

STATIONARY DISCS AND FINITE JET DETERMINATION FOR NON-DEGENERATE GENERIC REAL SUBMANIFOLDS

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ABSTRACT. Let $M \subset \mathbb{C}^N$ be a generic real submanifold of class \mathcal{C}^4 . In case M is Levi non-degenerate in the sense Tumanov, we construct stationary discs for M . If furthermore M satisfies an additional non-degeneracy condition, we apply the method of stationary discs to obtain 2-jet determination of CR automorphisms of M .

INTRODUCTION

An important part of the theory of functions of several complex variables going back to the pioneering work of Poincaré [32], is the understanding of biholomorphic equivalence of domains. In dimension $N = 1$, the Riemann mapping theorem asserts that domains have only topological invariants. A similar statement fails in higher dimension, which is connected with the fact that smooth boundaries of domains in \mathbb{C}^N , $N \geq 2$, possess infinitely many local biholomorphic invariants.

The simplest test case is the real hyperquadric Q given in \mathbb{C}^N by the equation

$$\Re w = \sum_{j=1}^{N-1} \pm |z_j|^2, \quad (z, w) \in \mathbb{C}^{N-1} \times \mathbb{C}.$$

From the infinite-dimensional family of all local biholomorphic self-maps of \mathbb{C}^N , only a finite-dimensional subfamily preserves Q . That family, known as the (*biholomorphic automorphism group*) of Q plays a fundamental role in understanding automorphism groups of more general real hypersurfaces satisfying a non-degeneracy condition called *Levi non-degeneracy*. For a real submanifold $M \subset \mathbb{C}^N$ and a point $p \in M$, we denote by $\text{Aut}(M, p)$ the set of germs of biholomorphic maps fixing p and such that $F(M) \subset M$. We recall here the classical statement contained in the work of Chern and Moser [16].

Theorem 0.1. [16] *If a real hypersurface $M \subset \mathbb{C}^N$ is real analytic Levi non-degenerate at a point $p \in M$, then elements of $\text{Aut}(M, p)$ are uniquely determined by their jets of order two at p .*

Let us remark that finite jet determination for holomorphic mappings between *real analytic* real hypersurfaces has attracted considerable attention these past years. For a survey, we refer for instance to the articles of Zaitsev [39] or Baouendi, Ebenfelt and Rothschild [3]. We also point out the works of Ebenfelt [18], Ebenfelt and Lamel [19], Kolar, Zaitsev and the third author [28] in case of \mathcal{C}^∞ real hypersurfaces. Finally the case of finitely smooth real hypersurfaces has been studied in [8, 9, 11].

The aim of this paper is to study the finite jet determination problem for finitely smooth real submanifolds $M \subset \mathbb{C}^N$ of codimension $d > 1$ using the method developed

by the first two authors in [8]. The present work is motivated in particular by the works of Beloshapka [7] and of the second and third authors [14]. More precisely, according to Beloshapka [7], if $M \subset \mathbb{C}^N$ is real analytic generic submanifold of codimension d and Levi non-degenerate at $p \in M$ in the sense of Beloshapka then elements of $\text{Aut}(M, p)$ are uniquely determined by their jets of order two at p . We emphasize that various not equivalent definitions for the generalization to real submanifolds $M \subset \mathbb{C}^N$ of codimension $d > 1$ of the notion of Levi non-degeneracy condition for a hypersurface have been introduced in the literature [4], [7], [35]. See also [13] for a survey of these different notions.

Important work on finite jet determination for holomorphic mappings between real submanifolds of higher codimension has also been done; we refer for instance to the articles of Zaitsev [38], Baouendi, Ebenfelt and Rotschild [1], Baouendi, Mir and Rotschild [5], Lamel and Mir [29], Juhlin [25], Juhlin and Lamel [26] for the real analytic case and Kim and Zaitsev [27] for the C^∞ case. We now state our main result.

Theorem 0.2. *Let $M \subset \mathbb{C}^N$ be a C^4 generic real submanifold. Assume that M is fully non-degenerate at $p \in M$. Then any germ at p of CR automorphism of M of class C^3 is uniquely determined by its jet of order two at p .*

We refer to Definition 1.2 for the notion of fully non-degenerate submanifold; this notion is closely related to the non-degeneracy condition introduced by Tumanov in [35]. Moreover, in case $M \subset \mathbb{C}^4$ the fully non-degeneracy condition is equivalent to the non-degeneracy condition in the sense of Beloshapka, and thus we have

Corollary 0.3. *Let $M \subset \mathbb{C}^4$ be a C^4 generic real submanifold. Assume that M is Levi non-degenerate at $p \in M$ in the sense of Beloshapka. Then any germ at p of CR automorphism of M of class C^3 is uniquely determined by its jet of order two at p .*

As a direct consequence of Theorem 0.2, we also have the following corollary.

Corollary 0.4. *Let $M \subset \mathbb{C}^N$ be a C^4 generic real submanifold. Assume that M is fully non-degenerate at $p \in M$. Then elements of $\text{Aut}(M, p)$ are uniquely determined by its jet of order two at p .*

Our approach is based on the study of an important family of invariant objects attached to real submanifolds, namely the stationary discs. These particular holomorphic discs were first introduced by Lempert in [30] as extremal discs for the Kobayashi metric of bounded smooth strongly convex domains in \mathbb{C}^n . The description of such discs and their applications were later on developed in more general settings such as strictly pseudoconvex hypersurfaces by Huang [24] and Pang [31] or in higher codimension by Tumanov [35]; see also the work of Sukhov and Tumanov [34] who construct stationary discs attached to small perturbations of $\mathbb{S}^3 \times \mathbb{S}^3 \subset \mathbb{C}^4$, where \mathbb{S}^3 denotes the unit sphere in \mathbb{C}^2 . Recently, the method of stationary discs has been particularly adapted to the study of jet determination problems for finitely smooth real hypersurfaces [8, 9, 11]. In order to prove Theorem 0.2, we construct stationary discs attached to M ; the idea of attaching such a disc to M is a boundary value problem, namely a nonlinear Riemann-Hilbert type problem. Inspired by the work of Forstnerič [20] and of Globevnik [21, 22] on analytic discs attached to totally real submanifolds, we analyze the existence and the structure

of solutions of such nonlinear Riemann-Hilbert type problems (see Theorem 2.1 and its Corollary 2.2). In order to study geometric properties of stationary discs, it is essential for the method developed in [8] to have the description of all stationary discs attached to the model hyperquadric. Our approach, instead, is based on the choice of a "good" initial stationary disc and is therefore more subtle. Finally we emphasize that the fully non-degeneracy condition imposed on M is natural in our context since it ensures that one can construct "enough" stationary discs with "good" geometric properties.

The paper is organized as follows. In section 1, we introduce and discuss the notion of fully non-degeneracy condition for a generic real submanifold in the light of the various conditions for non-degenerate real submanifolds introduced by Beloshapka and Tumanov. We also introduce the spaces of functions needed for the rest of the sequel and recall the definition and properties of stationary discs. In section 2, we first construct an initial stationary disc for the the quadric part of the generic real submanifold that will enable us to construct a "big enough" family of stationary discs. In section 3, we discuss crucial geometric properties of this family that will allow us to prove Theorem 0.2 in section 4.

1. PRELIMINARIES

We denote by Δ the unit disc in \mathbb{C} , by $\partial\Delta$ its boundary, and by $\mathbb{B} \subset \mathbb{C}^N$ the unit ball in \mathbb{C}^N .

1.1. Non-degenerate generic real submanifolds. Let $M \subset \mathbb{C}^N$ be a \mathcal{C}^2 generic real submanifold of real codimension $d \geq 1$ through p . Under these hypotheses, and after a local biholomorphic change of coordinates, we may assume that $p = 0$ and that $M \subset \mathbb{C}^N = \mathbb{C}_z^n \times \mathbb{C}_w^d$ is given locally by the following system of equations

$$(1.1) \quad \begin{cases} r_1 = \Re w_1 - {}^t \bar{z} A_1 z + O(3) = 0 \\ \vdots \\ r_d = \Re w_d - {}^t \bar{z} A_d z + O(3) = 0 \end{cases}$$

where A_1, \dots, A_d are Hermitian matrices of size n (see [2] and Section 7.2 [15] for more details). In the remainder $O(3)$, z is of weight one and $\Im w$ of weight two. We set $r = (r_1, \dots, r_d)$.

We recall the following definition introduced by Beloshapka in [6, 7].

Definition 1.1. A \mathcal{C}^2 generic real submanifold M of \mathbb{C}^N of codimension d and given by (1.1) is *Levi non-degenerate* at 0 in the sense of Beloshapka if the following two conditions are both satisfied

- (a) A_1, \dots, A_d are linearly independent (equivalently on \mathbb{R} or \mathbb{C})
- (b) $\bigcap_{j=1}^d \text{Ker} A_j = \{0\}$

The Levi non-degeneracy is a biholomorphic invariant notion (see for instance [13]). Note that the non-degeneracy condition introduced in [38, 2] is exactly condition (b). In our context, we need to work with a stronger notion of non-degeneracy.

