

EXPLICIT CONSTRUCTION OF STATIONARY DISCS AND ITS CONSEQUENCES FOR NONDEGENERATE QUADRICS

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ABSTRACT. We give an explicit construction of a key family of stationary discs attached to a nondegenerate model quadric in \mathbb{C}^N and derive a necessary condition for which (each lift) of those stationary discs is uniquely determined by its 1-jet at a given point via a local diffeomorphism. This unique 1-jet determination is a crucial step to deduce 2-jet determination for CR automorphisms of generic real submanifolds in \mathbb{C}^N .

1. INTRODUCTION

In the papers [9, 10, 29], the authors discuss the connection between the existence of stationary discs for generic real submanifolds in \mathbb{C}^N and the unique 2-jet determination of their CR automorphisms. Their approach is based on a method developed by the first author and Blanc-Centi in [8] which relies on the family of stationary discs introduced by Lempert [24]. These invariant discs and their use in mapping problems have attracted the attention of several authors (see for instance [17, 26, 28, 5]). As emphasized in [8] (see also [7, 29]), the stationary disc method is well adapted to study jet determination problems, and is, to our knowledge, the only approach in the literature which allows the treatment of (finitely) smooth CR automorphisms of finitely smooth submanifolds. Note that, under appropriate nondegeneracy conditions, if the submanifold is *real-analytic* then every C^1 CR automorphism is real-analytic (Theorem 2 in [30], Theorem 3.1 in [2]). In general, the unique 2-jet determination of CR automorphisms of real-analytic Levi nondegenerate hypersurfaces goes back to the works of Cartan [12], Tanaka [27], and Chern and Moser [11]. In the last twenty years, the question of jet determination for CR maps was pushed further in many important papers for real analytic submanifolds [7, 31, 1, 3, 15, 22, 18, 19, 25, 23] and in the C^∞ setting [13, 14, 20, 21]. Note that the question of 2-jet determination for real analytic submanifolds (of codimension $d > 1$) is not well understood [16].

Given a model quadric $Q \subset \mathbb{C}^N$ and an initial stationary disc f_0 , the key point of the stationary disc method is to construct "enough" discs, near f_0 , attached to deformations of Q , with "good" geometric properties. Such construction is usually done via an implicit function theorem for appropriate Banach spaces, and thus the choice of both the model quadric and the initial disc is crucial. While the construction of such discs only requires the submanifold to be strongly Levi nondegenerate (Definition 2.1), it is necessary to impose more nondegeneracy restrictions on the submanifold to ensure that the family of constructed discs enjoys good geometric properties. The papers [10, 29] actually suggest that this problem is related to the defect of the initial disc f_0 . In order to ensure the

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existence of such a nondefective disc, the authors rely on the explicit expression of - some - stationary discs attached to the given quadric. Tumanov provided in [28] their full description in case the quadric is strongly pseudoconvex with generating Levi form. In the present paper, we provide a similar description when the model quadric is merely strongly Levi nondegenerate (Theorem 3.1). Using this explicit family of discs, we then study conditions that ensure the unique 1-jet determination of their lift, and that their centers fill an open set in \mathbb{C}^N , *two key properties to deduce 2-jet determination for CR automorphisms*. As expected, such conditions are directly related to the geometry of the model quadric. More precisely, we show that if the constructed family of (lift of) discs is uniquely determined by their 1-jet at $\zeta = 1$, via a local diffeomorphism, then one of such discs must be nondefective (Theorem 4.5). Moreover, using the constructed explicit family of discs, we provide a geometric context to some of the previous nondegeneracy conditions introduced earlier in [9, 8] (Theorem 4.4 and Theorem 5.3).

2. PRELIMINARIES

We denote by Δ the unit disc in \mathbb{C} and by $\partial\Delta$ its boundary.

2.1. Strongly Levi nondegenerate generic submanifolds. Let $M \subset \mathbb{C}^{n+d}$ be a \mathcal{C}^4 generic real submanifold of real codimension $d \geq 1$ through $p = 0$ given locally by

$$(2.1) \quad \begin{cases} r_1 = \Re w_1 - {}^t \bar{z} A_1 z + O(3) = 0 \\ \vdots \\ r_d = \Re w_d - {}^t \bar{z} A_d z + O(3) = 0 \end{cases}$$

where A_1, \dots, A_d are Hermitian matrices of size n . In the remainder $O(3)$, the variables z and $\Im w$ are respectively of weight one and two. We set $r := (r_1, \dots, r_d)$. We associate to M its model quadric Q given by

$$(2.2) \quad \begin{cases} \rho_1 = \Re w_1 - {}^t \bar{z} A_1 z = 0 \\ \vdots \\ \rho_d = \Re w_d - {}^t \bar{z} A_d z = 0 \end{cases}$$

and we write $\rho := (\rho_1, \dots, \rho_d)$. The following notions of nondegeneracy, particularly the second one, are due to Tumanov [28].

Definition 2.1. Let M be a real submanifold given by (2.1).

- i. We say that M is *strongly Levi nondegenerate at 0* if there exists $b \in \mathbb{R}^d$ such that the matrix $\sum_{j=1}^d b_j A_j$ is invertible.
- ii. We say that M is *strongly pseudoconvex at 0* if there exists $b \in \mathbb{R}^d$ such that $\sum_{j=1}^d b_j A_j$ is positive definite.

We now recall the following types of nondegeneracy introduced in [9, 10].

Definition 2.2. Let M be a strongly Levi nondegenerate (at 0) real submanifold given by (2.1). Let $b \in \mathbb{R}^d$ be such that $A := \sum_{j=1}^d b_j A_j$ is invertible.

