STATIONARY DISCS FOR SMOOTH HYPERSURFACES OF FINITE TYPE AND FINITE JET DETERMINATION

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ABSTRACT. We construct a finitely dimensional manifold of invariant holomorphic discs attached to a certain class of smooth pseudconvex hypersurfaces of finite type, generalizing the notion of stationary discs. The discs we construct are determined by a finite jet at a given boundary point and their centers fill an open set. As a consequence, we obtain finite jet determination of CR diffeomorphisms between smooth hypersurfaces of this class.

1. INTRODUCTION

In the celebrated paper [34], L. Lempert introduced stationary discs as holomorphic discs attached to a given hypersurface, admitting a meromorphic lift to the cotangent bundle with at most one pole of order one at 0, and attached to the conormal bundle. He proved that, on a strongly convex domain, stationary discs coincide with extremal discs for the Kobayashi metric, and constructed an analogue of the Riemann mapping in higher dimension. Stationary discs were afterwards studied for more general classes of submanifolds, namely, strictly pseudoconvex hypersurfaces by X. Huang [28], in higher codimension by A. Tumanov [40], and in the almost complex setting by B. Coupet, H. Gaussier and A. Sukhov [13].

Due to their geometric properties, stationary discs are natural invariants to study dynamical properties of self-mappings [27, 28], mapping extension problems [35, 40], and jet determination problems [6]. Their existence relies on the study of a nonlinear Riemann-Hilbert boundary problem, whose solvability is strongly related to nondegeneracy properties of the given hypersurface. In such a case, its associated conormal bundle is totally real [41], and one can construct small stationary discs nearby a given one by developing a perturbation theory by introducing some integers invariant under homotopy, namely the partial indices and the Maslov index (see [22, 23, 11]). However, under no nondegeneracy conditions, it is not clear whether or not one can ensure the existence of smooth stationary discs, and not much seems to be known in that direction.

Surprisingly, when one allows the unique pole of the meromorphic lift to be of greater order, there might exist a lot of such discs, which still form a biholomorphically invariant family. Based on this observation, we define, for a given hypersurface of finite type, the notion of k_0-stationary discs, as attached holomorphic discs admitting a meromorphic lift with at most one pole of order k_0 at 0. In order to construct k_0-stationary discs nearby a given one, the associated Riemann-Hilbert problem seems to be no longer relevant since the conormal bundle of the given hypersurface is not anymore totally real. Instead, we introduce a nonlinear operator, whose construction is essentially based on Toeplitz and Hankel operators, and whose linearization involve Fredholm operators. The properties of the associated Fredholm indices such as their invariance under homotopy, ensure the existence of nearby small k_0-stationary discs attached to perturbed hypersurface, and the number of real variables parametrizing the perturbed discs is completely determined by those indices. Our main theorem can be stated as follows:

Theorem 1.1. *Let* $M \subset \mathbb{C}^2$ *be a real smooth hypersurface of finite type whose defining function is locally written as*

$$r(z, w) = -\operatorname{Re} w + P(z, \overline{z}) + O\left(|z|^{d+1}\right) + \operatorname{Im} w \, O\left(|z, \operatorname{Im} w|^{d-1}\right),$$

where

$$P(z, \overline{z}) = \sum_{j=d-k_0}^{k_0} \alpha_j z^j \overline{z}^{d-j}, \quad \alpha_j = \overline{\alpha}_{d-j}, \quad \alpha_{k_0} \neq 0, \quad \frac{d}{2} \leq k_0 \leq d - 1$$

is a homogeneous polynomial satisfying the open condition $\{P_{z\overline{z}} = 0\} = \{0\}$. *Then there exists a* $4k_0 - d + 3$ *dimensional biholomorphically invariant Banach manifold of small* k_0-*stationary discs attached to* M.

