ADDENDUM TO "STATIONARY DISCS AND FINITE JET DETERMINATION FOR NON-DEGENERATE GENERIC REAL SUBMANIFOLDS"

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Abstract. We previously introduced a new notion of non-degeneracy for generic real submanifolds in $\mathbb{C}^N$. The definition is however not complete and the purpose of this addendum is to complete it.

In the paper [1], we introduced the notion of *full non-degeneracy* of a generic real submanifold $M \subset \mathbb{C}^N$ (Definition 1.2 [1]). However, that definition must be completed as follows. Let $M \subset \mathbb{C}^N$ be a $\mathcal{C}^4$ generic real submanifold of real codimension $d \geq 1$ through $p = 0$. After a local biholomorphic change of coordinates, we assume that $M \subset \mathbb{C}^N = \mathbb{C}^n_z \times \mathbb{C}^d_w$ is given locally by

$$\begin{cases} r_1 = \Re e w_1 - {}^t\bar{z}A_1 z + O(3) = 0 \\ \quad\vdots \\ r_d = \Re e w_d - {}^t\bar{z}A_d z + O(3) = 0 \end{cases} \tag{0.1}$$

where $A_1, \ldots, A_d$ are Hermitian matrices of size n.

Definition 1. A $\mathcal{C}^4$ generic real submanifold M of $\mathbb{C}^N$ of codimension d given by (0.1) is *fully non-degenerate* at 0 if the following two conditions are satisfied

($\mathfrak{t}$) there exists a real linear combination $A := \sum_{j=1}^d c_j A_j$ that is invertible,

($\mathfrak{f}$) there exists $V \in \mathbb{C}^n$ such that if we set D to be the $n \times d$ matrix whose j^{th} column is $A_j V$, and $B = {}^t\overline{D}A^{-1}D$, then B and $\Re e B$ are invertible.

We discuss now the connection with Definition 1.2 [1]. The assumption ($\mathfrak{f}$) in Definition 1.2 [1] was the existence of a vector $V \in \mathbb{C}^n$ such that $W := \mathrm{span}_{\mathbb{C}}\{A_1V, \ldots, A_dV\}$ is of dimension d. While this is still needed, we also need two more conditions, namely

i. the Hermitian matrix A^{-1} is non-degenerate on W,

ii. the matrix $\Re e({}^t\overline{D}A^{-1}D)$ is invertible.

where D is the $n \times d$ matrix whose j^{th} column is A_jV. We point out that condition ($\mathfrak{f}$) in Definition 1.2 [1] together with the above condition i. is equivalent to the invertibility of ${}^t\overline{D}A^{-1}D$. Moreover, we note that in $\mathbb{C}^4$ the two notions of full non-degeneracy coincide. We now give an example of a quadric which is not fully non-degenerate in the present sense but which, however, is in the sense of Definition 1.2 [1].

Example 1. *The quadric in* $\mathbb{C}^9$ *given by*

$$\begin{cases} \Re e w_1 = |z_1|^2 \\ \Re e w_2 = |z_2|^2 \\ \Re e w_3 = 2\Re e(z_1\overline{z_4} + z_2\overline{z_3}) \end{cases}$$

satisfies the conditions of Definition 1.2 [1] *but is not fully non-degenerate at the origin.*

It is important to note that all results in [1] remain valid under the present notion of full non-degeneracy. More precisely, only two lemmas must be taken care of, namely Lemma 3.4 and Lemma 3.7 [1]. The proof of Lemma 3.4 [1] is unchanged until Equation (3.9) p. 929 which is in fact

$$g(\zeta) = -4\Re e\left(\overline{B_1}A^{-1}X\right) + 2\overline{B_1}A^{-1}Y - 2\overline{B_1}A^{-1}Y\zeta.$$

Accordingly, we have

$$\overline{B_1}A^{-1}Y \;=\; 2\underbrace{\begin{pmatrix} \overline{(A_1)^1 V} & \dots & \overline{(A_1)^n V} \\ \vdots & & \vdots \\ \overline{(A_d)^1 V} & \dots & \overline{(A_d)^n V} \end{pmatrix}}_{{}^t\overline{D}} A^{-1} \underbrace{\begin{pmatrix} {}^tV\overline{(A_1)_1} & \dots & {}^tV\overline{(A_d)_1} \\ \vdots & & \vdots \\ {}^tV\overline{(A_1)_n} & \dots & {}^tV\overline{(A_d)_n} \end{pmatrix}}_{D} \begin{pmatrix} \overline{a_1} \\ \vdots \\ \overline{a_d} \end{pmatrix} = 0.$$

Since ${}^t\overline{D}A^{-1}D$ is invertible, we obtain $a_1 = \ldots = a_d = 0$. The rest of the proof follows exactly the same lines as the original proof of Lemma 3.4 [1].

As for Lemma 3.7 [1], its proof relies heavily on the one of Lemma 3.4 [1]. We have

$$\begin{cases} \tilde{g}_j = a_j - \overline{a_j}\zeta, \;\; a_j \in \mathbb{C} \\ h = A^{-1}(X + Y\zeta), \\ g(\zeta) = -4\Re e\left(\overline{B_1}A^{-1}X\right) + 2\overline{B_1}A^{-1}Y - 2\overline{B_1}A^{-1}Y\zeta. \end{cases}$$

Since $\boldsymbol{f}$ has a trivial 1-jet at 1, we must have

$$\begin{cases} a_j \in \mathbb{R} \\ X = -Y \\ \Re e\left(\overline{B_1}A^{-1}X\right) = \Re e\left(\overline{B_1}A^{-1}Y\right) = 0. \end{cases}$$

It follows that

$$\Re e\left(\overline{B_1}A^{-1}Y\right) = \Re e({}^t\overline{D}A^{-1}D)\begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix} = 0,$$

and since $\Re e({}^t\overline{D}A^{-1}D)$ is invertible, we have $a_1 = \ldots = a_d = 0$. The rest of the proof follows exactly the same lines as the original proof of Lemma 3.7 [1].

References

[1] F. Bertrand, L. Blanc-Centi, F. Meylan, *Stationary discs and finite jet determination for non-degenerate generic real submanifolds*, Adv. Math. **343** (2019), 910-934.

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