- i. We say that M is *\mathfrak{D} -nondegenerate at 0* if there exists $V \in \mathbb{C}^n$ such that, if D_0 denotes the $n \times d$ matrix whose j^{th} column is $A_j V$, then $\Re({}^t \bar{D}_0 A^{-1} D_0)$ is invertible.

ii. If in addition ${}^t\bar{D}_0 A^{-1} D_0$ is invertible, then we say that M is *fully nondegenerate* at 0.

We will need the following factorization lemma which generalizes a classical factorization theorem due to Lempert (Theorem B p. 442 in [24]) for positive definite matrix.

Lemma 2.3. *Let M be a strongly Levi nondegenerate (at 0) real submanifold given by (2.1). Let $b \in \mathbb{R}^d$ be such that $A := \sum_{j=1}^d b_j A_j$ and let $a \in \mathbb{C}^d$. Consider the quadratic matrix equation*

$$(2.3) \quad PX^2 + AX + {}^t\bar{P} = 0$$

where $P := \sum_{j=1}^d a_j A_j$ and X is a $n \times n$ matrix. Then for a sufficiently small, we have

$$\sum_{j=1}^d (a_j \bar{\zeta} + b_j + \bar{a}_j \zeta) A_j = (I - \bar{\zeta} {}^t\bar{X}) B (I - \zeta X), \quad \zeta \in \partial\Delta.$$

where X is the unique $n \times n$ matrix solution of (2.3) such that $\|X\| < 1$ and where B is an invertible Hermitian matrix of size n .

Proof. We consider a small enough and X the unique matrix solution of (2.3) with $\|X\| < 1$. Following Tumanov [28], we define the invertible $n \times n$ matrix

$$(2.4) \quad B := A + PX.$$

Note that using (2.3), we obtain directly

$$(2.5) \quad \begin{cases} BX = -{}^t\bar{P} \\ A = B + {}^t\bar{X} {}^t\bar{B} X. \end{cases}$$

We claim that B is hermitian. Indeed, due to (2.4) and (2.5) we have

$$B - {}^t\bar{B} = {}^t\bar{X} (B - {}^t\bar{B}) X.$$

Therefore, for any positive integer k

$$B - {}^t\bar{B} = {}^t\bar{X}^k (B - {}^t\bar{B}) X^k,$$

which implies $B = {}^t\bar{B}$ since $\|X\| < 1$. The claim is proved and the factorization follows directly from (2.5). \square

Without loss of generality, we assume that $\|X\| < 1$ for the rest of the paper. Inspired by the work of Tumanov in the strongly pseudoconvex case [28], we now introduce the following definition.

Definition 2.4. Let M be a strongly Levi nondegenerate (at 0) real submanifold given by (2.1). Let $b \in \mathbb{R}^d$ be such that $\sum_{j=1}^d b_j A_j$ is invertible and let $V \in \mathbb{C}^n$. Consider $a \in \mathbb{C}^d$ sufficiently small and the solution X of (2.3) with $\|X\| < 1$. We say that M is *stationary minimal at 0 for (a, b, V)* if the matrices A_1, \dots, A_d restricted to the orbit space

$$\mathcal{O}_{X,V} := \text{span}_{\mathbb{R}}\{V, XV, X^2V, \dots, X^kV, \dots\}$$

are \mathbb{R} -linearly independent.

Note that since $X = 0$ when $a = 0$, M is stationary minimal at 0 for $(0, b, V)$ if and only if the space $\text{span}_{\mathbb{R}}\{A_1V, \dots, A_dV\}$ is of real dimension d . It follows that if M is \mathfrak{D} -nondegenerate then M is stationary minimal for $(0, b, V)$. The converse holds in case M is strongly pseudoconvex; this point was in fact already observed by the authors in [10]. It is important to point out that the above definition is independent of the choice of holomorphic coordinates. Also, if M is stationary minimal at 0 then M is of finite type at 0 with 2 the only Hörmander number, that is, the matrices A_1, \dots, A_d are linearly independent. The converse is true in case M is strongly pseudoconvex [29].

Remark 2.5. Tumanov observed in [28] (see the proof of Lemma 6.7 [28]) that the spaces $\mathcal{O}_{X,V}$ and $\mathcal{O}_{X,(I-X)V}$ coincide. In particular, the submanifold M is stationary minimal at 0 for (a, b, V) if and only if it is stationary minimal at 0 for $(a, b, (I - X)^kV)$ for any integer k .

We now state

Lemma 2.6. *Let M be a strongly Levi nondegenerate submanifold given by (2.1). Let $b \in \mathbb{R}^d$ be such that $\sum_{j=1}^d b_j A_j$ is invertible and let $V \in \mathbb{C}^n$. Consider $a \in \mathbb{C}^d$ sufficiently small and the solution X of (2.3) with $\|X\| < 1$. Then the following statements are equivalent:*

- i. The submanifold M is stationary minimal at 0 for (a, b, V) .*
- ii. Assume that $\lambda_1, \dots, \lambda_d \in \mathbb{R}$ are such that $\sum_{j=1}^d \lambda_j A_j X^r V = 0$ for all $r = 0, 1, 2, \dots$, then $\lambda_1 = \dots = \lambda_d = 0$.*
- iii. The matrix $\Re e \left(\sum_{r=0}^{\infty} {}^t \bar{V} {}^t \bar{X}^r A_j A_s X^r V \right)_{j,s}$ is positive definite.*

Proof. The equivalence between the first two statements follows directly from Definition 2.4. We will only prove the equivalence between the last two statements.

