). \tag{1}$$

In the present paper we will also need:

Lemma 2.1. *The following inclusion holds*

$$\mathcal{C}^{k+1+\alpha,k+1}(\overline{\Omega}, I) \subset \mathcal{C}^k(\overline{\Omega} \times I).$$

Proof. Let $f \in \mathcal{C}^{k+1+\alpha,k+1}(\overline{\Omega}, I)$. We first note that f is k-times differentiable. Now let $p, q \geq 0$ be two integers with $p+q = k$ and let $(x,t), (x',t') \in \overline{\Omega} \times I$ sufficiently close to each other. We have

$$\begin{aligned}
|\partial_x^p\partial_t^q f(x,t) &- \partial_x^p\partial_t^q f(x',t')| \\
&\leq |\partial_x^p\partial_t^q f(x,t) - \partial_x^p\partial_t^q f(x,t')| + |\partial_x^p\partial_t^q f(x,t') - \partial_x^p\partial_t^q f(x',t')| \\
&\leq \sup_{s\in I} |\partial_x^p\partial_t^{q+1} f(x,s)||t-t'| + \sup_{y\in[x,x']} |\partial_x^{p+1}\partial_t^q f(y,t')|\|x-x'\| \\
&\leq \sup_{s\in I} |\partial_t^{q+1} f(\cdot,s)|_{k+1+\alpha}|t-t'| + \sup_{s\in I} |\partial_t^q f(\cdot,s)|_{k+1+\alpha}\|x-x'\| \\
&\leq |f|_{k+1+\alpha,k+1}|t-t'| + |f|_{k+1+\alpha,k+1}\|x-x'\|,
\end{aligned}$$

which proves the lemma. $\square$

3. A further result and proof of the main theorem

In order to prove Theorem 1.1, we shall make use of the following result which is valid for finitely smooth real hypersurfaces in $\mathbb{C}^2$. We use the following convention: We write $O(|z|^k)$ (resp. $O(t^k)$) to denote a function of class at least $\mathcal{C}^k$ which is bounded by $|z|^k$ (resp. t^k) up to a multiplicative constant.

Theorem 3.1. *Let $S \subset \mathbb{C}^2$ be a $\mathcal{C}^{8+\alpha}$-smooth real hypersurface through the origin, with $\alpha > 0$, whose defining equation (near the origin) can be written in the form*

$$\varrho(z, w, \overline{z}, \overline{w}) = \operatorname{Re} w - |z|^2 + A z^2 \overline{z}^4 + \overline{A} z^4 \overline{z}^2 + \operatorname{Im} w\, h(z, \overline{z}, \operatorname{Im} w) + g(z, \overline{z}),$$

with $g(z, \overline{z}) = O(|z|^7)$. If the discs $\Omega_t = \{w = t^2\} \cap \{\varrho > 0\}$, for small $t \in \mathbb{R}$, are stationary, then $A = 0$.

Theorem 1.1 is a straightforward consequence of Theorem 3.1: the result of Huang [12] already mentioned in the introduction shows that the extremal discs we consider are actually stationary, and the results of Chern and Moser summarized in subsection 2.1 show that there exists a spherical neighbourhood of 0 in S. The rest of this section is devoted to the proof of Theorem 3.1, which is going to be developed in a series of lemmas.

First note that if for any $t \in \mathbb{R}$ we write $S_t = \{w = t^2\} \cap S$, then for $|t| > 0$ small enough, S_t is a closed curve (of class $\mathcal{C}^{8+\alpha}$) contained in S, bounding Ω_t. For t small enough, we will have that $(0, t^2) \in \Omega_t$. We denote by π^1 the projection onto the first coordinate and for all $|t| > 0$ small enough, we consider $R_t : \Delta \to \pi^1(\Omega_t)$ the (uniquely determined) Riemann map such that $R_t(0) = 0$, $R_t'(0) > 0$. Define $f_t : \Delta \to \mathbb{C}^2$ as $f_t(\zeta) = (R_t(\zeta), t^2)$. By construction, each f_t is an analytic disc attached to S. In the following, for the sake of notational simplicity, we will identify Ω_t with $\pi^1(\Omega_t)$ (as well as S_t with $\pi^1(S_t)$).