Similarly to [6], the discs constructed in Theorem 1.1 are particularly adapted to study jet determination of CR-diffeomorphisms. As we know from results of [10, 12, 37], germs of biholomorphisms preserving a real-analytic Levi nondegenerate hypersurface M in $\mathbb{C}^n$ are uniquely determined by their 2-jet at $p \in M$. In $\mathbb{C}^2$, P. Ebenfelt, B. Lamel and D. Zaitsev [20] proved that finite jet determination of germs at a point p of local biholomorphisms preserving a real analytic hypersurface M holds if and only if M is not Levi flat at p, and obtained a 2-jet determination for real-analytic hypersurfaces of finite type (see also [2, 3, 33, 32]). In the present paper, we consider the situation of smooth hypersurfaces of finite type in $\mathbb{C}^2$. We first point out that the finite jet determination of smooth CR mappings of smooth CR submanifolds of finite type has been considered in [18, 31] (see also [19]). Their approach is based on the method of complete differential systems, introduced by C.K. Han [24, 25], and the order of the jet determination obtained in [31] is independent of the type.

However, the assumption of finite nondegeneracy, as well as the ($\mathcal{C}^\infty$) smoothness of both the mapping and the submanifold, seem to be essential for their method to work. Recently, the first co-author and L. Blanc-Centi [6] obtained a 2-jet determination for merely $\mathcal{C}^4$ Levi nondegenerate hypersurfaces in $\mathbb{C}^n$ by considering invariance - under one-sided biholomorphisms - and geometric properties of stationary discs attached to such hypersurfaces. Following this approach, we obtain:

Theorem 1.2. *Let $M \subset \mathbb{C}^2$ be a real $\mathcal{C}^{d/2+k_0+5}$ pseudoconvex hypersurface of finite type whose defining function is locally written as in Theorem 1.1. Then the germs at $p = 0 \in M$ of CR-diffeomorphisms H of class $\mathcal{C}^{d/2+k_0+3}$ such that $H(M) = M$ are uniquely determined by their $k_0 - d/2 + 2$-jet at p.*

Notice that when the essential type of M is exactly half of the type, namely when $P(z, \bar{z}) = |z|^d$, we obtain the 2-jet determination of germs of CR-diffeomorphisms at 0. It is worth emphasizing that the fact that we can treat CR-diffeomorphisms as opposed to just biholomorphisms is due to the method of stationary discs together with the one-sided holomorphic extendability of CR-diffeomorphisms [4, 39, 38]. In principle, one might expect a higher jet determination for CR-diffeomorphisms than for biholomorphisms, although they coincide in the real-analytic case [30, 16, 17]. Thus, the question whether or not the order of the jet determination we obtain in Theorem 1.2 is optimal, seems to be linked with the possible existence of one-sided biholomorphisms that do not extend across the hypersurface M.

Since any biholomorhisms between two smooth bounded pseudoconvex domains of finite type extends up to the boundary [5, 15], one can apply Theorem 1.2 in order to obtain a boundary version of H. Cartan's uniqueness theorem.

Theorem 1.3. *Let D, $D' \subset \mathbb{C}^2$ be two smooth bounded pseudoconvex domains of finite type. Assume that bD admits a local defining function centered at $p \in bD$ written as in Theorem 1.1. If H_1 and H_2 are two biholomorhisms from D onto D' with the same $(k_0 - d/2 + 2)$-jet at p, they coincide.*

The paper is organized as follows. In Section 2, we discuss the properties of the Banach spaces of functions and operators that we will need, and we define the notion of k_0-stationary discs. Section 3 is devoted to the construction of k_0-stationary discs In Section 4, we study the geometric properties of k_0-stationary discs. Finally, we prove Theorem 1.2 and other finite jet determination results in Section 5.

2. PRELIMINARIES

We denote by Δ the unit disc in $\mathbb{C}$. We denote by (z, w) the standard coordinates in $\mathbb{C}^2$.

2.1. **Spaces of functions.** Let k be an integer and let $0 < \alpha < 1$. We denote by $\mathcal{C}^{k,\alpha} = \mathcal{C}^{k,\alpha}(b\Delta, \mathbb{R})$ the space of real-valued functions defined on $b\Delta$ of class $\mathcal{C}^{k,\alpha}$. The space $\mathcal{C}^{k,\alpha}$ is endowed with its usual norm

$$\|f\|_{\mathcal{C}^{k,\alpha}} = \sum_{j=0}^{k} \|f^{(j)}\|_\infty + \sup_{\zeta \neq \eta \in b\Delta} \frac{\|f^{(k)}(\zeta) - f^{(k)}(\eta)\|}{|\zeta - \eta|^\alpha},$$