Suppose that *ii.* is satisfied. Let $W = (w_1, \dots, w_d) \in \mathbb{C}^d$. We have

$$(2.6) \quad {}^t \bar{W} \Re e \left(\sum_{r=0}^{\infty} {}^t \bar{V} {}^t \bar{X}^r A_j A_s X^r V \right)_{j,s} W = \sum_{r=0}^{\infty} \|D_r W\|^2 + \sum_{r=0}^{\infty} \|\bar{D}_r W\|^2 \geq 0$$

where D_r is the $n \times d$ matrix whose s^{th} column is $A_s X^r V$. If $D_r W = \bar{D}_r W = 0$ for all nonnegative integer r then

$$\sum_{j=1}^d \Re e(w_j) A_j X^r V = \sum_{j=1}^d \Im m(w_j) A_j X^r V = 0,$$

which implies that $W = 0$ by *ii.*

Assume now that the matrix $\Re e \left(\sum_{r=0}^{\infty} {}^t \bar{V} {}^t \bar{X}^r A_j A_s X^r V \right)_{j,s}$ is positive definite. Let $W \in \mathbb{R}^d$ be such that $D_r W = 0$ for all nonnegative integer r . According to (2.6), it follows that ${}^t \bar{W} \Re e \left(\sum_{r=0}^{\infty} {}^t \bar{V} {}^t \bar{X}^r A_j A_s X^r V \right)_{j,s} W$ and thus $W = 0$. \square

The next lemma shows that the notion of stationary minimality is open.

Lemma 2.7. *Let M be a strongly Levi nondegenerate submanifold given by (2.1). Assume that M is stationary minimal at 0 for some (a_0, b_0, V) . Then M is stationary minimal at 0 for (a, b, V) for (a, b) sufficiently close to (a_0, b_0) .*

It is important to note that the solution matrix X of (2.3) depends on $a \in \mathbb{C}^d$ and $b \in \mathbb{R}^d$.

Proof. Let $s \leq 2n$ be such that

$$\mathcal{O}_{X(a_0, b_0), V} = \text{span}_{\mathbb{R}}\{V, X(a_0, b_0)V, X(a_0, b_0)^2V, \dots, X(a_0, b_0)^sV\}.$$

Define the vector

$$\tilde{V}(a, b) := (V, X(a, b)V, X(a, b)^2V, \dots, X(a, b)^sV) \in \mathbb{C}^{sn}$$

and the following $sn \times sn$ matrix, $j = 1 \dots, d$,

$$\tilde{A}_j := \begin{pmatrix} A_j & & & (0) \\ & A_j & & \\ & & \ddots & \\ (0) & & & A_j \end{pmatrix}.$$

The vectors $\tilde{A}_1 \tilde{V}(a_0, b_0), \dots, \tilde{A}_d \tilde{V}(a_0, b_0)$ are \mathbb{R} -linearly independent. Hence the vectors $\tilde{A}_1 \tilde{V}(a, b), \dots, \tilde{A}_d \tilde{V}(a, b)$ are \mathbb{R} -linearly independent for (a, b) in a neighborhood of (a_0, b_0) and the conclusion follows. \square

In the same vein, we note the following result interesting on its own.

Lemma 2.8. *Let M be a strongly Levi nondegenerate submanifold given by (2.1). Assume that M is stationary minimal at 0 for (a, b, V) . Then M is stationary minimal at 0 for $(\lambda a, \lambda b, V)$ for any $\lambda \in \mathbb{R} \setminus \{0\}$.*

Proof. This is due to the homogeneity of Equation (2.3) which implies that $X(\lambda a, \lambda b) = X(a, b)$. \square

We end this section with the following lemma whose proof is straightforward.

Lemma 2.9. *Let Q be a strongly Levi nondegenerate quadric given by (2.2) and let $b_0 \in \mathbb{R}^d$ be such that $\sum_{j=1}^d b_{0j} A_j$ is invertible. Let $a \in \mathbb{C}^d$ be small enough and let X be the unique $n \times n$ matrix solution of (2.3) (with $b = b_0 - a - \bar{a}$) with $\|X\| < 1$. Then for any $s = 1, \dots, d$, we have*

$$\begin{cases} \frac{\partial X}{\partial a_s}(0) = 0 \\ \frac{\partial X}{\partial \bar{a}_s}(0) = \frac{\partial X}{\partial \Re e a_s}(0) = - \left(\sum_{k=1}^d b_{0k} A_k \right)^{-1} A_s. \end{cases}$$

In what follows, we denote by $X_{\Re e a_s}$ the derivative $\frac{\partial X}{\partial \Re e a_s}$, $s = 1, \dots, d$.

2.2. Stationary discs. Let M be a \mathcal{C}^4 generic real submanifold of \mathbb{C}^N of codimension d given by (2.1). A holomorphic disc $f : \Delta \rightarrow \mathbb{C}^N$ continuous up to $\partial\Delta$ is *attached to a* M if $f(\partial\Delta) \subset M$. The following definition is due to Lempert [24] for hypersurfaces and to Tumanov [28] for higher codimension submanifolds.

Definition 2.10. A holomorphic disc $f : \Delta \rightarrow \mathbb{C}^N$ continuous up to $\partial\Delta$ and attached to M is *stationary for* M if there exists a holomorphic lift $\mathbf{f} = (f, \tilde{f})$ of f to the cotangent bundle $T^*\mathbb{C}^N$, continuous up to $\partial\Delta$ and such that for all $\zeta \in \partial\Delta$, $\mathbf{f}(\zeta) \in \mathcal{N}M(\zeta)$ where

$$\mathcal{N}M(\zeta) := \{(z, w, \tilde{z}, \tilde{w}) \in T^*\mathbb{C}^N \mid (z, w) \in M, (\tilde{z}, \tilde{w}) \in \zeta N_{(z,w)}^*M \setminus \{0\}\},$$

and where

$$N_{(z,w)}^*M = \text{span}_{\mathbb{R}}\{\partial r_1(z, w), \dots, \partial r_d(z, w)\}$$

is the conormal fiber at (z, w) of M . The set of these lifts $\mathbf{f} = (f, \tilde{f})$, with f nonconstant, is denoted by $\mathcal{S}(M)$.