By definition, f_t is stationary if and only if there exists a continuous function $a_t : b\Delta \to \mathbb{R}^+$ and holomorphic functions $\widetilde{z}_t, \widetilde{w}_t \in \mathcal{O}(\Delta) \cap C(\overline{\Delta})$ satisfying

$$\begin{aligned}
\widetilde{z}_t(\zeta) &= \zeta a_t(\zeta) \frac{\partial \varrho}{\partial z}(R_t(\zeta), t^2, \overline{R_t(\zeta)}, t^2) \\
\widetilde{w}_t(\zeta) &= \zeta a_t(\zeta) \frac{\partial \varrho}{\partial w}(R_t(\zeta), t^2, \overline{R_t(\zeta)}, t^2)
\end{aligned} \tag{2}$$

for all $\zeta \in b\Delta$.

Let now $R_t^{-1} : \Omega_t \to \Delta$ be the inverse of the Riemann map. Note that R_t^{-1} is smooth of class $\mathcal{C}^{8+\alpha}$ up to the boundary S_t by Kellogg's theorem [14] (see e.g. the book of Pommerenke [19]). We also note that we can write $R_t^{-1}(z) = z e^{\varphi_t(z)}$ for a suitable holomorphic function $\varphi_t : \Omega_t \to \Delta$, where φ_t is again smooth of class $\mathcal{C}^{8+\alpha}$ up to S_t. Applying (2) for $\zeta = R_t^{-1}(z)$ we obtain

$$\widetilde{z}_t(R_t^{-1}(z)) = ze^{\varphi_t(z)}a_t(R_t^{-1}(z))\frac{\partial \varrho}{\partial z}(z,t^2,\overline{z},t^2)$$
$$\widetilde{w}_t(R_t^{-1}(z)) = ze^{\varphi_t(z)}a_t(R_t^{-1}(z))\frac{\partial \varrho}{\partial w}(z,t^2,\overline{z},t^2)$$

for all $z \in b\Omega_t = S_t$. Putting $b_t(z) = a_t(R_t^{-1}(z))$, $Z_t(z) = e^{-\varphi_t(z)}\widetilde{z}_t(R_t^{-1}(z))$ and $W_t(z) = e^{-\varphi_t(z)}\widetilde{w}_t(R_t^{-1}(z))$ we can rewrite the system as

$$\begin{aligned} Z_t(z) &= zb_t(z)\frac{\partial \varrho}{\partial z}(z,t^2,\overline{z},t^2) \\ W_t(z) &= zb_t(z)\frac{\partial \varrho}{\partial w}(z,t^2,\overline{z},t^2) \end{aligned} \tag{3}$$

for $z \in S_t$. Here b_t is a continuous positive function on S_t and the functions Z_t, W_t extend holomorphically to Ω_t.

We will use systematically the following fact: a continuous function $f : S_t \to \mathbb{C}$ extends holomorphically to Ω_t if and only if it satisfies the *moment conditions*

$$\int_{S_t} z^m f(z)dz = 0 \quad \text{for all } m \geq 0.$$

Though this fact is well-known, we provide a proof for the convenience of the reader. Denote by Cf the Cauchy transform

$$Cf(z) = \frac{1}{2\pi i}\int_{S_t}\frac{f(\zeta)}{\zeta - z}d\zeta.$$

By Plemelj's formula, f extends holomorphically if and only if $Cf(z) = 0$ for $z \notin \overline{\Omega}_t$. For any fixed z outside $\overline{\Omega}_t$, $1/(\zeta - z)$ can be approximated uniformly by polynomials on S_t by Runge's theorem. Since the moment conditions mean that the integral of f against any holomorphic polynomial vanishes, we deduce that f extends holomorphically whenever it satisfies the moment conditions. The opposite implication is a consequence of Cauchy's integral formula.

Consider the scaling $\Lambda_t : \mathbb{C} \to \mathbb{C}$ defined by $\Lambda_t(z) = z/t$. We set $\widetilde{\Omega}_t = \Lambda_t(\Omega_t)$, $\widetilde{S}_t = \Lambda_t(S_t)$, and $\widetilde{\Omega}_0 = \Delta$ (with $\widetilde{S}_0 = b\Delta$). A change of variables in the above integral implies that $f : S_t \to \mathbb{C}$ extends holomorphically to Ω_t if and only if it satisfies the moment conditions

$$\int_{\widetilde{S}_t} z^m f(tz)dz = 0 \quad \text{for all } m \geq 0.$$

In order to compute integrals of this kind, we use polar coordinates (r,θ) and parametrize the curve $\widetilde{S}_t$ according to the following Lemma.