where $\|f^{(j)}\|_\infty = \max_{\partial\Delta} \|f^{(j)}\|$. We set $\mathcal{C}_{\mathbb{C}}^{k,\alpha} = \mathcal{C}^{k,\alpha} + i\mathcal{C}^{k,\alpha}$. Hence $f \in \mathcal{C}_{\mathbb{C}}^{k,\alpha}$ if and only if $\operatorname{Re} f, \operatorname{Im} f \in \mathcal{C}^{k,\alpha}$. The space $\mathcal{C}_{\mathbb{C}}^{k,\alpha}$ is equipped with the norm

$$\|f\|_{\mathcal{C}_{\mathbb{C}}^{k,\alpha}} = \|\operatorname{Re} f\|_{\mathcal{C}^{k,\alpha}} + \|\operatorname{Im} f\|_{\mathcal{C}^{k,\alpha}}$$

We denote by $\mathcal{A}^{k,\alpha}$ the subspace of $\mathcal{C}_{\mathbb{C}}^{k,\alpha}$ consisting of functions $f : \overline{\Delta} \to \mathbb{C}$, holomorphic on Δ with trace on $b\Delta$ belonging to $\mathcal{C}_{\mathbb{C}}^{k,\alpha}$. We define $(1 - \zeta)\mathcal{A}^{k,\alpha}$ to be the subspace of $\mathcal{C}_{\mathbb{C}}^{k,\alpha}$ of functions f that can be written as $f = (1 - \zeta)\tilde{f}$, with $\tilde{f} \in \mathcal{A}^{k,\alpha}$. We equip $(1 - \zeta)\mathcal{A}^{k,\alpha}$ with the following norm

$$\|(1 - \zeta)\tilde{f}\|_{(1-\zeta)\mathcal{A}^{k,\alpha}} = \|\tilde{f}\|_{\mathcal{C}_{\mathbb{C}}^{k,\alpha}}.$$

Hence $(1 - \zeta)\mathcal{A}^{k,\alpha}$ is a Banach space. Notice that the inclusion of $(1 - \zeta)\mathcal{A}^{k,\alpha}$ into $\mathcal{A}^{k,\alpha}$ is a bounded linear operator. We also point out that the linear operator $L : \mathcal{A}^{k,\alpha} \to (1 - \zeta)\mathcal{A}^{k,\alpha}$, defined by

$$L(\tilde{f}) = (1 - \zeta)\tilde{f}$$

is an isometry.

Finally, we denote by $\mathcal{C}_0^{k,\alpha}$ the subspace of $\mathcal{C}^{k,\alpha}$ consisting of elements that can be written as $(1 - \zeta)v$ with $v \in \mathcal{C}_{\mathbb{C}}^{k,\alpha}$. We equip $\mathcal{C}_0^{k,\alpha}$ with the norm

$$\|(1 - \zeta)v\|_{\mathcal{C}_0^{k,\alpha}} = \|v\|_{\mathcal{C}_{\mathbb{C}}^{k,\alpha}}.$$

Notice that $\mathcal{C}_0^{k,\alpha}$ is a Banach space.

2.2. **Pseudoconvex hypersurfaces of finite type.** In this section, we recall some facts about pseudoconvex hypersurface of finite type in $\mathbb{C}^2$. Let $M = \{r = 0\}$ be a smooth pseudoconvex hypersurface defined in a neighborhood of the origin C^2.

Definition 2.1. Let $f : (\Delta, 0) \to (\mathbb{C}^2, 0)$ be a holomorphic disc satisfying $f(0) = 0$. The order of contact $\delta_0(M, f)$ with M at the origin is the degree of the first term in the Taylor expansion of $r \circ f$. We denote by $\delta(f)$ the multiplicity of f at the origin.

We now define the D'Angelo type and the regular type of the real hypersurface M at the origin.