We note that a disc $f \in \mathcal{S}(M)$ if there exist d real valued functions $c_1, \dots, c_d : \partial\Delta \rightarrow \mathbb{R}$ such that $\sum_{j=1}^d c_j(\zeta) \partial r_j(0) \neq 0$ for all $\zeta \in \partial\Delta$ and such that the map

$$\zeta \mapsto \zeta \sum_{j=1}^d c_j(\zeta) \partial r_j \left(f(\zeta), \overline{f(\zeta)} \right)$$

defined on $\partial\Delta$ extends holomorphically on Δ .

3. EXPLICIT CONSTRUCTION OF STATIONARY DISCS FOR QUADRIC SUBMANIFOLDS

3.1. Explicit construction of stationary discs. Let $Q \subset \mathbb{C}^N$ be a quadric submanifold of real codimension d given by (2.2). In the recent papers [9] with Blanc-Centi, and [10], we worked with a special family of lifts $\mathbf{f} = (h, g, \tilde{h}, \tilde{g}) \in \mathcal{S}(Q)$ of the form

$$(3.1) \quad \mathbf{f} = \left((1 - \zeta)V, 2(1 - \zeta) {}^t\bar{V}A_1V, \dots, 2(1 - \zeta) {}^t\bar{V}A_dV, (1 - \zeta) {}^t\bar{V} \sum_{j=1}^b b_j A_j, \frac{\zeta}{2}b \right),$$

where $V \in \mathbb{C}^n$ and $b \in \mathbb{R}^d$ is such that $\sum_{j=1}^d b_j A_j$ is invertible. This special family of lift can be used to obtain unique jet determination properties for \mathfrak{D} -nondegenerate submanifolds. Nevertheless, this class of submanifolds is the largest one can treat by working with discs of the form (3.1). Therefore, in order to study jet determination problems for larger classes of strongly Levi nondegenerate submanifolds, it is crucial to work with more (explicit) stationary discs. This is precisely the purpose of Theorem 3.1 in which we describe explicitly stationary discs attached to Q .

Before stating the main theorem, we need to introduce the following. Let $b \in \mathbb{R}^d, a \in \mathbb{C}^d, P$ and X be as Lemma 2.3. Denote by $\mathcal{M}_n(\mathbb{C})$ the space of square matrices of size n with complex coefficients. Consider the linear map $\psi : \mathcal{M}_n(\mathbb{C}) \rightarrow \mathcal{M}_n(\mathbb{C})$ defined by

$$(3.2) \quad \psi(M) = M - {}^t\bar{X}MX.$$

Due to the fact that a is small and $\|X\| < 1$, the map ψ is invertible with inverse

$$\psi^{-1}(M) = \sum_{r=0}^{\infty} {}^t\bar{X}^r M X^r.$$

Note that M is Hermitian if and only if $\psi(M)$ is Hermitian.

In Theorem 3.1, we focus on lifts of discs attached to a fixed point in the cotangent bundle. We fix $b_0 \in \mathbb{R}^d$ such that $\sum_{j=1}^d b_{0j} A_j$ is invertible and we define $\mathcal{S}_0(Q) \subset \mathcal{S}(Q)$ to be the subset of lifts whose value at $\zeta = 1$ is $(0, 0, 0, b_0/2)$. Consider an initial disc $\mathbf{f}_0 \in \mathcal{S}_0(Q)$ given by

$$\mathbf{f}_0 = \left((1 - \zeta)V_0, 2(1 - \zeta)^t \bar{V}_0 A_1 V_0, \dots, 2(1 - \zeta)^t \bar{V} A_d V_0, (1 - \zeta)^t \bar{V}_0 \left(\sum b_{0j} A_j \right), \frac{\zeta}{2} b_0 \right),$$

with $V_0 \in \mathbb{C}^n$. We then obtain the following explicit expression for lifts of stationary discs near \mathbf{f}_0 .

Theorem 3.1. *Let Q be a strongly Levi nondegenerate quadric given by (2.2). Then stationary discs $f = (h, g)$ with lifts $\mathbf{f} = (h, g, \tilde{h}, \tilde{g}) \in \mathcal{S}_0(Q)$ near \mathbf{f}_0 are exactly of the form*

$$(3.3) \quad \begin{cases} h(\zeta) = V - \zeta(I - \zeta X)^{-1}(I - X)V \\ g_j(\zeta) = {}^t \bar{V} A_j V - 2 {}^t \bar{V} A_j \zeta (I - \zeta X)^{-1}(I - X)V + \\ \quad {}^t \bar{V} (I - {}^t \bar{X}) K_j (I + 2\zeta X (I - \zeta X)^{-1})(I - X)V + {}^t \bar{V} ({}^t \bar{X} K_j - K_j X) V \end{cases}$$

where $V \in \mathbb{C}^n$ (close to V_0), $a \in \mathbb{C}^d$ is sufficiently small, X is the unique $n \times n$ matrix solution of (2.3) (with $b = b_0 - a - \bar{a}$) with $\|X\| < 1$, and $K_j = \psi^{-1}(A_j)$ is Hermitian, $j=1, \dots, d$.