Lemma 3.2. *If we parametrize the curve $\widetilde{S}_t$ as $\theta \mapsto r(\theta,t)e^{i\theta}$, then the function r is of class $\mathcal{C}^{6+\alpha}$ in both variables in a full neighbourhood of $(0,0) \in \mathbb{R}^2$, and can be written as*

$$r(\theta,t) = 1 + k(\theta)t^4 + r_5(\theta,t)$$

where $k(\theta) = \operatorname{Re}(Ae^{-2i\theta})$ and $r_5(\theta,t) = O(|t|^5)$.

Proof. The function r satisfies $\varrho(tr(\theta,t)e^{i\theta},t^2,tr(\theta,t)e^{-i\theta},t^2)\equiv 0$, i.e.

$$t^2-t^2r^2(\theta,t)+2k(\theta)t^6r^6(\theta,t)+g(tr(\theta,t)e^{i\theta},tr(\theta,t)e^{-i\theta})=0. \tag{4}$$

Since $g(z,\overline{z})=O(|z|^7)$ we have $g(tr(\theta,t)e^{i\theta},tr(\theta,t)e^{-i\theta})=t^2G(\theta,t)$ where G is of class $\mathcal{C}^{6+\alpha}$ and satisfies $G(\theta,t)=O(t^5)$. This allows us to rewrite (4) as

$$1-r^2(\theta,t)+2k(\theta)t^4r^6(\theta,t)+G(\theta,t)=0.$$

The implicit function theorem allows us to solve this equation with a unique r of class $\mathcal{C}^{6+\alpha}$ satisfying $r(\theta,0)=1$. Taking successive derivatives it is immediate that $\frac{\partial^j r}{\partial t^j}(\theta,0)=0$ for $j=1,2,3$ and

$$-2r(\theta,t)\frac{\partial^4 r}{\partial t^4}(\theta,t)+4!\cdot 2k(\theta)+O(t)=0,$$

so that $\frac{\partial^4 r}{\partial t^4}(\theta,0)=4!k(\theta)$. This concludes the proof of the lemma. $\square$

The lemma above allows us to extend the boundary parametrization of $\widetilde{S}_t$ to the interior of the unit disc, to obtain a family of diffeomorphisms $\Gamma_t=\Gamma(\cdot,t):\overline{\Delta}\to\overline{\widetilde{\Omega}_t}$ which is $\mathcal{C}^{6+\alpha}$ in both variables z and t and equal to the identity for $t=0$. We shall (if necessary) rescale with a map of the form $(z,w)\mapsto(\lambda z,\lambda^2 w)$ to have that $\Gamma\in\mathcal{C}^{6+\alpha}(\overline{\Delta}\times I)$, where $I=[-1,1]$.

Lemma 3.3. *The map $R_t^{-1}\circ\Lambda_t^{-1}\circ\Gamma_t:\overline{\Delta}\to\overline{\Delta}$ is in $\mathcal{C}^5(\overline{\Delta}\times I)$ and is, for $t=0$, the identity.*

Proof. Since the family of diffeomorphisms $\Gamma_t:\overline{\Delta}\to\overline{\widetilde{\Omega}_t}$ is $\mathcal{C}^{6+\alpha}$ in both variables z and t, we can apply (1) to obtain that $\Gamma\in\mathcal{C}^{6+\alpha,6}(\overline{\Delta}\times I)$. By [3, Corollary 9.4], we have $R_t^{-1}\circ\Lambda_t^{-1}\circ\Gamma_t\in\mathcal{C}^{6+\alpha,6}(\overline{\Delta}\times I)$ and by Lemma 2.1 it follows that $R_t^{-1}\circ\Lambda_t^{-1}\circ\Gamma_t\in\mathcal{C}^5(\overline{\Delta}\times I)$.