Definition 1.2. A \mathcal{C}^2 generic real submanifold M of \mathbb{C}^N of codimension d given by Equations (1.1) is *fully non-degenerate* at 0 if the following two conditions are both satisfied

- (f) there exists $V \in \mathbb{C}^n$ such that $\text{span}_{\mathbb{C}}\{A_1V, \dots, A_dV\}$ is of dimension d ,
- (t) if there exists a real linear combination $\sum_{j=1}^d c_j A_j$ that is invertible.

Note that (f) implies that $d \leq n$ and also implies (a). We point out that in case $d = 2$ it can be checked that (f) and (a) are equivalent, but in general they are not equivalent as illustrated by the following example.

Example 1.3. *The quadric in \mathbb{C}^8 given by*

$$\begin{cases} \Re w_1 = |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 \\ \Re w_2 = |z_1|^2 \\ \Re w_3 = |z_2|^2 \\ \Re w_4 = |z_1|^2 + \Re(\bar{z}_1 z_2) \end{cases}$$

is Levi non-degenerate at the origin but does not satisfy (f).

Condition (f) is tightly related to the existence of analytic discs whose centers fill an open set (see Proposition 3.3) and determined by their 1-jet (see Proposition 3.6). Recall that condition (t) was introduced by Tumanov [35] and is essential for the construction of stationary disc (see Theorem 2.1); for this reason, we say that M is *Levi non-degenerate at 0 in the sense of Tumanov* if it satisfies condition (t). Condition (t) implies (b) but not (a), except in the hypersurface case, since one can choose A_1 invertible and $A_2 = \dots = A_d = 0$. Moreover, as pointed out in [13], the Levi non-degeneracy in the sense of Beloshapka does not necessarily imply (t). Finally we note that in case $M \subset \mathbb{C}^4$, the fully non-degeneracy condition and the Levi non-degeneracy in the sense of Beloshapka coincide.

1.2. Stationary discs. We first introduce the spaces of functions we need. Let $k \geq 0$ be an integer and let $0 < \alpha < 1$. We denote by $\mathcal{C}^{k,\alpha} = \mathcal{C}^{k,\alpha}(\partial\Delta, \mathbb{R})$ the space of real-valued functions defined on $\partial\Delta$ of class $\mathcal{C}^{k,\alpha}$. The space $\mathcal{C}^{k,\alpha}$ is endowed with its usual norm

$$\|f\|_{\mathcal{C}^{k,\alpha}} = \sum_{j=0}^k \|f^{(j)}\|_{\infty} + \sup_{\zeta \neq \eta \in \partial\Delta} \frac{\|f^{(k)}(\zeta) - f^{(k)}(\eta)\|}{|\zeta - \eta|^\alpha},$$

where $\|f^{(j)}\|_{\infty} = \max_{\partial\Delta} \|f^{(j)}\|$. We set $\mathcal{C}_{\mathbb{C}}^{k,\alpha} = \mathcal{C}^{k,\alpha} + i\mathcal{C}^{k,\alpha}$. The space $\mathcal{C}_{\mathbb{C}}^{k,\alpha}$ is equipped with the norm

$$\|f\|_{\mathcal{C}_{\mathbb{C}}^{k,\alpha}} = \|\Re f\|_{\mathcal{C}^{k,\alpha}} + \|\Im f\|_{\mathcal{C}^{k,\alpha}}$$

We denote by $\mathcal{A}^{k,\alpha}$ the subspace of *analytic discs* in $\mathcal{C}_{\mathbb{C}}^{k,\alpha}$ consisting of functions $f : \bar{\Delta} \rightarrow \mathbb{C}$, holomorphic on Δ with trace on $\partial\Delta$ belonging to $\mathcal{C}_{\mathbb{C}}^{k,\alpha}$.

Let M be a \mathcal{C}^2 generic real submanifold of \mathbb{C}^N of codimension d given by (1.1). An analytic disc $f \in (\mathcal{A}^{k,\alpha})^N$ is *attached to M* if $f(\partial\Delta) \subset M$. Following Lempert [30] and Tumanov [35] we define

Definition 1.4. A holomorphic disc $f : \Delta \rightarrow \mathbb{C}^N$ continuous up to $\partial\Delta$ and attached to M is *stationary for M* if there exists a holomorphic lift $\mathbf{f} = (f, \tilde{f})$ of f to the cotangent bundle $T^*\mathbb{C}^N$, continuous up to $\partial\Delta$ and such that for all $\zeta \in \partial\Delta$, $\mathbf{f}(\zeta) \in \mathcal{N}M(\zeta)$ where

$$(1.2) \quad \mathcal{N}M(\zeta) := \{(z, w, \tilde{z}, \tilde{w}) \in T^*\mathbb{C}^N \mid (z, w) \in M, (\tilde{z}, \tilde{w}) \in \zeta N_{(z,w)}^*M \setminus \{0\}\},$$

and where

$$N_{(z,w)}^*M = \text{span}_{\mathbb{R}}\{\partial r_1(z, w), \dots, \partial r_d(z, w)\}$$

is the conormal fiber at (z, w) of M . The set of these lifts $\mathbf{f} = (f, \tilde{f})$, with f non-constant, is denoted by $\mathcal{S}(M)$.

Note that equivalently, an analytic disc $f \in (\mathcal{A}^{k,\alpha})^N$ attached to M is stationary for M if there exists d real valued functions $c_1, \dots, c_d : \partial\Delta \rightarrow \mathbb{R}$ such that $\sum_{j=1}^d c_j(\zeta) \partial r_j(0) \neq 0$ for all $\zeta \in \partial\Delta$ and such that the map

$$\zeta \mapsto \zeta \sum_{j=1}^d c_j(\zeta) \partial r_j(f(\zeta), \overline{f(\zeta)})$$

defined on $\partial\Delta$ extends holomorphically on Δ .

The set of such small discs is invariant under CR automorphisms; recall that if F is a CR automorphism of M and f an analytic disc attached to M then the map $F \circ f$ defined on $\partial\Delta$ extends holomorphically to Δ (see Proposition 6.2.2 in [2] or Theorem 1 p. 200 in [15]). Moreover recall the following essential result due to Webster [37] in the hypersurface case and to Tumanov [35] for higher codimension submanifolds.

Proposition 1.5 ([35]). *Let M be a \mathcal{C}^2 generic real submanifold of \mathbb{C}^N of codimension d with local defining function of the form (1.1). Then M is Levi non-degenerate at 0 in the sense of Tumanov if and only if the conormal bundle N^*M is totally real at $(0, \sum_{j=1}^d c_j \partial r_j(0))$, where the c_1, \dots, c_d are such that $\sum_{j=1}^d c_j A_j$ is invertible.*

We end this section with an important remark on the smoothness of stationary discs. Let M be a \mathcal{C}^4 generic real submanifold of \mathbb{C}^N of codimension d with local defining function of the form (1.1). Assume that M is Levi non-degenerate at 0 in the sense of Tumanov (t). Consider a lift of stationary disc $\mathbf{f} = (f, \tilde{f})$ for M satisfying $\mathbf{f}(1) = (0, \sum_{j=1}^d c_j \partial r_j(0))$ where $\sum_{j=1}^d c_j A_j$ is invertible. It follows from Proposition 1.5 and from Chirka (Theorem 33 in [17]) that such discs are of class $\mathcal{C}^{2,\alpha}$ for any $0 < \alpha < 1$ near $\zeta = 1$.

1.3. Partial indices and Maslov index. The construction of stationary discs attached to a generic real submanifold of \mathbb{C}^N (see Theorem 2.1) relies on a non-linear Riemann-Hilbert problem whose study is related to certain geometric integers, namely the partial indices and the Maslov index, associated to the linearized problem. In this section, we recall the definition of these integers. We denote by $Gl_N(\mathbb{C})$ the general linear group on \mathbb{C}^N . Let $G : \partial\Delta \rightarrow Gl_N(\mathbb{C})$ be a smooth map. We consider a Birkhoff factorization (see

Section 3 [21] or [36]) of $-\overline{G}^{-1}G$ on $\partial\Delta$:

$$-\overline{G(\zeta)}^{-1}G(\zeta) = B^+(\zeta) \begin{pmatrix} \zeta^{\kappa_1} & & & (0) \\ & \zeta^{\kappa_2} & & \\ & & \ddots & \\ (0) & & & \zeta^{\kappa_N} \end{pmatrix} B^-(\zeta),$$

where $\zeta \in \partial\Delta$, $B^+ : \bar{\Delta} \rightarrow Gl_N(\mathbb{C})$ and $B^- : (\mathbb{C} \cup \infty) \setminus \Delta \rightarrow Gl_N(\mathbb{C})$ are smooth maps, holomorphic on Δ and $\mathbb{C} \setminus \bar{\Delta}$ respectively. The integers $\kappa_1, \dots, \kappa_N$ are called the *partial indices* of $-\overline{G}^{-1}G$ and their sum $\kappa := \sum_{j=1}^N \kappa_j$ the *Maslov index* of $-\overline{G}^{-1}G$. Recall that the Maslov index κ is equal to the winding number of the function

$$\zeta \mapsto \det \left(-\overline{G(\zeta)}^{-1}G(\zeta) \right)$$

at the origin ([22], see also Lemma B.1 [12] for a proof); here $\det \left(-\overline{G(\zeta)}^{-1}G(\zeta) \right)$ denotes the determinant of $-\overline{G(\zeta)}^{-1}G(\zeta)$.