Proof. Let $\mathbf{f} = (h, g, \tilde{h}, \tilde{g}) \in \mathcal{S}_0(Q)$ be a lift of stationary disc. Consider d real valued functions $c_1, \dots, c_d : \partial\Delta \rightarrow \mathbb{R}$ such that $\sum_{j=1}^d c_j(\zeta) \partial\rho_j(0) \neq 0$ for all $\zeta \in \partial\Delta$ and such that the map $\zeta \mapsto \zeta \sum_{j=1}^d c_j(\zeta) \partial\rho_j(f(\zeta), f(\zeta))$ defined on $\partial\Delta$ extends holomorphically on Δ . It follows in particular that each function c_j is of the form

$$c_j(\zeta) = a_j \bar{\zeta} + b_j + \bar{a}_j \zeta,$$

where $a_j \in \mathbb{C}$ and $b_j \in \mathbb{R}$. We set $a = (a_1, \dots, a_d)$. So the lift components of \mathbf{f} are of the form

$$(3.4) \quad \tilde{h}(\zeta) = -\zeta {}^t \bar{h}(\zeta) \left(\sum_{j=1}^d (a_j \bar{\zeta} + b_j + \bar{a}_j \zeta) A_j \right)$$

and

$$(3.5) \quad \tilde{g}(\zeta) = \frac{a + b\zeta + \bar{a}\zeta^2}{2}$$

with $b = b_0 - a - \bar{a} \in \mathbb{R}^d$ to ensure that $\tilde{g}(1) = b_0/2$.

Consider now $a \in \mathbb{C}^d$ small enough and the corresponding solution X of (2.3) with $\|X\| < 1$. Using (3.4) and Lemma 2.3, we obtain, by definition of the stationarity, that the map

$$\zeta \mapsto \zeta {}^t \bar{h}(\zeta) (I - \bar{\zeta} {}^t \bar{X}) B (I - \zeta X)$$

defined on $\partial\Delta$ extends holomorphically to the unit disc. Therefore the map

$$\zeta \mapsto \zeta(I - \bar{\zeta} \bar{X})\overline{h(\zeta)}$$

extends holomorphically to the unit disc. Writing $h(\zeta) = \sum_{j=0}^{\infty} \alpha_j \zeta^j$, this implies directly that $\alpha_j = X^{j-1} \alpha_1$ for $j \geq 2$ and so the component h is precisely of the form

$$h(\zeta) = h(0) + \zeta(I - \zeta X)^{-1} h'(0).$$

Since $h(1) = 0$ we obtain $h'(0) = -(I - X)h(0)$, and so the first part of (3.3) follows with $V = h(0)$. The form of the component g is obtained by using the fact that the disc is attached to the quadric Q and thus satisfies $\Re g_j = {}^t \bar{h} A_j h$ for $j = 1, \dots, d$. The computation is straightforward and leads to

$$\begin{aligned} g_j(\zeta) &= {}^t \bar{V} A_j V - 2 {}^t \bar{V} A_j \zeta (I - \zeta X)^{-1} (I - X) V + \\ & {}^t \bar{V} (I - {}^t \bar{X}) K_j (I + 2\zeta X (I - \zeta X)^{-1}) (I - X) V + i y_j \end{aligned}$$

with $y_j \in \mathbb{R}$ and $K_j = \psi^{-1}(A_j)$, where Ψ is defined in (3.2). Finally, since $g(1) = 0$ and using the fact that $\psi(K_j) = A_j$, we obtain

$$i y_j = {}^t \bar{V} ({}^t \bar{X} K_j - K_j X) V.$$

This achieves the proof of the theorem. \square

Remark 3.2. The above theorem shows that, for a strongly Levi nondegenerate quadric given by (2.2), $\mathcal{S}_0(Q)$ is parametrized by $a \in \mathbb{C}^d$ and $V \in \mathbb{C}^n$ near \mathbf{f}_0 , that is, $2n + 2d = 2N$ real parameters. That result was obtained via an implicit function theorem in [9] and explicitly in the case of strongly pseudoconvex quadric with generating Levi form in [28]. Also, note that in case $a = (0, \dots, 0) \in \mathbb{C}^d$, one recovers the special family of lifts given by (3.1).

In the above theorem, although it is important that the parameter $a \in \mathbb{C}^d$ is sufficiently small, no condition is needed for the parameter $V \in \mathbb{C}^n$. We only require V close to a given V_0 to make sure that the constructed family of lifts is in a neighborhood of the initial disc \mathbf{f}_0 .

3.2. Nondefective stationary discs. In what follows, we discuss the notion of defect of a stationary disc and its relation with Definition 2.4. Following [4], a stationary disc f is *defective* if it admits a lift $\mathbf{f} = (f, \tilde{f}) : \Delta \rightarrow T^*\mathbb{C}^N$ such that $1/\zeta \cdot \mathbf{f} = (f, \tilde{f}/\zeta)$ is holomorphic on Δ . The disc is *nondefective* in case it is not defective. For a quadric $Q \subset \mathbb{C}^N$ of the form (2.2), in view of (3.4) and (3.5), a stationary disc $f = (h, g)$ for Q is defective if there exists $c = (c_1, \dots, c_d) \in \mathbb{R}^d \setminus \{0\}$ such that the map

$$\zeta \mapsto c \partial_z \rho(f(\zeta)) = \sum_{j=1}^d c_j \partial_z \rho_j(f(\zeta)) = -{}^t \bar{h}(\zeta) \sum_{j=1}^d c_j A_j$$

defined on $\partial\Delta$ extends holomorphically on Δ . In [10], we observed that a stationary disc of the form

$$f = ((1 - \zeta)V, 2(1 - \zeta) {}^t \bar{V} A_1 V, \dots, 2(1 - \zeta) {}^t \bar{V} A_d V),$$

where $V \in \mathbb{C}^n$ and with lift of the special form (3.1), is nondefective if and only if Q is stationary minimal at 0 for $(0, b, V)$ (see Lemma 3.3 in [10]). In general, we have

Proposition 3.3. *Let Q be a strongly Levi nondegenerate quadric given by (2.2) and let $b_0 \in \mathbb{R}^d$ be such that $\sum_{j=1}^d b_{0j} A_j$ is invertible. Consider a stationary disc f of the form (3.3) with lift $\in \mathcal{S}_0(Q)$. The following statements are equivalent:*