Since $\widetilde{\Omega}_0=\Delta$ and since the Riemann map $R_t^{-1}\circ\Lambda_t^{-1}:\overline{\widetilde{\Omega}_t}\to\overline{\Delta}$ is chosen in such a way that $R_t(0)=0$, $R_t'(0)>0$ and so $R_t^{-1}\circ\Lambda_t^{-1}(0)=0$ and $\left(R_t^{-1}\circ\Lambda_t^{-1}\right)'(0)>0$, it follows that $R_t^{-1}\circ\Lambda_t^{-1}$ is the identity for $t=0$. Finally note that by definition, Γ_0 is the identity. $\square$

We will now apply the moment conditions to the second equation in (3):

$$\int_{\widetilde{S}_t} z^m\left(zb_t(tz)\frac{\partial\varrho}{\partial w}(tz,t^2,t\overline{z},t^2)\right)dz=0 \quad \text{for all } m\geq 0$$

or equivalently

$$\int_{\widetilde{S}_t} z^j b_t(tz)\frac{\partial\varrho}{\partial w}(tz,t^2,t\overline{z},t^2)dz=0 \quad \text{for all } j\geq 1.$$

Computing $\partial\varrho/\partial w$ we get

$$\frac{\partial\varrho}{\partial w}(z,w,\overline{z},\overline{w})=\frac{1}{2}-\frac{i}{2}h(z,\overline{z},\operatorname{Im} w)+\operatorname{Im} w\frac{\partial}{\partial w}(h(z,\overline{z},\operatorname{Im} w))$$

so that

$$\frac{\partial\varrho}{\partial w}(tz,t^2,t\overline{z},t^2)=\frac{1}{2}-\frac{i}{2}h(tz,t\overline{z},0).$$

Hence $b_t(tz)$ must satisfy

$$\int_{\widetilde{S}_t} z^j b_t(tz)\left(\frac{1}{2} - \frac{i}{2}h(tz, t\overline{z}, 0)\right) dz = 0 \quad \text{for all } j \geq 1.$$

Using the parametrization $\theta \mapsto r(\theta, t)e^{i\theta}$ for $\widetilde{S}_t$ the integral becomes

$$\int_0^{2\pi} r^j e^{ij\theta} b_t(tre^{i\theta})\left(\frac{1}{2} - \frac{i}{2}h(tre^{i\theta}, tre^{-i\theta}, 0)\right)\left(\frac{\partial r}{\partial\theta} + ir\right) e^{i\theta} d\theta = 0 \quad \text{for all } j \geq 1, \tag{5}$$

where we write $r = r(\theta, t)$ for brevity.

For a continuous function $a_t : b\Delta \to \mathbb{R}^+$ satisfying (2), we define

$$c(\theta, t) = b_t(tr(\theta, t)e^{i\theta}) = b_t(\Lambda_t^{-1} \circ \Gamma_t(e^{i\theta})) = a_t(R_t^{-1} \circ \Lambda_t^{-1} \circ \Gamma_t(e^{i\theta})).$$

Lemma 3.4. *There is a choice of a_t such that the function $c(\theta, t)$ is $\mathcal{C}^4$ in a neighbourhood of $[0, 2\pi] \times \{0\}$ and satisfies $\int_0^{2\pi} c(\theta, t)dt = 1$ for all $t \neq 0$ small enough.*

We note that the normalization condition can of course be assumed because the sign of c is fixed. The point of the Lemma is the smoothness of the function a_t.

Proof. Recall that by Lemma 3.3, $R_t^{-1} \circ \Lambda_t^{-1} \circ \Gamma_t$ is of class $\mathcal{C}^5$. From Pang [18], if $f_t(\zeta) = (R_t(\zeta), t^2)$ is stationary and satisfies (2) for a continuous function $a_t : b\Delta \to \mathbb{R}^+$, then a_t is a positive multiple of $\widehat{a_t}$, which is defined for $\zeta \in b\Delta$ by

$$\frac{1}{\widehat{a_t}(\zeta)} = \zeta\partial\varrho(f_t(\zeta)) \cdot f_t'(\zeta).$$

First the map

$$\partial\varrho(f_t(R_t^{-1} \circ \Lambda_t^{-1} \circ \Gamma_t(e^{i\theta}))) = \partial\varrho(t\Gamma_t(e^{i\theta}), t^2)$$

is $\mathcal{C}^{6+\alpha}$. Note that by the chain rule, we have

$$\begin{aligned}
\frac{d}{d\theta}\left(t\Gamma_t(e^{i\theta})\right) &= \frac{d}{d\theta}R_t \circ \left(R_t^{-1} \circ \Lambda_t^{-1} \circ \Gamma_t(e^{i\theta})\right) \\
&= R_t'\left(R_t^{-1} \circ \Lambda_t^{-1} \circ \Gamma_t(e^{i\theta})\right)\frac{d}{d\theta}\left(R_t^{-1} \circ \Lambda_t^{-1} \circ \Gamma_t(e^{i\theta})\right)
\end{aligned}$$