2. CONSTRUCTION OF STATIONARY DISCS

In this section, we construct lifts of stationary discs attached to small perturbations of a quadric submanifold, non-degenerate in the sense of Tumanov (t). In particular we show that the set of such lifts form a finite dimensional submanifold of the Banach space of analytic discs.

2.1. Construction of stationary discs in the quadric case. Consider a quadric submanifold $Q \subset \mathbb{C}^N = \mathbb{C}_z^n \times \mathbb{C}_w^d$ of real codimension d

$$(2.1) \quad \begin{cases} \rho_1 = \Re w_1 - {}^t \bar{z} A_1 z = 0 \\ \vdots \\ \rho_d = \Re w_d - {}^t \bar{z} A_d z = 0 \end{cases}$$

where A_1, \dots, A_d are hermitian matrices of size n . We set $\rho = (\rho_1, \dots, \rho_d)$. We will determine a special family of lifts of stationary discs in $\mathcal{S}(Q)$. We suppose the quadric Q to be Levi non-degenerate at 0 in the sense of Tumanov (t). Note that the Levi non-degeneracy condition (t) is not strictly needed to determine some stationary discs for Q but will be essential later in order to construct discs attached to small perturbations of Q (see Theorem 2.1). We have

$$\begin{cases} \partial\rho_1 = (\partial_z \rho_1, \partial_w \rho_1) = \left(-{}^t \bar{z} (A_1)_1, \dots, -{}^t \bar{z} (A_1)_n, \frac{1}{2}, 0, \dots, 0 \right) \\ \vdots \\ \partial\rho_d = (\partial_z \rho_d, \partial_w \rho_d) = \left(-{}^t \bar{z} (A_d)_1, \dots, -{}^t \bar{z} (A_d)_n, 0, \dots, 0, \frac{1}{2} \right) \end{cases}$$

where $(A_j)_l$ denotes, for $j = 1, \dots, d$ and $l = 1, \dots, n$, the l^{th} column of A_j . Consider a disc $f = (h, g) \in (\mathcal{A}^{k, \alpha})^{n+d}$, with $h \in (\mathcal{A}^{k, \alpha})^n$ and $g \in (\mathcal{A}^{k, \alpha})^d$, and d real valued functions $c_1, \dots, c_d : \partial\Delta \rightarrow \mathbb{R}$ such that $\sum_{j=1}^d c_j(\zeta) \partial\rho_j(0) \neq 0$. Note that if the map

$\zeta \mapsto \zeta \sum_{j=1}^d c_j(\zeta) \partial \rho_j(f(\zeta), \overline{f(\zeta)})$ defined on $\partial \Delta$ extends holomorphically on Δ then we must have $c_j(\zeta) = a_j \bar{\zeta} + b_j + \overline{a_j} \zeta$ with $a_j \in \mathbb{C}$ and $b_j \in \mathbb{R}$. We restrict to the case where $c = (c_1, \dots, c_d)$ is a real constant vector-valued function and chosen such that $A := \sum_{j=1}^d c_j A_j$ is invertible. It follows that

$$\zeta \sum_{j=1}^d c_j(\zeta) \partial \rho_j(f(\zeta), \overline{f(\zeta)}) = \left(-\zeta {}^t \overline{h(\zeta)} A, \frac{\zeta}{2} c \right).$$

Consequently, h must be of the form

$$(2.2) \quad h(\zeta) = V + \zeta W,$$

with $V \in \mathbb{C}^n$, $W \in \mathbb{C}^n \setminus \{0\}$ and g is thus of the form

$$(2.3) \quad g_j(\zeta) = {}^t \overline{V} A_j V + {}^t \overline{W} A_j W + 2 {}^t \overline{V} A_j W \zeta + i y_j$$

with $y_j \in \mathbb{R}$, $j = 1, \dots, d$. Note that, in such case, we have

$$(2.4) \quad \tilde{h}(\zeta) = -\zeta {}^t \overline{h(\zeta)} A$$

and

$$(2.5) \quad \tilde{g}(\zeta) = \frac{\zeta}{2} c.$$

2.2. Defining equations of the conormal bundle and Riemann-Hilbert problem. Let $Q \subset \mathbb{C}^{n+d}$ be a quadric submanifold of real codimension d defined by $\{\rho = 0\}$ (2.1). Denote by $(z, w, \tilde{z}, \tilde{w})$ the coordinates on $T^* \mathbb{C}^{n+d}$, with $\tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_n)$ and $\tilde{w} = (\tilde{w}_1, \dots, \tilde{w}_d)$. The $2n + 2d$ real defining equations for $\mathcal{N}Q(\zeta)$, $\zeta \in \partial \Delta$, (see (1.2) for its definition) are given by

$$\left\{ \begin{array}{ll} \tilde{\rho}_1(\zeta)(z, w, \tilde{z}, \tilde{w}) & = \frac{w_1 + \overline{w_1}}{2} - {}^t \tilde{z} A_1 z = 0, \\ \vdots & \vdots \\ \tilde{\rho}_d(\zeta)(z, w, \tilde{z}, \tilde{w}) & = \frac{w_d + \overline{w_d}}{2} - {}^t \tilde{z} A_d z = 0, \\ \tilde{\rho}_{d+1}(\zeta)(z, w, \tilde{z}, \tilde{w}) & = \left(\tilde{z}_1 + 2 {}^t \tilde{z} \sum_{j=1}^d \tilde{w}_j (A_j)_1 \right) + \left(\overline{\tilde{z}_1} + 2 {}^t z \sum_{j=1}^d \overline{\tilde{w}_j (A_j)_1} \right) = 0, \\ \vdots & \vdots \\ \tilde{\rho}_{d+n}(\zeta)(z, w, \tilde{z}, \tilde{w}) & = \left(\tilde{z}_n + 2 {}^t \tilde{z} \sum_{j=1}^d \tilde{w}_j (A_j)_n \right) + \left(\overline{\tilde{z}_n} + 2 {}^t z \sum_{j=1}^d \overline{\tilde{w}_j (A_j)_n} \right) = 0, \\ \tilde{\rho}_{d+n+1}(\zeta)(z, w, \tilde{z}, \tilde{w}) & = i \left(\tilde{z}_1 + 2 {}^t \tilde{z} \sum_{j=1}^d \tilde{w}_j (A_j)_1 \right) - i \left(\overline{\tilde{z}_1} + 2 {}^t z \sum_{j=1}^d \overline{\tilde{w}_j (A_j)_1} \right) = 0, \\ \vdots & \vdots \\ \tilde{\rho}_{2n+d}(\zeta)(z, w, \tilde{z}, \tilde{w}) & = i \left(\tilde{z}_n + 2 {}^t \tilde{z} \sum_{j=1}^d \tilde{w}_j (A_j)_n \right) - i \left(\overline{\tilde{z}_n} + 2 {}^t z \sum_{j=1}^d \overline{\tilde{w}_j (A_j)_n} \right) = 0, \\ \tilde{\rho}_{2n+d+1}(\zeta)(z, w, \tilde{z}, \tilde{w}) & = i \frac{\tilde{w}_1}{\zeta} - i \zeta \overline{\tilde{w}_1} = 0, \\ \vdots & \vdots \\ \tilde{\rho}_{2n+2d}(\zeta)(z, w, \tilde{z}, \tilde{w}) & = i \frac{\tilde{w}_d}{\zeta} - i \zeta \overline{\tilde{w}_d} = 0, \end{array} \right.$$

where $(A_j)_l$ denotes, for $j = 1, \dots, d$ and $l = 1, \dots, n$, the l^{th} column of A_j . We set $\tilde{\rho} := (\tilde{\rho}_1, \dots, \tilde{\rho}_{2n+2d})$. For a general quadric $M = \{r = 0\}$ with $r = (r_1, \dots, r_d)$ of the form (1.1), we denote by \tilde{r} the corresponding defining functions of $\mathcal{NM}(\zeta)$. This allows to consider lifts of stationary discs as solutions of a nonlinear Riemann-Hilbert type problem. Indeed, an analytic disc $\mathbf{f} \in (\mathcal{A}^{k,\alpha})^{2n+2d}$ is the lift of a stationary disc for $M = \{r = 0\}$ if and only if

$$(2.6) \quad \tilde{r}(\mathbf{f}) = 0 \quad \text{on } \partial\Delta.$$

The next section is devoted to the study of this problem.

2.3. Construction of stationary discs. We recall that the set of lifts of non-constant stationary discs attached to a generic real submanifold $M \subset \mathbb{C}^N$ is denoted by $\mathcal{S}(M)$. We have the following theorem which is crucial in our approach of the jet determination problem.

Theorem 2.1. *Let $Q \subset \mathbb{C}^{n+d}$ be a quadric submanifold of real codimension d defined by $\{\rho = 0\}$ (2.1). Assume that Q is Levi non-degenerate at 0 in the sense of Tumanov (t). Consider an initial lift of a stationary disc, $\mathbf{f}_0 = (h_0, g_0, \tilde{h}_0, \tilde{g}_0)$ of the form (2.2), (2.3), (2.4), (2.5) where c_1, \dots, c_d are chosen such that the matrix $A := \sum_{j=1}^d c_j A_j$ is invertible. Then there exist open neighborhoods U of ρ in $(\mathcal{C}^4(\mathbb{B}))^d$ and V of 0 in \mathbb{R}^{4n+4d} , a real number $\varepsilon > 0$ and a map*

$$\mathcal{F} : U \times V \rightarrow (\mathcal{A}^{1,\alpha})^{2n+2d}$$

of class \mathcal{C}^1 such that:

- i. $\mathcal{F}(\rho, 0) = \mathbf{f}_0$,
- ii. for all $r \in U$, the map

$$\mathcal{F}(r, \cdot) : V \rightarrow \{\mathbf{f} \in \mathcal{S}(\{r = 0\}) \mid \|\mathbf{f} - \mathbf{f}_0\|_{1,\alpha} < \varepsilon\}$$

is one-to-one and onto.