- i. The disc f is nondefective.*
- ii. The quadric Q is stationary minimal at 0 for $(a, b_0 - a - \bar{a}, h'(0))$.*
- iii. The quadric Q is stationary minimal at 0 for $(a, b_0 - a - \bar{a}, h'(1))$.*
- iv. The quadric Q is stationary minimal at 0 for $(a, b_0 - a - \bar{a}, h(0))$.*

Proof. The equivalence of the statements *ii.*, *iii.* and *iv.* follows from Remark 2.5 and the fact that $h'(0) = -(I - X)V$ and $h'(1) = -(I - X)^{-1}V$ and $h(0) = V$. We then prove that *i.* implies *iv.*. Assume that there exist $(\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d \setminus \{(0, 0, \dots, 0)\}$ such that $\sum_{j=1}^d \lambda_j A_j X^r V = 0$ for all $r = 0, 1, 2, \dots$. We claim that the disc f is defective since it admits a lift $\mathbf{f} = (h, g, 0, \zeta \lambda_1/2, \dots, \zeta \lambda_d/2)$ such that $1/\zeta \cdot \mathbf{f}$ is holomorphic on Δ . Indeed, we have

$$\begin{aligned} {}^t \overline{h(\zeta)} \sum_{j=1}^d \lambda_j A_j &= \underbrace{{}^t \overline{V} \sum_{j=1}^d \lambda_j A_j - \bar{\zeta} {}^t \overline{V} (I - \bar{\zeta} {}^t \overline{X})^{-1} (I - {}^t \overline{X}) \sum_{j=1}^d \lambda_j A_j}_{=0} \\ &= -\bar{\zeta} \sum_{r=0}^{\infty} \bar{\zeta}^r \underbrace{{}^t \overline{V} {}^t \overline{X}^r \sum_{j=1}^d \lambda_j A_j}_{=0} + \bar{\zeta} \sum_{r=0}^{\infty} \bar{\zeta}^r \underbrace{{}^t \overline{V} {}^t \overline{X}^{r+1} \sum_{j=1}^d \lambda_j A_j}_{=0} = 0. \end{aligned}$$

We now prove that *ii.* implies *i.*. Assume that f is defective. There exist $\lambda_1, \dots, \lambda_d \in \mathbb{R}$ such that ${}^t \overline{h(\zeta)} \sum_{j=1}^d \lambda_j A_j$ extends holomorphically on Δ . Set $\tilde{V} = (I - X)V$. Since

$${}^t \overline{h(\zeta)} \sum_{j=1}^d \lambda_j A_j = {}^t \overline{V} \sum_{j=1}^d \lambda_j A_j - \sum_{r=0}^{\infty} \bar{\zeta}^{r+1} {}^t \overline{(X^r \tilde{V})} \sum_{j=1}^d \lambda_j A_j$$

we have ${}^t \overline{(X^r \tilde{V})} \sum_{j=1}^d \lambda_j A_j = 0$ for all $r = 0, 1, 2, \dots$ which shows that Q is not stationary minimal at 0 for $(a, b_0 - a - \bar{a}, \tilde{V})$. \square

4. 1-JET DETERMINATION OF STATIONARY DISCS

Let Q be a strongly Levi nondegenerate quadric given by (2.2). Consider the 1-jet map

$$j_1 : \mathbf{f} \mapsto (\mathbf{f}(1), \mathbf{f}'(1))$$

at $\zeta = 1$. We focus on lifts $\mathbf{f} \in \mathcal{S}_0(Q)$. Since $\mathbf{f}(1) = (0, 0, 0, b_0/2)$ where $b_0 \in \mathbb{R}^d$ is fixed, we identify the 1-jet map with the derivative map $\mathbf{f} \mapsto \mathbf{f}'(1)$ at $\zeta = 1$. This map may be expressed explicitly in view of Theorem 3.1 and Remark 3.2:

Proposition 4.1. *In the context of Theorem 3.1, the 1-jet map*

$$j_1 : \mathbb{C}^d \times \mathbb{C}^n \rightarrow \mathbb{C}^n \times \mathbb{R}^d \times \mathbb{C}^d$$

at $\zeta = 1$ is given by

$$(a, V) \mapsto \mathbf{f} \mapsto (h'(1), g'(1), \tilde{g}'(1)) = \left(-(I - X)^{-1}V, -2 {}^t\bar{V}K_1V, \dots, -2 {}^t\bar{V}K_dV, \frac{1}{2}(b_0 - 2i\Im ma) \right).$$

Note that the component

$$\tilde{h}'(1) = {}^t\overline{h'(1)} \left(\sum_{j=1}^d (a_j + b_j + \bar{a}_j) A_j \right) = {}^t\overline{h'(1)} \left(\sum_{j=1}^d b_{0j} A_j \right)$$

is omitted since the information it carries is redundant due to the invertibility of the matrix $\sum_{j=1}^d b_{0j} A_j$.

Proof of Proposition 4.1. We first note that the 1-jet map of $\zeta(I - \zeta X)^{-1}$ at $\zeta = 1$ is given by $(I - X)^{-2}$. It follows directly that using the form of h given by (3.3), we obtain

$$h'(1) = -(I - X)^{-1}V$$

and

$$g'_j(1) = 2 {}^t\bar{V}(-A_j + K_j X - {}^t\bar{X}K_j X)(I - X)^{-1}V = -2 {}^t\bar{V}K_j V$$

for $j = 1, \dots, d$. We also have, using the expression (3.5),

$$\tilde{g}'(1) = \frac{1}{2}(b_0 + \bar{a} - a).$$

This achieves the proof of the lemma. \square

After performing changes of variables in both the source and the target spaces, the 1-jet map j_1 from the previous corollary may be written as

$$(4.1) \quad j_1 : (a, V) \mapsto (V, {}^t\bar{V}(I - {}^t\bar{X})K_1(I - X)V, \dots, {}^t\bar{V}(I - {}^t\bar{X})K_d(I - X)V, \Im ma).$$

Due to the form of the differential map of j_1 at $(a, V) \in \mathbb{C}^d \times \mathbb{C}^n$ we obtain directly

Corollary 4.2. *The 1-jet map j_1 is a local diffeomorphism at $(a, V) \in \mathbb{C}^d \times \mathbb{C}^n$ if and only if the following $d \times d$ matrix*

$$\left(\frac{\partial}{\partial \Re a_s} {}^t\bar{V}(I - {}^t\bar{X})K_j(I - X)V \right)_{j,s}$$

is invertible.