since R_t is holomorphic. It follows that

$$f_t'(R_t^{-1} \circ \Lambda_t^{-1} \circ \Gamma_t(e^{i\theta})) = \left(\frac{\frac{d}{d\theta}\left(t\Gamma_t(e^{i\theta})\right)}{\frac{d}{d\theta}\left(R_t^{-1} \circ \Lambda_t^{-1} \circ \Gamma_t(e^{i\theta})\right)}, 0\right).$$

Since $\left(R_t^{-1} \circ \Lambda_t^{-1} \circ \Gamma_t(e^{i\theta})\right)$ is $\mathcal{C}^5$ and equal to $e^{i\theta} + O(t)$ by Lemma 3.3, the function $\frac{d}{d\theta}\left(R_t^{-1} \circ \Lambda_t^{-1} \circ \Gamma_t(e^{i\theta})\right)$ is $\mathcal{C}^4$ and equal to $ie^{i\theta}$ for $t = 0$. This shows that the function $f_t'(R_t^{-1} \circ \Lambda_t^{-1} \circ \Gamma_t(e^{i\theta}))$ is $\mathcal{C}^4$ and therefore that

$1/\widehat{a}_t(R_t^{-1}\circ\Lambda_t^{-1}\circ\Gamma_t(e^{i\theta}))$ is $\mathcal{C}^4$. Finally, note that since $t\Gamma_t(e^{i\theta}) = te^{i\theta}+O(t^2)$ and $\partial_z\varrho(z,w) = \overline{z}+O(|z|^5)$, we have

$$\frac{\frac{d}{d\theta}\left(t\Gamma_t(e^{i\theta})\right)}{\frac{d}{d\theta}\left(R_t^{-1}\circ\Lambda_t^{-1}\circ\Gamma_t(e^{i\theta})\right)} = \frac{ite^{i\theta}+O(t^2)}{ie^{i\theta}+O(t)} = t+O(t^2),$$

and

$$\partial_z\varrho(R_t^{-1}\circ\Lambda_t^{-1}\circ\Gamma_t(e^{i\theta}),t^2) = te^{-i\theta}+O(t^2)$$

from which it follows directly that

$$\frac{1}{\widehat{a}_t(R_t^{-1}\circ\Lambda_t^{-1}\circ\Gamma_t(e^{i\theta}))} = t^2+O(t^3).$$

The function $\tilde{a}_t = t^2\widehat{a}_t$ satisfies all of the required properties and can be rescaled so that $\int_0^{2\pi} c(\theta,t)dt = 1$ for all $t\neq 0$ small enough without changing the smoothness of c. $\square$

Since $h(z,\overline{z},0) = O(|z|^6)$, using Lemma 3.2 we deduce that $h(tre^{i\theta},tre^{-i\theta},0) = O(t^6)$, and furthermore

$$r(\theta,t)^j = 1+jk(\theta)t^4+O(t^5), \tag{6}$$

$$\frac{\partial r}{\partial\theta}(\theta,t) = \frac{dk}{d\theta}(\theta)t^4+O(t^5). \tag{7}$$

Thus we can rewrite (5) as

$$\int_0^{2\pi} ce^{i(j+1)\theta}(1+jkt^4+O(t^5))\left(\frac{1}{2}+O(t^6)\right)\left(i+\left(\frac{dk}{d\theta}+ik\right)t^4+O(t^5)\right)d\theta = 0$$

for all $j\geq 1$. Further developing the products we obtain

$$\int_0^{2\pi} e^{i(j+1)\theta}c(\theta,t)\left(i+\left(i(j+1)k(\theta)+\frac{dk}{d\theta}(\theta)\right)t^4+O(t^5)\right)d\theta = 0 \tag{8}$$

for all $j\geq 1$. For all $|t|$ small enough we expand the function $c(\cdot,t)$ in its Fourier series $c(\theta,t) = \sum_{k=-\infty}^{+\infty}\gamma_k(t)e^{ik\theta}$, where γ_k is $\mathcal{C}^4$ for all $k\in\mathbb{Z}$, $\gamma_{-k} = \overline{\gamma}_k$ and $\gamma_0(t)\equiv 1$ due to our normalization. We insert this series in (8) and ignore for the moment the precise expression of the factor multiplying t^4:

$$\int_0^{2\pi} e^{i(j+1)\theta}\sum_{k=-\infty}^{+\infty}\gamma_k(t)e^{ik\theta}d\theta = O(t^4),$$

which means that

$$\overline{\gamma}_{j+1}(t) = O(t^4)\quad\text{for all } j\geq 1. \tag{9}$$

Next, we write also $k(\theta) = Ae^{-2i\theta}/2+\overline{A}e^{2i\theta}/2$ and $\frac{dk}{d\theta}(\theta) = -iAe^{-2i\theta}+i\overline{A}e^{2i\theta}$ as Fourier polynomials, so that

$$i(j+1)k(\theta)+\frac{dk}{d\theta}(\theta) = i\frac{j-1}{2}Ae^{-2i\theta}+i\frac{j+3}{2}\overline{A}e^{2i\theta}$$

and we take the fourth derivative of (8) with respect to t:

$$\sum_{\ell=0}^{4}\binom{4}{\ell}\int_0^{2\pi}\frac{\partial^\ell c}{\partial t^\ell}e^{i(j+1)\theta}\cdot$$

$$\cdot\left(\delta_4^\ell+\frac{4!t^\ell}{\ell!}\left(\frac{j-1}{2}Ae^{-2i\theta}+\frac{j+3}{2}\overline{A}e^{2i\theta}\right)+O(t^{\ell+1})\right)d\theta=0$$

where $\delta_4^\ell=1$ if $\ell=4$ and $\delta_4^\ell=0$ otherwise, and we divided by the common factor i. Replacing $\frac{\partial^\ell c}{\partial t^\ell}(\theta,t)$ with its Fourier series, we see that

$$\sum_{\ell=0}^{4}\binom{4}{\ell}\cdot$$

$$\cdot\left(\frac{d^\ell\overline{\gamma}_{j+1}}{dt^\ell}(t)\delta_4^\ell+\frac{4!}{\ell!}\left(\frac{j-1}{2}A\frac{d^\ell\overline{\gamma}_{j-1}}{dt^\ell}(t)+\frac{j+3}{2}\overline{A}\frac{d^\ell\overline{\gamma}_{j+3}}{dt^\ell}(t)\right)t^\ell+O(t^{\ell+1})\right)=0$$

for all $j\geq 1$. In particular for $j=1$

$$\sum_{\ell=0}^{4}\binom{4}{\ell}\left(\frac{d^\ell\overline{\gamma}_2}{dt^\ell}(t)\delta_4^\ell+\frac{4!}{\ell!}\left(2\overline{A}\frac{d^\ell\overline{\gamma}_4}{dt^\ell}(t)\right)t^\ell+O(t^{\ell+1})\right)=0.$$

All the terms in the previous sum are $O(t)$ except for $\ell=0$ and $\ell=4$:

$$\frac{d^4\overline{\gamma}_2}{dt^4}(t)+4!\cdot 2\overline{A}\overline{\gamma}_4(t)=O(t),$$

which by (9) implies that

$$\frac{d^4\overline{\gamma}_2}{dt^4}(t)=O(t). \tag{10}$$

We now turn to the first equation in (3). The moment conditions read

$$\int_{\widetilde{S}_t}z^jb_t(tz)\frac{\partial\varrho}{\partial z}(tz,t^2,t\overline{z},t^2)dz=0\quad\text{for all } j\geq 1,$$

and since

$$\frac{\partial\varrho}{\partial z}(z,w,\overline{z},\overline{w})=-\overline{z}+2Az\overline{z}^4+4\overline{A}z^3\overline{z}^2+\operatorname{Im}w\,\frac{\partial h}{\partial z}(z,\overline{z},\operatorname{Im}w)+\frac{\partial g}{\partial z}(z,\overline{z}),\ \text{ so that}$$

$$\frac{\partial\varrho}{\partial z}(tz,t^2,t\overline{z},t^2)=-t\overline{z}+2At^5z\overline{z}^4+4\overline{A}t^5z^3\overline{z}^2+\frac{\partial g}{\partial z}(tz,t\overline{z}),$$