As a direct corollary, we obtain

Corollary 2.2. *Under the assumptions of Theorem 2.1, there exist open neighborhoods U of ρ in $(\mathcal{C}^4(\mathbb{B}))^d$ and a real number $\varepsilon > 0$ such that*

$$\{\mathbf{f} \in \mathcal{S}(\{r = 0\}) \mid \|\mathbf{f} - \mathbf{f}_0\|_{1,\alpha} < \varepsilon\}$$

forms a \mathcal{C}^1 real submanifold of dimension $4n + 4d$ of the Banach space of analytic discs.

Remark 2.3. Working with the Banach spaces $\mathcal{C}^4(\mathbb{B})$ and $\mathcal{A}^{1,\alpha}$ is crucial for our approach, which is based on the implicit function theorem. The required smoothness is indeed necessary for the below map F (see (2.7)) to be \mathcal{C}^1 .

Remark 2.4. In [34], Sukhov and Tumanov prove Theorem 2.1 in the special case the model quadric is $\mathbb{S}^3 \times \mathbb{S}^3 \subset \mathbb{C}^4$, where \mathbb{S}^3 denotes the unit sphere in \mathbb{C}^2 (see Corollary 3.2 and Theorem 3.4 [34]). Their approach also relies on the study of the corresponding Riemann-Hilbert problem using the methods developed by [20, 21, 22]. See also [35, 33].

Proof of Theorem 2.1. In a neighborhood of (ρ, \mathbf{f}_0) in $(\mathcal{C}^4(\mathbb{B}))^d \times (\mathcal{A}^{1,\alpha})^{2n+2d}$, we define the following map between Banach spaces

$$F : (\mathcal{C}^4(\mathbb{B}))^d \times (\mathcal{A}^{1,\alpha})^{2n+2d} \rightarrow (\mathcal{C}^{1,\alpha})^{2n+2d}$$

by

$$(2.7) \quad F(r, \mathbf{f}) := \tilde{r}(\mathbf{f}).$$

Here we use the notation

$$\tilde{r}(\mathbf{f})(\zeta) = \tilde{r}(\zeta)(\mathbf{f}(\zeta)), \quad \zeta \in \partial\Delta.$$

The map F is of class \mathcal{C}^1 (see Lemma 5.1 in [23] and Lemma 6.1 and Lemma 11.2 in [21] which generalizes to $\mathcal{C}^{1,\alpha}$). Recall that an analytic disc $\mathbf{f} \in (\mathcal{A}^{1,\alpha})^{2n+2d}$ is the lift of a stationary disc for $\{r = 0\}$ if and only if it solves the nonlinear Riemann-Hilbert problem (2.6). In other words, for any fixed $r \in (\mathcal{C}^4(\mathbb{B}))^d$, the zero set of $F(r, \cdot)$ coincides with $\mathcal{S}(\{r = 0\})$. In order to prove Theorem 2.1, we apply the implicit function theorem to the map F . We need to consider the partial derivative of F with respect to $(\mathcal{A}^{1,\alpha})^{2n+2d}$ at (ρ, \mathbf{f}_0)

$$(2.8) \quad \partial_2 F(\rho, \mathbf{f}_0)\mathbf{f} = 2\Re e \left[\overline{G(\zeta)}\mathbf{f} \right]$$

where $G(\zeta)$ is the following complex valued square matrix of size $2n + 2d$

$$G(\zeta) := (\tilde{\rho}_{\bar{z}}(\mathbf{f}_0), \tilde{\rho}_{\bar{w}}(\mathbf{f}_0), \tilde{\rho}_{\bar{z}}(\mathbf{f}_0), \tilde{\rho}_{\bar{w}}(\mathbf{f}_0)).$$

Recall that the non-degeneracy in the sense of Tumanov is equivalent to the fact that the conormal bundle is totally real (see Proposition 1.5); with respect to the choice of the initial disc, this ensures that the matrix $G(\zeta)$ is invertible for all $\zeta \in \partial\Delta$. For sake of clarity, this fact will be proved again below. We need to show that (see p. 39 [22])

- i. the map $\partial_2 F(\rho, \mathbf{f}_0)$ is onto, and
- ii. the real dimension of the kernel of $\partial_2 F(\rho, \mathbf{f}_0)$ is $4n + 4d$.

2.3.1. *Surjectivity of $\partial_2 F(\rho, \mathbf{f}_0)$.* It is more convenient to reorder coordinates and consider $(w, z, \tilde{z}, \tilde{w})$ instead of $(z, w, \tilde{z}, \tilde{w})$. Accordingly, discs \mathbf{f} are of the form $(g, h, \tilde{h}, \tilde{g})$. We still denote by $G(\zeta)$ the corresponding reordered matrix, namely

$$G(\zeta) := (\tilde{\rho}_{\bar{w}}(\mathbf{f}_0), \tilde{\rho}_{\bar{z}}(\mathbf{f}_0), \tilde{\rho}_{\bar{z}}(\mathbf{f}_0), \tilde{\rho}_{\bar{w}}(\mathbf{f}_0)).$$

The matrix $G(\zeta)$ is square of size $2n + 2d$, upper block triangular and given by

$$(2.9) \quad G(\zeta) = \begin{pmatrix} \frac{1}{2}I_d & & (*) \\ & G_2(\zeta) & \\ (0) & & -i\zeta I_d \end{pmatrix},$$

where I_d denotes the identity matrix of size d and $G_2(\zeta)$ is the following square matrix of size $2n$

$$(2.10) \quad G_2 = \begin{pmatrix} 2 \sum_{j=1}^d \tilde{g}_j(A_j)_{11} & \cdots & 2 \sum_{j=1}^d \tilde{g}_j(A_j)_{n1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \\ 2 \sum_{j=1}^d \tilde{g}_j(A_j)_{1n} & \cdots & 2 \sum_{j=1}^d \tilde{g}_j(A_j)_{nn} & 0 & \cdots & 1 \\ 2i \sum_{j=1}^d \tilde{g}_j(A_j)_{11} & \cdots & 2i \sum_{j=1}^d \tilde{g}_j(A_j)_{n1} & -i & \cdots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \\ 2i \sum_{j=1}^d \tilde{g}_j(A_j)_{1n} & \cdots & 2 \sum_{j=1}^d \tilde{g}_j(A_j)_{nn} & 0 & \cdots & -i \end{pmatrix}.$$

where $\tilde{g}_j = c_j/2\zeta$ and $(A_j)_{kl}$, $k = 1, \dots, n$, $l = 1, \dots, n$, denotes the kl coefficient of A_j . Recall that the real constants c_1, \dots, c_d are chosen such that $A := \sum_{j=1}^d c_j A_j$ is invertible. It follows that

$$G_2(\zeta) = \begin{pmatrix} \zeta {}^t A & I_n \\ i\zeta {}^t A & -iI_n \end{pmatrix}$$

is invertible on $\partial\Delta$, and that, accordingly, so is $G(\zeta)$. Due to the expression of $G(\zeta)$, in order to show its surjectivity, it is enough to show that the map

$$L_1 : (\mathcal{A}^{1,\alpha})^{2n} \rightarrow (\mathcal{C}^{1,\alpha})^{2n}$$

defined by $L_1 = 2\Re e \left[\overline{G_2(\zeta)} \cdot \right]$ is surjective. For this purpose, we will show that the partial indices k_1, \dots, k_{2n} of $-\overline{G_2^{-1}}G_2$ are nonnegative (see [21] or Section 4 in [22]). Right multiplication by the constant matrix $\begin{pmatrix} {}^t A^{-1} & 0 \\ 0 & I_n \end{pmatrix}$ does not change the partial indices, and gives us the matrix

$$\begin{pmatrix} \zeta I_n & I_n \\ i\zeta I_n & -iI_n \end{pmatrix}$$

After permuting rows and columns, which also does not change the partial indices, we obtain

$$(2.11) \quad G_2^b = \begin{pmatrix} R & & 0 \\ & \ddots & \\ 0 & & R \end{pmatrix}, \text{ with } R(\zeta) = \begin{pmatrix} \zeta & 1 \\ i\zeta & -i \end{pmatrix}.$$

By a direct computation we have

$$-\overline{(G_2^b)^{-1}}G_2^b = \begin{pmatrix} P & & 0 \\ & \ddots & \\ 0 & & P \end{pmatrix} \text{ with } P(\zeta) = -\begin{pmatrix} 0 & \zeta \\ \zeta & 0 \end{pmatrix},$$

which, for instance, decomposes as which decomposes as

$$(2.12) \quad P(\zeta) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that the partial indices of $-\overline{(G_2^b)^{-1}}G_2^b$ are all equal to one and that, therefore, the map $\partial_2 F(\rho, \mathbf{f}_0)$ is onto.