In what follows, we investigate the invertibility of that matrix. We denote by A_H the Hermitian part of a square matrix A , namely

$$A_H := \frac{1}{2}(A + {}^t\bar{A}).$$

Note that $\psi^{-1}(A_H) = (\psi^{-1}(A))_H$, where ψ is defined by (3.2). We need the following lemma.

Lemma 4.3. *Let Q be a strongly Levi nondegenerate quadric given by (2.2) and let $b_0 \in \mathbb{R}^d$ be such that $\sum_{j=1}^d b_{0j}A_j$ is invertible. Let $a \in \mathbb{C}^d$ be small enough and let X be the unique $n \times n$ matrix solution of (2.3) (with $b = b_0 - a - \bar{a}$) such that $\|X\| < 1$. Then for any $s = 1, \dots, d$, we have*

$$(4.2) \quad \frac{\partial}{\partial \Re a_s} (I - {}^t\bar{X})K_j(I - X) = -2 \left((I - {}^t\bar{X})^2 \psi^{-1}(K_j X_{\Re a_s}) \right)_H.$$

The term $(I - {}^t\bar{X})^2 \psi^{-1}(K_j X_{\Re a_s})$ depends on a , and in particular we have, for $a = 0$,

$$\left(I - {}^t\bar{X}(0) \right)^2 K_j(0) X_{\Re a_s}(0) = -A_j \left(\sum_{k=1}^d b_{0k} A_k \right)^{-1} A_s.$$

Proof. Since $\psi(K_j) = A_j$, we have

$$\frac{\partial}{\partial \Re a_s} (I - {}^t\bar{X})K_j(I - X) = \frac{\partial}{\partial \Re a_s} (2K_j - K_j X - {}^t\bar{X}K_j) = 2 \frac{\partial}{\partial \Re a_s} (K_j - K_j X)_H$$

and

$$\psi \left(\frac{\partial}{\partial \Re a_s} K_j \right) = 2 ({}^t\bar{X} K_j X_{\Re a_s})_H.$$

Inverting ψ leads to

$$\frac{\partial}{\partial \Re a_s} K_j = 2\psi^{-1} ({}^t\bar{X} K_j X_{\Re a_s})_H = 2 \left({}^t\bar{X} \underbrace{\psi^{-1}(K_j X_{\Re a_s})}_{=: B} \right)_H.$$

We also have

$$\frac{\partial}{\partial \Re a_s} K_j X = \left(\frac{\partial}{\partial \Re a_s} K_j \right) X + K_j X_{\Re a_s}.$$

It follows that

$$\begin{aligned} \frac{\partial}{\partial \Re a_s} (I - {}^t\bar{X})K_j(I - X) &= 2 \frac{\partial}{\partial \Re a_s} (K_j - K_j X)_H \\ &= 2 \left(2({}^t\bar{X}B)_H - 2({}^t\bar{X}B)_H X - K_j X_{\Re a_s} \right)_H \\ &= 2 \left(2{}^t\bar{X}B - {}^t\bar{B}X^2 - \underbrace{{}^t\bar{X}BX - K_j X_{\Re a_s}}_{=: B} \right)_H \\ &= 2 \left(2{}^t\bar{X}B - {}^t\bar{X}^2 B - B \right)_H \\ &= -2 \left((I - {}^t\bar{X})^2 B \right)_H. \end{aligned}$$

This concludes the proof of (4.2). The proof of the second statement of Lemma 4.3 follows directly from Lemma 2.9 and the fact that ψ is the identity when $a = 0$. \square

As a direct consequence, we obtain

Theorem 4.4. *Let Q be a strongly Levi nondegenerate quadric given by (2.2) and let $b_0 \in \mathbb{R}^d$ be such that $\sum_{j=1}^d b_{0j}A_j$ is invertible. Then the 1-jet map j_1 (4.1) is a local diffeomorphism at $(0, V)$ if and only if the $d \times d$ matrix*

$$\Re e \left({}^t\bar{V}A_j \left(\sum_{k=1}^d b_{0k}A_k \right)^{-1} A_s V \right)_{j,s}$$

is invertible. In other words, the 1-jet map j_1 is a local diffeomorphism at $(0, V)$ if and only if the quadric Q is \mathfrak{D} -nondegenerate (with V).

Proof. According to Lemma 4.3, we have for any $s = 1, \dots, d$,

$$\frac{\partial}{\partial \Re a_s} ((I - {}^t\bar{X})K_j(I - X))(0) = 2 \left(A_j \left(\sum_{k=1}^d b_{0k}A_k \right)^{-1} A_s \right)_H.$$

Now note that for any $V \in \mathbb{C}^n$ and any $n \times n$ matrix, we have

$$(4.3) \quad {}^t\bar{V}A_H V = \Re e({}^t\bar{V}AV).$$

Thus

$$\frac{\partial}{\partial \Re a_s} {}^t\bar{V}(I - {}^t\bar{X})K_j(I - X)V = 2\Re e \left({}^t\bar{V}A_j \left(\sum_{k=1}^d b_{0k}A_k \right)^{-1} A_s V \right).$$

□

We want to emphasize that the \mathfrak{D} -nondegeneracy of Q (see Definition 2.2) is not a purely technical condition. In fact, it is important to note that Theorem 4.4 shows the geometric and adapted nature of this nondegeneracy condition. In general, it is important to find necessary and sufficient conditions (more trackable and geometric than the one given in Corollary 4.2) to ensure that the 1-jet map j_1 is a local diffeomorphism. In the next theorem, we show that the stationary minimality of the quadric Q is necessary. In a forthcoming paper, we will address and study the question of the sufficient condition.