by using the parametrization $\theta\to re^{i\theta}$ the integral turns into

$$\int_0^{2\pi}e^{i(j+1)\theta}b_t\left(-tr^{j+1}e^{-i\theta}+t^5r^{j+5}(2Ae^{-3i\theta}+4\overline{A}e^{i\theta})+r^j\frac{\partial g}{\partial z}\right)\left(\frac{\partial r}{\partial\theta}+ir\right)d\theta=0$$

for all $j\geq 1$. Since $\frac{\partial g}{\partial z}(z,\overline{z})=O(|z|^6)$ we have $r^j\frac{\partial g}{\partial z}(tre^{i\theta},tre^{-i\theta})=O(t^6)$. We use now (6) and recall that $b_t(tre^{i\theta})=c(\theta,t)$ to rewrite the previous equation as

$$\int_0^{2\pi}e^{i(j+1)\theta}c(\theta,t)\left(-(t+(j+1)k(\theta)t^5)e^{-i\theta}+t^5(2Ae^{-3i\theta}+4\overline{A}e^{i\theta})+O(t^6)\right)\cdot$$

$$\cdot\left(i+\left(\frac{dk}{d\theta}(\theta)+ik(\theta)\right)t^4+O(t^5)\right)d\theta=0\quad\text{for all } j\geq 1.$$

We substitute the expression of $k(\theta)$ and divide by $-t$ to obtain

$$\int_0^{2\pi} e^{i(j+1)\theta}c(\theta,t)\left(e^{-i\theta}+\left(\frac{j-3}{2}Ae^{-3i\theta}+\frac{j-7}{2}\overline{A}e^{i\theta}\right)t^4+O(t^5)\right)\cdot$$
$$\cdot\left(i+\left(-\frac{i}{2}Ae^{-2i\theta}+\frac{3i}{2}\overline{A}e^{2i\theta}\right)t^4+O(t^5)\right)d\theta=0\quad\text{for all } j\geq 1,$$

and after carrying out the products and dividing by i,

$$\int_0^{2\pi} c(\theta,t)\left(e^{ij\theta}+\frac{j-4}{2}\left(Ae^{i(j-2)\theta}+\overline{A}e^{i(j+2)\theta}\right)t^4+O(t^5)\right)d\theta=0$$

for all $j\geq 1$. Once again we differentiate under the integral sign with respect to t four times:

$$\sum_{\ell=0}^{4}\binom{4}{\ell}\int_0^{2\pi}\sum_{k=-\infty}^{+\infty}\frac{d^\ell\gamma_k}{dt^\ell}(t)e^{ik\theta}\cdot$$
$$\cdot\left(\delta_4^\ell e^{ij\theta}+\frac{4!}{\ell!}\frac{j-4}{2}\left(Ae^{i(j-2)\theta}+\overline{A}e^{i(j+2)\theta}\right)t^\ell+O(t^{\ell+1})\right)d\theta=0$$

which translates into

$$\sum_{\ell=0}^{4}\binom{4}{\ell}\left(\frac{d^\ell\overline{\gamma}_j}{dt^\ell}(t)\delta_4^\ell+\frac{4!}{\ell!}\frac{j-4}{2}\left(A\frac{d^\ell\overline{\gamma}_{j-2}}{dt^\ell}(t)+\overline{A}\frac{d^\ell\overline{\gamma}_{j+2}}{dt^\ell}(t)\right)t^\ell+O(t^{\ell+1})\right)=0.$$

Taking $j=2$ we have

$$\sum_{\ell=0}^{4}\binom{4}{\ell}\left(\frac{d^\ell\overline{\gamma}_2}{dt^\ell}(t)\delta_4^\ell-\frac{4!}{\ell!}\left(A\frac{d^\ell\overline{\gamma}_0}{dt^\ell}(t)+\overline{A}\frac{d^\ell\overline{\gamma}_4}{dt^\ell}(t)\right)t^\ell+O(t^{\ell+1})\right)=0;$$

except for $\ell=4$ and $\ell=0$ every term is $O(t)$, hence we get

$$\frac{d^4\overline{\gamma}_2}{dt^4}(t)-4!\left(A\overline{\gamma}_0(t)+\overline{A}\overline{\gamma}_4(t)\right)=O(t).$$

Recalling that $\gamma_0\equiv 1$ due to our normalization, and using (9), (10), we deduce that $A=O(t)$. This is only possible if $A=0$.

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