2.3.2. *Kernel of $\partial_2 F(\rho, \mathbf{f}_0)$.* Recall (see [21] or Section 5 in [22]) that the real dimension of the kernel of $\partial_2 F(\rho, \mathbf{f}_0)$ is given by $\kappa + 2n + 2d$, where κ is the Maslov index of $-\overline{G(\zeta)^{-1}}G(\zeta)$ and is equal to the winding number of the function $\zeta \mapsto \det \left(-\overline{G(\zeta)^{-1}}G(\zeta) \right)$ at the origin. A direct computation - using the form of $G(\zeta)$ and in particular $G_2(\zeta)$ - shows that $\kappa = 2n + 2d$. □

3. GEOMETRIC PROPERTIES OF STATIONARY DISCS

In [8], the first two authors describe geometric properties of stationary discs attached to small deformations of a given Levi non-degenerate hyperquadric; it is important for that approach to have an explicit description of all stationary discs attached to the model hypersurface. In our case, however, it does not seem possible to obtain an explicit description of all stationary discs attached to the model quadric in general, and therefore, it is more subtle to obtain that centers of stationary discs fill an open set (Proposition 3.3) and that their lifts are determined by their 1-jet at 1 (Proposition 3.6).

3.1. Discs with pointwise constraints. In Section 3.2, we wish to show that centers of stationary discs fill an open set. This will be achieved by showing that the map $\mathbf{f} \mapsto f(0) \in \mathbb{C}^{n+d}$ restricted to a smaller family of discs is a diffeomorphism onto its image. For dimensional reasons, it is essential to impose pointwise constraints on stationary discs in order to obtain a $2n + 2d$ real dimensional family (see Theorem 3.1).

We consider a quadric submanifold $Q \subset \mathbb{C}^{n+d}$ of real codimension d given by $\{\rho = 0\}$ (2.1), Levi non-degenerate at 0 in the sense of Tumanov (t) and an initial lift of a stationary disc $\mathbf{f}_0 = (h_0, g_0, \tilde{h}_0, \tilde{g}_0)$ of the form (2.2), (2.3), (2.4), (2.5) where c_1, \dots, c_d are chosen such that the matrix $A := \sum_{j=1}^d c_j A_j$ is invertible. We wish to show that for r close enough to ρ and for some positive ε , the set

$$\left\{ \mathbf{f} \in \mathcal{S}(\{r = 0\}), \|\mathbf{f} - \mathbf{f}_0\|_{1,\alpha} < \varepsilon, f(1) = 0, \tilde{g}(1) = \left(\frac{c_1}{2}, \dots, \frac{c_d}{2} \right) \right\}$$

is a real submanifold of dimension $2n + 2d$ of the Banach space of analytic discs. To this end we introduce spaces of analytic discs with prescribed pointwise constraints. We denote by $\mathcal{A}_0^{1,\alpha}$ the subspace of $\mathcal{C}_{\mathbb{C}}^{1,\alpha}$ of functions of the form $(1 - \zeta)f$, with $f \in \mathcal{A}^{1,\alpha}$. We equip $\mathcal{A}_0^{1,\alpha}$ with the following norm

$$(3.1) \quad \|(1 - \zeta)f\|_{\mathcal{A}_0^{1,\alpha}} = \|f\|_{\mathcal{C}_{\mathbb{C}}^{1,\alpha}}$$

which makes it a Banach space and isomorphic to $\mathcal{A}^{1,\alpha}$. We also denote by $\mathcal{C}_0^{1,\alpha}$ the subspace of $\mathcal{C}^{1,\alpha}$ of functions of the form $(1 - \zeta)v$ with $v \in \mathcal{C}_{\mathbb{C}}^{1,\alpha}$. The space $\mathcal{C}_0^{1,\alpha}$ is equipped with the norm

$$\|(1 - \zeta)f\|_{\mathcal{C}_0^{1,\alpha}} = \|f\|_{\mathcal{C}_{\mathbb{C}}^{1,\alpha}}.$$

Note that $\mathcal{C}_0^{k,\alpha}$ is a Banach space.

Recall that the initial lift of stationary disc $\mathbf{f}_0 = (h_0, g_0, \tilde{h}_0, \tilde{g}_0)$ that we are considering is of the form (2.2), (2.3), (2.4), (2.5) where $c = (c_1, \dots, c_d)$ is chosen such that the matrix

$\sum_{j=1}^d c_j A_j$ is invertible. Assume furthermore that $W = -V$; more explicitly

$$(3.2) \quad \mathbf{f}_0 = \left((1 - \zeta)V, 2(1 - \zeta)^t \bar{V} A_1 V, \dots, 2(1 - \zeta)^t \bar{V} A_d V, (1 - \zeta)^t \bar{V} A, \frac{\zeta}{2} c \right).$$

We define the affine space \mathcal{A} to be the subset of $(\mathcal{A}^{1,\alpha})^{2n+2d}$ of discs of the form

$$\left((1 - \zeta)h, (1 - \zeta)g, (1 - \zeta)\tilde{h}, (1 - \zeta)\tilde{g} + \frac{\zeta}{2}c \right),$$

and we set

$$\mathcal{S}_0(\{r = 0\}) = \mathcal{S}(\{r = 0\}) \cap \mathcal{A}.$$

Notice that $\mathbf{f}_0 \in \mathcal{S}_0(\{r = 0\})$. We have

Theorem 3.1. *Let $Q \subset \mathbb{C}^{n+d}$ be a quadric submanifold of real codimension d given by (2.1), Levi non-degenerate at 0 in the sense of Tumanov (t). Let $\mathbf{f}_0 = (h_0, g_0, \tilde{h}_0, \tilde{g}_0)$ a lift of a stationary disc for Q of the form (3.2). Then there exist open neighborhoods U of ρ in $(\mathcal{C}^4(\mathbb{B}))^d$ and V of 0 in \mathbb{R}^{2n+2d} , a real number $\varepsilon > 0$ and a map*

$$\mathcal{F}_0 : U \times V \rightarrow \mathcal{A}$$

of class \mathcal{C}^1 such that:

- i. $\mathcal{F}_0(\rho, 0) = \mathbf{f}_0$,
- ii. for all $r \in U$ the map

$$\mathcal{F}_0(r, \cdot) : V \rightarrow \{\mathbf{f} \in \mathcal{S}_0(\{r = 0\}) \mid \|\mathbf{f} - \mathbf{f}_0\|_{\mathcal{A}_0^{1,\alpha}} < \varepsilon\}$$

is one-to-one and onto.

We will need the following theorem from [10].

Theorem 3.2 (Theorem 2.4 [10]). *Let $G : \partial\Delta \rightarrow GL_N(\mathbb{C})$ be a smooth map of the form*

$$G(\zeta) = \begin{pmatrix} G_1(\zeta) & & & (*) \\ & G_2(\zeta) & & \\ & & \ddots & \\ (0) & & & G_r(\zeta) \end{pmatrix},$$

where $G_j(\zeta) \in GL_{N_j}(\mathbb{C})$ for all $j = 1, \dots, r$, for all $\zeta \in \partial\Delta$. Let $0 < \alpha < 1$. Consider the following operator

$$L : (\mathcal{A}_0^{1,\alpha})^N \rightarrow (\mathcal{C}_0^{1,\alpha})^N$$

defined by

$$L(\mathbf{f}) = 2\Re [\overline{G}\mathbf{f}].$$

For $j = 1, \dots, r$ we denote by κ_l^j , $l = 1, \dots, N_j$, the partial indices of $-\overline{G_j}^{-1}G_j$ and by κ the Maslov index of $-\overline{G}^{-1}G$. We have

- (i) If $\kappa_l^j \geq 0$ for all $l = 1, \dots, N_j$ and $j = 1, \dots, r$ then the map L is onto.
- (ii) Assume that L is onto. Then the kernel of L has real dimension κ .

Proof of Theorem 3.1. The proof is basically a repetition of the one of Theorem 2.1. We consider the corresponding map between Banach spaces

$$F_0 : (\mathcal{C}^4(\mathbb{B}))^d \times (\mathcal{A}_0^{1,\alpha})^{2n+2d} \rightarrow (\mathcal{C}_0^{1,\alpha})^{2n+2d}$$

defined by

$$(3.3) \quad F_0(r, \mathbf{f}) := \tilde{r} \left(h, g, \tilde{h}, \tilde{g} + \frac{\zeta}{2}c \right),$$

where $\mathbf{f} = (h, g, \tilde{h}, \tilde{g})$. Its differential $\partial_2 F_0(\rho, \mathbf{f}_0^*)$ is still of the form (2.8) where G is given by (2.9); here \mathbf{f}_0^* denotes the disc

$$(3.4) \quad \mathbf{f}_0^* = ((1 - \zeta)V, 2(1 - \zeta)^t \bar{V} A_1 V, \dots, 2(1 - \zeta)^t \bar{V} A_d V, (1 - \zeta)^t \bar{V} A, 0).$$

The surjectivity and real dimension of its kernel are now insured by Theorem 3.2; more precisely, strictly following the notations of Theorem 3.2, we have $r = 3$, $G_1(\zeta) = \frac{1}{2}I_d$ with d indices all equal to zero, $G_2(\zeta)$ given by (2.10) with $2n$ indices all equal to 1 and $G_3(\zeta) = -i\zeta I_d$ with d indices all equal to two. Therefore the differential $\partial_2 F_0(\rho, \mathbf{f}_0^*)$ is surjective and the real dimension of its kernel is $\kappa = 2n + 2d$. \square

3.2. Filling an open set. We now prove that in case the quadric Q is fully non-degenerate at 0, the family of stationary discs covers an open set of \mathbb{C}^{n+d} . We point out that the fact that Q satisfies condition (f), namely that there exists $V \in \mathbb{C}^n$ such that $\text{span}_{\mathbb{C}}\{A_1 V, \dots, A_d V\}$ is of dimension d , is essential here.