Theorem 4.5. *Let Q be a strongly Levi nondegenerate quadric given by (2.2) and let $b_0 \in \mathbb{R}^d$ be such that $\sum_{j=1}^d b_{0j}A_j$ is invertible. Assume that the 1-jet map j_1 (4.1) is a local diffeomorphism at some (a_0, V) , for a_0 sufficiently small. Then Q is stationary minimal at 0 for $(a_0, b_0 - a_0 - \bar{a}_0, V)$.*

Proof. Assume that the 1-jet map j_1 is a local diffeomorphism at (a_0, V) and suppose by contradiction that Q is not stationary minimal at 0 for $(a_0, b_0 - a_0 - \bar{a}_0, V)$. According to Remark 2.5, Q is not stationary minimal at 0 for $(a_0, b_0 - a_0 - \bar{a}_0, V')$ where $V' = (I - X)^2V$, where X is the unique $n \times n$ matrix solution of (2.3) (with $b = b_0 - a_0 - \bar{a}_0$) such that $\|X\| < 1$. In particular, there exists $W \in \mathbb{R}^d \setminus \{0\}$ such that for all integer

$r \geq 0$, $D_r W = 0$, where D_r is the $n \times d$ matrix whose j^{th} column is $A_j X^r V'$. According to Lemma 4.3 and (4.3), we may rewrite $-\frac{1}{2} \frac{\partial}{\partial \Re a_s} {}^t \bar{V} (I - {}^t \bar{X}) K_j (I - X) V$ as follows

$$\begin{aligned} {}^t \bar{V} \left(\sum_{r=0}^{\infty} {}^t \bar{X}^r (I - {}^t \bar{X})^2 K_j X_{\Re a_s} X^r \right)_{\mathcal{H}} V &= \Re \left(\sum_{r=0}^{\infty} {}^t \bar{V}' {}^t \bar{X}^r K_j X_{\Re a_s} X^r V \right) \\ &= \sum_{r=0}^{\infty} \sum_{l=0}^{\infty} \Re \left({}^t \bar{V}' {}^t \bar{X}^{r+l} A_j X^l X_{\Re a_s} X^r V \right) \end{aligned}$$

It follows that ${}^t W \left(\frac{\partial}{\partial \Re a_s} {}^t \bar{V} (I - {}^t \bar{X}) K_j (I - X) V \right)_{j,s}$ only involves terms of the form ${}^t W {}^t \bar{D}_{r+l}$ or ${}^t W {}^t D_{r+l}$ and is thus equal to zero. According to Corollary 4.2, this is a contradiction. \square

5. FILLING PROPERTIES OF STATIONARY DISCS

Let Q be a strongly Levi nondegenerate quadric given by (2.2). In this section, we consider the center evaluation map

$$\Psi : \mathbf{f} \mapsto f(0) = (h(0), g(0)),$$

where $\mathbf{f} \in \mathcal{S}_0(Q)$. We obtain immediately from Theorem 3.1 the following explicit expression of Ψ :

Proposition 5.1. *In the context of Theorem 3.1, the center evaluation map*

$$\Psi : \mathbb{C}^d \times \mathbb{C}^n \rightarrow \mathbb{C}^n \times \mathbb{C}^d$$

at $\zeta = 0$ is given by

$$(a, V) \mapsto \mathbf{f} \mapsto f(0) = (V, {}^t \bar{V} 2K_j (I - X) V).$$

According to that explicit form, we have

Corollary 5.2. *The center evaluation map Ψ is a local diffeomorphism at $(a, V) \in \mathbb{C}^d \times \mathbb{C}^n$ if and only if the following $d \times d$ matrix*

$$\left(\frac{\partial}{\partial \Re a_s} {}^t \bar{V} K_j (I - X) V \right)_{j,s}$$

is invertible.

In the next theorem, we investigate the invertibility of that matrix in the case of a strongly Levi nondegenerate quadric for $a = 0$ and any $V \in \mathbb{C}^n$. We have

Theorem 5.3. *Let Q be a strongly Levi nondegenerate quadric given by (2.2) and let $b_0 \in \mathbb{R}^d$ be such that $\sum_{j=1}^d b_{0j} A_j$ is invertible. Then the center evaluation map Ψ is a local diffeomorphism at $(0, V)$ if and only if the $d \times d$ matrix*

$$\left({}^t \bar{V} {}^t \bar{A}_j \left(\sum_{k=1}^d b_{0k} A_k \right)^{-1} A_s V \right)_{j,s}$$

is invertible.

It is remarkable that in Theorem 5.3, the condition under which the center evaluation map is a local diffeomorphism at $(0, V)$ is precisely the invertibility condition in ii. of Definition 2.2 of full nondegeneracy. This illustrates the relevance of this notion of nondegeneracy and its relation with the geometric properties of stationary discs with lift of the form (3.1).

Proof of Theorem 5.3. Using Lemma 2.9 and the proof of Lemma 4.3, we obtain for any $j, s = 1, \dots, d$,

$$\begin{aligned} \frac{\partial}{\partial \Re a_s} (K_j(I - X))(0) &= \frac{\partial K_j}{\partial \Re a_s}(0) - A_j \frac{\partial X}{\partial \Re a_s}(0) \\ &= \underbrace{2\psi^{-1} \left({}^t \bar{X} K_j X_{\Re a_s} \right)_H(0)}_{=0} + A_j \left(\sum_{k=1}^d b_{0k} A_k \right)^{-1} A_s \end{aligned}$$

and the proof follows. \square

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