Proposition 3.3. *Let $Q \subset \mathbb{C}^{n+d}$ be a quadric submanifold of real codimension d given by (2.1), fully non-degenerate at 0. Consider an initial disc \mathbf{f}_0 of the form*

$$\mathbf{f}_0 = \left((1 - \zeta)V, 2(1 - \zeta)^t \bar{V} A_1 V, \dots, 2(1 - \zeta)^t \bar{V} A_d V, (1 - \zeta)^t \bar{V} A, \frac{\zeta}{2}c \right)$$

where V is given by (f) and $c = (c_1, \dots, c_d)$ is chosen such that the matrix $\sum_{j=1}^d c_j A_j$ is invertible. Then there exist an open neighborhood U of ρ in $(\mathcal{C}^4(\mathbb{B}))^d$ and a positive ε such that for all $r \in U$ the set

$$\{f(0) \mid \mathbf{f} \in \mathcal{S}_0(\{r = 0\}), \|\mathbf{f} - \mathbf{f}_0\|_{\mathcal{A}_0^{1,\alpha}} < \varepsilon\}$$

contains an open set of \mathbb{C}^{n+d} .

We will prove that there exist an open neighborhood U of ρ in $(\mathcal{C}^4(\mathbb{B}))^d$ and a positive ε such that for all $r \in U$ the map

$$\Psi_r : \{\mathbf{f} \in \mathcal{S}_0(\{r = 0\}), \|\mathbf{f} - \mathbf{f}_0\|_{\mathcal{A}_0^{1,\alpha}} < \varepsilon\} \rightarrow \mathbb{C}^{n+d}$$

defined by

$$\Psi_r(\mathbf{f}) = f(0)$$

is a diffeomorphism onto its image. We consider the corresponding differential

$$d_{\mathbf{f}_0} \Psi_\rho : T_{\mathbf{f}_0} \mathcal{S}_0(Q) \rightarrow \mathbb{C}^{n+d}$$

with $d_{\mathbf{f}_0} \Psi_\rho(\mathbf{f}) = f(0)$. Recall that according to Theorem 3.1, the tangent space $T_{\mathbf{f}_0} \mathcal{S}_0(Q)$ is the kernel of the differential $\partial_2 F_0(\rho, \mathbf{f}_0^*) = 2\Re e \left[\overline{G(\zeta)} \cdot \right]$ where F_0 , \mathbf{f}_0^* and G are

respectively defined in (3.3), (3.4) and (2.9), and is of dimension $2n + 2d$. Proposition 3.3 is then a consequence of the following lemma.

Lemma 3.4. *The linear map $d_{\mathbf{f}_0} \Psi_\rho : T_{\mathbf{f}_0} \mathcal{S}_0(Q) \rightarrow \mathbb{C}^{n+d}$ is injective.*

Proof. We follow strictly the notations introduced in the proof of Theorem 2.1. Let $\mathbf{f} = (1 - \zeta)(h, g, \tilde{h}, \tilde{g})$ be an element of $T_{\mathbf{f}_0} \mathcal{S}_0(Q)$ in the kernel of $d_{\mathbf{f}_0} \Psi_\rho$, that is, satisfying $f(0) = 0$. Since the tangent space $T_{\mathbf{f}_0} \mathcal{S}_0(Q)$ is the kernel of the differential $\partial_2 F_0(\rho, \mathbf{f}_0^*) = 2\Re e \left[\overline{G(\zeta)} \cdot \right]$, we have

$$(3.5) \quad \overline{G(\zeta)} \mathbf{f} + G(\zeta) \overline{\mathbf{f}} = 0,$$

where G is of the form (2.9); more explicitly G is of the form

$$(3.6) \quad G(\zeta) = \begin{pmatrix} \frac{1}{2} I_d & B(\zeta) & 0 \\ 0 & G_2(\zeta) & C(\zeta) \\ 0 & 0 & -i\zeta I_d \end{pmatrix}, \zeta \in \partial\Delta,$$

where I_d denotes the identity matrix of size d , $B(\zeta)$ is the following matrix of size $d \times 2n$

$$(3.7) \quad B = \begin{pmatrix} -(A_1)^1 h_0(\zeta) & \dots & -(A_1)^n h_0(\zeta) & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ -(A_d)^1 h_0(\zeta) & \dots & -(A_d)^n h_0(\zeta) & 0 & \dots & 0 \end{pmatrix} = (1 - \zeta) B_1,$$

and $C(\zeta)$ is the following matrix of size $2n \times d$

$$C = \begin{pmatrix} 2^t h_0(\overline{A_1}_1) & \dots & 2^t h_0(\overline{A_d}_1) \\ \vdots & & \vdots \\ 2^t h_0(\overline{A_1}_n) & \dots & 2^t h_0(\overline{A_d}_n) \\ -2i^t h_0(\overline{A_1}_1) & \dots & -2i^t h_0(\overline{A_d}_1) \\ \vdots & & \vdots \\ -2i^t h_0(\overline{A_1}_n) & \dots & -2i^t h_0(\overline{A_d}_n) \end{pmatrix}$$

where we recall that $(A_j)_l$ denotes, for $j = 1, \dots, d$ and $l = 1, \dots, n$, the l^{th} column of A_j . The last d rows of Equation (3.5) are of the form

$$i\bar{\zeta}(1 - \zeta)\tilde{g}_j(\zeta) - i\zeta\overline{(1 - \zeta)\tilde{g}_j(\zeta)} = 0, \quad j = 1, \dots, n,$$

from which it follows that

$$\tilde{g}_j = a_j - \bar{a}_j \zeta, \quad a_j \in \mathbb{C}.$$

Solving the system backward, the previous $2n$ rows of Equation (3.5) are of the form

$$\overline{G_2(\zeta)} \begin{pmatrix} (1 - \zeta)h \\ (1 - \zeta)\tilde{h} \end{pmatrix} + G_2(\zeta) \begin{pmatrix} \overline{(1 - \zeta)h} \\ \overline{(1 - \zeta)\tilde{h}} \end{pmatrix} + \overline{C(\zeta)}(1 - \zeta)\tilde{g} + C(\zeta)\overline{(1 - \zeta)\tilde{g}} = 0.$$

Following the operations that lead to consider G_2^\flat (see Equation (2.11)), we obtain n systems

$$\overline{R(\zeta)} \begin{pmatrix} (1 - \zeta)h_k \\ (1 - \zeta)\tilde{h}_k \end{pmatrix} + R(\zeta) \begin{pmatrix} \overline{(1 - \zeta)h_k} \\ \overline{(1 - \zeta)\tilde{h}_k} \end{pmatrix} + \overline{C_k(\zeta)}(1 - \zeta)\tilde{g} + C_k(\zeta)\overline{(1 - \zeta)\tilde{g}} = 0$$

with the abuse of notation $h = {}^t\bar{A}h$ and where

$$C_k = \begin{pmatrix} 2^t h_0(\overline{A_1})_k & \dots & 2^t h_0(\overline{A_d})_k \\ -2i {}^t h_0(A_1)_k & \dots & -2i {}^t h_0(A_d)_k \end{pmatrix}.$$

Equivalently, we have

$$\begin{aligned} \begin{pmatrix} (1-\zeta)h_k \\ (1-\zeta)\tilde{h}_k \end{pmatrix} &= -\overline{R(\zeta)^{-1}}R(\zeta) \begin{pmatrix} \overline{(1-\zeta)h_k} \\ (1-\zeta)\tilde{h}_k \end{pmatrix} - \overline{R(\zeta)^{-1}} \left(\overline{C_k(\zeta)}(1-\zeta)\tilde{g} + C_k(\zeta)\overline{(1-\zeta)\tilde{g}} \right) \\ &= -\begin{pmatrix} 0 & \zeta \\ \zeta & 0 \end{pmatrix} \begin{pmatrix} \overline{(1-\zeta)h_k} \\ (1-\zeta)\tilde{h}_k \end{pmatrix} - \overline{R(\zeta)^{-1}} \left(\overline{C_k(\zeta)}(1-\zeta)\tilde{g} + C_k(\zeta)\overline{(1-\zeta)\tilde{g}} \right). \end{aligned}$$

Recall that

$$-\begin{pmatrix} 0 & \zeta \\ \zeta & 0 \end{pmatrix} = \Theta^{-1} \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix} \bar{\Theta}$$

with

$$\Theta = \begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}.$$

Therefore

$$\Theta \begin{pmatrix} (1-\zeta)h_k \\ (1-\zeta)\tilde{h}_k \end{pmatrix} = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix} \overline{\begin{pmatrix} (1-\zeta)h_k \\ (1-\zeta)\tilde{h}_k \end{pmatrix}} - \Theta \overline{R(\zeta)^{-1}} \left(\overline{C_k(\zeta)}(1-\zeta)\tilde{g} + C_k(\zeta)\overline{(1-\zeta)\tilde{g}} \right).$$

Note that

$$\Theta \overline{R(\zeta)^{-1}} = \frac{1}{2} \begin{pmatrix} -(1-\zeta) & i(1+\zeta) \\ i(1+\zeta) & 1-\zeta \end{pmatrix}.$$

Setting $\begin{pmatrix} u_k \\ \tilde{u}_k \end{pmatrix} = \Theta \begin{pmatrix} h_k \\ \tilde{h}_k \end{pmatrix}$, we have

$$\begin{pmatrix} (1-\zeta)u_k \\ (1-\zeta)\tilde{u}_k \end{pmatrix} = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix} \begin{pmatrix} \overline{(1-\zeta)u_k} \\ (1-\zeta)\tilde{u}_k \end{pmatrix} - \Theta \overline{R(\zeta)^{-1}} \left(\overline{C_k(\zeta)}(1-\zeta)\tilde{g} + C_k(\zeta)\overline{(1-\zeta)\tilde{g}} \right).$$

and therefore, dividing by $1-\zeta$,

$$\begin{pmatrix} u_k \\ \tilde{u}_k \end{pmatrix} = -\begin{pmatrix} \overline{u_k} \\ \tilde{u}_k \end{pmatrix} - \Theta \overline{R(\zeta)^{-1}} \left(\overline{C_k(\zeta)}\tilde{g} - C_k(\zeta)\overline{\zeta\tilde{g}} \right),$$

that is

$$\begin{pmatrix} u_k + \overline{u_k} \\ \tilde{u}_k + \overline{\tilde{u}_k} \end{pmatrix} = -\Theta \overline{R(\zeta)^{-1}} \left(\overline{C_k(\zeta)}\tilde{g} - C_k(\zeta)\overline{\zeta\tilde{g}} \right).$$

A direct computation shows that

$$\overline{C_k(\zeta)}\tilde{g} - C_k(\zeta)\overline{\zeta\tilde{g}} = \begin{pmatrix} v_k(\zeta) - \overline{\zeta v_k(\zeta)} \\ i(v_k(\zeta) + \zeta v_k(\zeta)) \end{pmatrix}$$

where

$$v_k(\zeta) = 2 \sum_{j=1}^d {}^t \overline{h_0(A_j)}_k \tilde{g}_j = 2 \sum_{j=1}^d {}^t \overline{V(A_j)}_k (1 - \bar{\zeta})(a_j - \bar{a}_j \zeta),$$

and

$$-\Theta \overline{R(\zeta)}^{-1} \left(\overline{C_k(\zeta)} \tilde{g} - C_k(\zeta) \zeta \tilde{g} \right) = \begin{pmatrix} v_k(\zeta) + \overline{v_k(\zeta)} \\ -i \left(v_k(\zeta) - \overline{v_k(\zeta)} \right) \end{pmatrix}.$$

Therefore

$$\begin{cases} u_k + \overline{u_k} = v_k(\zeta) + \overline{v_k(\zeta)} \\ \tilde{u}_k + \overline{\tilde{u}_k} = -i \left(v_k(\zeta) - \overline{v_k(\zeta)} \right) \end{cases}$$

from which it follows that

$$\begin{cases} u_k = 4 \sum_{j=1}^d \Re e({}^t \overline{V(A_j)_k}) \Re e(a_j) + i y_k - 4 \sum_{j=1}^d \Re e({}^t \overline{V(A_j)_k}) \overline{a_j} \zeta \\ \tilde{u}_k = 4 \sum_{j=1}^d \Im m({}^t \overline{V(A_j)_k}) \Re e(a_j) + i \tilde{y}_k - 4 \sum_{j=1}^d \Im m({}^t \overline{V(A_j)_k}) \overline{a_j} \zeta \end{cases}$$

where $y_k, \tilde{y}_k \in \mathbb{R}$, and thus, since $h_k = 1/2(u_k - i\tilde{u}_k)$,

$$(3.8) \quad h_k(\zeta) = 2 \sum_{j=1}^d {}^t \overline{V(A_j)_k} \Re e(a_j) + \frac{\tilde{y}_k}{2} + i \frac{y_k}{2} - 2 \sum_{j=1}^d {}^t \overline{V(A_j)_k} \overline{a_j} \zeta.$$

We write $h = X + Y\zeta$. Now, the first d rows of Equation (3.5) give rise to

$$\frac{1}{2}((1 - \zeta)g + \overline{(1 - \zeta)g}) = -\overline{B(\zeta)}(1 - \zeta)h - B(\zeta)\overline{(1 - \zeta)h}$$

and so

$$\begin{aligned} \frac{1}{2}(g - \overline{\zeta}g) &= -\overline{B(\zeta)}h + B(\zeta)\zeta\overline{h} \\ &= -(1 - \overline{\zeta})(\overline{B_1}h + B_1\overline{h}). \end{aligned}$$

Recall that B and B_1 are defined in (3.7); we use the inconsequential abuse of notation $B = (B, 0)$ since B just acts on the h component of the discs. This implies directly that

$$(3.9) \quad g(\zeta) = -4\Re e(\overline{B_1}X) + 2\overline{B_1}Y - 2\overline{B_1}Y\zeta.$$

Now since $h(0) = 0$ and $g(0) = 0$ we have from (3.8) and (3.9)

$$\begin{cases} X = 0 \\ \overline{B_1}Y = 0. \end{cases}$$

We will prove using the second set of equations, namely $\overline{B_1}Y = 0$, that $a_1 = \dots = a_d = 0$.

We have

$$\begin{aligned} \overline{B_1}Y &= 2 \begin{pmatrix} \overline{(A_1)^1 V} & \dots & \overline{(A_1)^n V} \\ \vdots & & \vdots \\ \overline{(A_d)^1 V} & \dots & \overline{(A_d)^n V} \end{pmatrix} \begin{pmatrix} \sum_{j=1}^d {}^t \overline{V(A_j)_1} \overline{a_j} \\ \vdots \\ \sum_{j=1}^d {}^t \overline{V(A_j)_n} \overline{a_j} \end{pmatrix} \\ &= 2 \underbrace{\begin{pmatrix} \overline{(A_1)^1 V} & \dots & \overline{(A_1)^n V} \\ \vdots & & \vdots \\ \overline{(A_d)^1 V} & \dots & \overline{(A_d)^n V} \end{pmatrix}}_{D_1} \underbrace{\begin{pmatrix} {}^t \overline{V(A_1)_1} & \dots & {}^t \overline{V(A_d)_1} \\ \vdots & & \vdots \\ {}^t \overline{V(A_1)_n} & \dots & {}^t \overline{V(A_d)_n} \end{pmatrix}}_{D_2} \begin{pmatrix} \overline{a_1} \\ \vdots \\ \overline{a_d} \end{pmatrix}. \end{aligned}$$

Recall that we have assumed Q to satisfy condition (f) and thus $d \leq n$; in case $d > n$, D_1 and D_2 have rank less than $d-1$ and thus the $d \times d$ matrix $D_1 D_2$ cannot be invertible. In such case, the linear map $d_{\mathbf{f}} \Psi_\rho : T_{\mathbf{f}_0} \mathcal{S}_0(Q) \rightarrow \mathbb{C}^{n+d}$ is not injective; note also that this will also be the case if A_1, \dots, A_d are not linearly independent. Note that $D_1 = {}^t \overline{D_2}$. Since V is given by condition (f), it follows that D_1 is of rank d and therefore $D_1 D_2 = {}^t \overline{D_2} D_2$ is positive definite. It follows that $a_1 = \dots = a_d = 0$. Since $X = 0$, we also have $y_1 = \dots = y_d = \tilde{y}_1 = \dots = \tilde{y}_d = 0$. This proves that $d_{\mathbf{f}} \Psi_\rho : T_{\mathbf{f}_0} \mathcal{S}_0(Q) \rightarrow \mathbb{C}^{n+d}$ is injective. \square

Remark 3.5. Proposition 3.3 provides an open set $O \subset \mathbb{C}^{n+d}$ such that

$$O \subset \{f(0) \mid \mathbf{f} \in \mathcal{S}_0(\{r_t = 0\}), \|\mathbf{f} - \mathbf{f}_0\|_{\mathcal{A}_0^{1,\alpha}} < \varepsilon\}.$$

It follows directly from the proof that for any point $q \in O$, there exists a unique lift of stationary disc \mathbf{f} such that $f(0) = q$.

3.3. Injectivity of the jet map. Consider the linear jet map

$$j_1 : (\mathcal{A}^{1,\alpha})^{2n+2d} \rightarrow \mathbb{C}^{2(2n+2d)}$$

mapping \mathbf{f} to its 1-jet at $\zeta = 1$, namely

$$j_1(\mathbf{f}) = \left(\mathbf{f}(1), \frac{\partial \mathbf{f}}{\partial \zeta}(1) \right) \in \mathbb{C}^{2(2n+2d)}.$$

Proposition 3.6. *Let $Q \subset \mathbb{C}^{n+d}$ be a quadric submanifold of real codimension d given by (2.1), fully non-degenerate at 0. Consider an initial disc \mathbf{f}_0 of the form*

$$\mathbf{f}_0 = \left((1 - \zeta)V, 2(1 - \zeta) {}^t \overline{V} A_1 V, \dots, 2(1 - \zeta) {}^t \overline{V} A_d V, (1 - \zeta) {}^t \overline{V} A, \frac{\zeta}{2} c \right)$$

where V is given by (f) and $c = (c_1, \dots, c_d)$ is chosen such that the matrix $\sum_{j=1}^d c_j A_j$ is invertible. Then there exist an open neighborhood U of ρ in $(\mathbb{C}^4(\mathbb{B}))^d$ and a positive ε such that for all $r \in U$ the map j_1 is injective on $\{\mathbf{f} \in \mathcal{S}_0(\{r = 0\}) \mid \|\mathbf{f} - \mathbf{f}_0\|_{1,\alpha} < \varepsilon\}$; in other words, such discs are determined by their 1-jet at 1.

Note that by the implicit function theorem, it is enough to prove that the restriction of j_1 to the tangent space $T_{\mathbf{f}_0} \mathcal{S}_0(Q)$ of $\mathcal{S}_0(Q)$ at the point \mathbf{f}_0 is injective. Moreover, the tangent space $T_{\mathbf{f}_0} \mathcal{S}_0(Q)$ is the kernel of the differential $\partial_2 F_0(\rho, \mathbf{f}_0^*) = 2\Re \left[\overline{G(\zeta)} \cdot \right]$ where F_0 , \mathbf{f}_0^* and G are respectively defined in (3.3), (3.4) and (2.9). Proposition 3.6 is then a consequence of the following lemma.

Lemma 3.7. *The restriction of j_1 to the kernel of $2\Re \left[\overline{G(\zeta)} \cdot \right]$ is injective.*

Proof. We follow strictly the notations introduced in the proofs of Theorem 2.1 and Lemma 3.4. The main step of the proof is to describe the kernel of $2\Re \left[\overline{G(\zeta)} \cdot \right]$; this was exactly done in Lemma 3.4. Let $\mathbf{f} = (1 - \zeta)(h, g, \tilde{h}, \tilde{g})$ be an element of the kernel

of $2\Re \left[\overline{G(\zeta)} \cdot \right]$ with trivial 1-jet at 1. We have

$$\begin{cases} \tilde{g}_j = a_j - \bar{a}_j \zeta, & a_j \in \mathbb{C} \\ h = X + Y \zeta \\ g(\zeta) = -4\Re(\bar{B}_1 X) + 2\bar{B}_1 Y - 2\bar{B}_1 Y \zeta. \end{cases}$$

Since \mathbf{f} has a trivial 1-jet at 1, we must have

$$\begin{cases} a_j \in \mathbb{R} \\ X = -Y \\ \Re(\bar{B}_1 X) = 0. \end{cases}$$

It follows that

$$\bar{B}_1 X = -\bar{B}_1 Y = -{}^t \bar{D}_2 D_2 \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix} \in \mathbb{R}$$

and since ${}^t \bar{D}_2 D_2$ is positive definite, we must have $a_1 = \dots = a_d = 0$. This implies directly that $g = \tilde{g} = 0$ and $h = 0$. Equation (3.8) implies that $y_k = \tilde{y}_k = 0$ and thus $u_k = \tilde{u}_k = 0$ which finally leads to $\tilde{h} = 0$. \square

4. JET DETERMINATION OF CR AUTOMORPHISMS

Let k be a positive integer. Let $M \subset \mathbb{C}^N$ be a \mathcal{C}^4 generic real submanifold and let $p \in M$. We denote by $\text{Aut}^k(M, p)$ the set of germs at p of CR automorphisms F of M of class \mathcal{C}^k ; in particular we have $F(p) = p$ and $F(M) \subset M$.

Theorem 4.1. *Let $M \subset \mathbb{C}^N$ be a \mathcal{C}^4 generic real submanifold. Assume that M is fully non-degenerate at $p \in M$. Then elements of $\text{Aut}^3(M, p)$ are uniquely determined by their 2-jet at p .*

In order to prove Theorem 4.1, we need a technical lemma. Let $M \subset \mathbb{C}^N$ be a \mathcal{C}^4 generic real submanifold given by $\{r = 0\}$ (1.1), and let Q be its associated quadric part defined by $\{\rho = 0\}$ (2.1). We recall the following anisotropic dilation $\Lambda_t : \mathbb{C}^{n+d} \rightarrow \mathbb{C}^{n+d}$ given by

$$\Lambda_t(z, w) = (tz, t^2w).$$

We set $M_t = \Lambda_t^{-1}(M)$, $r_t = \frac{1}{t^2}r \circ \Lambda_t$ and $F_t = \Lambda_t^{-1} \circ F \circ \Lambda_t$, where $F \in \text{Aut}^3(M, p)$. We also recall that for an analytic disc $\mathbf{f} = (f, \tilde{f}) \in (\mathcal{A}^{1,\alpha})^{2n+2d}$ where $0 < \alpha < 1$, we have

$$(F_t)_* \mathbf{f}(\zeta) = \left(F_t \circ f(\zeta), \tilde{f}(\zeta) (d_{f(\zeta)} F_t)^{-1} \right)$$

for $\zeta \in \Delta$, and $F_t \circ f(\zeta)$ is well defined thanks to Proposition 6.2.2 [2]. The following lemma follows from the same arguments used in Section 5.1 [8] or Section 5.1 [9].

Lemma 4.2. *We have the following:*

- i. Let U be a neighborhood of ρ in \mathcal{C}^4 topology. Then for t small enough, $r_t \in U$.*

ii. Let $F \in \text{Aut}^3(M, p)$ with a trivial 2-jet. There exists a positive constant K such that for all $\mathbf{f} = ((1 - \zeta)h, (1 - \zeta)g, (1 - \zeta)\tilde{h}, (1 - \zeta)\tilde{g} + c\zeta) \in \mathcal{A}$ we have $(F_t)_*\mathbf{f} \in \mathcal{A}$ and

$$\|(F_t)_*\mathbf{f} - \mathbf{f}\|_{\mathcal{A}_0^{1,\alpha}} < tK \left(\max\{\|h\|_{1,\alpha}, \|g\|_{1,\alpha}, \|\tilde{h}\|_{1,\alpha}, \|\tilde{g}\|_{1,\alpha}\} \right)^3.$$

Proof of Theorem 4.1. Let $M \subset \mathbb{C}^N$ be a \mathcal{C}^4 generic real submanifold, fully non-degenerate at $p \in M$. We may assume that $p = 0$ and that M is locally given by $\{r = 0\}$ (1.1). Denote by Q the associated quadric part of M defined by $\{\rho = 0\}$ (2.1). Let $F \in \text{Aut}^3(M, 0)$ with a trivial 2-jet at 0. We wish to show that F is the identity.

Since M satisfies (f) at 0, there exists $V \in \mathbb{C}^n$ such that $\text{span}_{\mathbb{C}}\{A_1V, \dots, A_dV\}$ is of dimension d . Consider an initial lift of stationary disc \mathbf{f}_0 of the form

$$\mathbf{f}_0 = \left((1 - \zeta)V, 2(1 - \zeta)^t \bar{V} A_1 V, \dots, 2(1 - \zeta)^t \bar{V} A_d V, (1 - \zeta)^t \bar{V} A, \frac{\zeta}{2} c \right)$$

where c_1, \dots, c_d are chosen such that $\sum_{j=1}^d c_j A_j$ is invertible. Denote by U the neighborhood of ρ in \mathcal{C}^4 topology obtained in Theorem 3.1. According to Lemma 4.2, for t small enough, the defining functions $r_t = \frac{1}{t^2} r \circ \Lambda_t \in U$. Reducing U , and therefore t , if necessary, Proposition 3.3 provides an open set $O \subset \mathbb{C}^{n+d}$ such that

$$O \subset \{f(0) \mid \mathbf{f} \in \mathcal{S}_0(\{r_t = 0\}), \|\mathbf{f} - \mathbf{f}_0\|_{\mathcal{A}_0^{1,\alpha}} < \varepsilon/2\}.$$

We will show that $F_t = \Lambda_t^{-1} \circ F \circ \Lambda$ is equal to the identity on the open set O . Let $q \in O$ and let \mathbf{f} be the (unique) lift of stationary disc in $\mathcal{S}_0(\{r_t = 0\})$ with $\|\mathbf{f} - \mathbf{f}_0\|_{\mathcal{A}_0^{1,\alpha}} < \varepsilon/2$ and such that $f(0) = q$. By invariance and since F_t has a trivial 2-jet, we have $(F_t)_*\mathbf{f} \in \mathcal{S}_0(\{r_t = 0\})$ and by Lemma 4.2 we have $\|(F_t)_*\mathbf{f} - \mathbf{f}_0\|_{\mathcal{A}_0^{1,\alpha}} < \varepsilon$ for t small enough. Moreover, the discs $(F_t)_*\mathbf{f}$ and \mathbf{f} have the same 1-jet. By Proposition 3.6 we have $(F_t)_*\mathbf{f} = \mathbf{f}$ and therefore $F_t \circ f(0) = f(0)$, that is $F_t(q) = q$. This achieves the proof of Theorem 4.1. \square

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