Definition 2.2.

i) The D'Angelo type of M at the origin is defined by:

$$\Delta^1(M,0) := \sup\left\{\frac{\delta_0(M,f)}{\delta(f)},\ f:(\Delta,0)\to(\mathbb{C}^2,0),\ \text{holomorphic}\right\}.$$

The point 0 is a point of finite D'Angelo type d if $\Delta^1(M,0) = d < +\infty$.

ii) The regular type of M at origin is defined by:

$$\Delta^1_{\text{reg}}(M,0) := \sup\left\{\delta_0(M,f),\ f:(\Delta,0)\to(\mathbb{C}^2,0),\ \text{holomorphic}, d_0u \neq 0\right\}.$$

The type condition as defined in part 1 of Definition 2.2 was introduced by J.-P. D'Angelo [14] who proved that this coincides with the regular type in complex manifolds of dimension two. Following [9] and [21], a smooth pseudoconvex hypersurface M of finite type d can be locally written as $M = \{r = 0\}$ with

$$r(z,w) = -\operatorname{Re} w + P(z,\bar{z}) + O\left(|z|^{d+1} + |\operatorname{Im} w||z|^{\frac{d}{2}+1} + |\operatorname{Im} w|^2\right),$$

where P is a subharmonic homogeneous polynomial of degree d, not identically zero, with no harmonic terms. Notice then that d is necessarily even. In this paper, we will restrict our attention smooth pseudoconvex hypersurfaces $M = \{r = 0\}$ of finite type d, where

$$r(z,w) = -\operatorname{Re} w + P(z,\bar{z}) + \theta_1(\operatorname{Im} w) + O\left(|z|^{d+1}\right) + \operatorname{Im} w\, O\left(|z, \operatorname{Im} w|^{d-1}\right),$$

such that $\{P_{z\bar{z}} = 0\} = \{0\}$ and where $\theta_1(\operatorname{Im} w) = O(|\operatorname{Im} w|^2)$. Furthermore, up to another change of coordinates, one can assume

$$r(z,w) = -\operatorname{Re} w + P(z,\bar{z}) + O\left(|z|^{d+1}\right) + \operatorname{Im} w\, O\left(|z, \operatorname{Im} w|^{d-1}\right),$$

Notice that the set of points in M of type greater than two is contained in $M \cap \{z = 0\}$. Given a polynomial P satisfying the above conditions, the following hypersurface will be called a *model hypersurface*, $S = \{\rho = 0\}$ with

$$\rho(z,w) = -\operatorname{Re} w + P(z,\bar{z}).$$

To fix some notations, we write

$$P(z,\bar{z}) = \sum_{j=d-k_0}^{k_0} \alpha_j z^j \bar{z}^{d-j},\quad \alpha_j = \bar{\alpha}_{d-j},\quad \alpha_{k_0} \neq 0.$$

where $d/2 \leq k_0 \leq d-1$. The integer $d - k_0$ is *the essential type* of the model hypersurface.

2.3. k_0-**stationary discs.** Let $M = \{r = 0\}$ be a smooth pseudoconvex hypersurface defined in a neighborhood of the origin in $\mathbb{C}^2$. Let k be an integer and let $0 < \alpha < 1$. A holomorphic disc $f \in (\mathcal{A}^{k,\alpha})^2$ is *attached* to M if $f(b\Delta) \subset M$.

Definition 2.3. A holomorphic disc $f \in (\mathcal{A}^{k,\alpha})^2$ attached to $M = \{r = 0\}$ is k_0-stationary if there exists a continuous function $c : b\Delta \to \mathbb{R}^*$ such that the map $\zeta^{k_0} c(\zeta) \partial r(f(\zeta))$, defined on $b\Delta$, extends as a map in $(\mathcal{A}^{k,\alpha})^2$. The map (c, f) is called a lift of f.

We first prove that such a set of holomorphic discs is invariant under biholomorphisms.

Proposition 2.4. *Let $M \subset \mathbb{C}^2$ be a smooth pseudoconvex real hypersurface of finite type containing 0. Let H be a biholomorphism in $\mathbb{C}^2$ sending M to a real hypersurface M'. If $f : \Delta \to \mathbb{C}^2$ is a k_0-stationary disc attached to M then the disc $H \circ f$ is a k_0-stationary disc attached to M'. More precisely, if (c, f) is a lift of f attached to M then $(c, H \circ f)$ is a lift of $H \circ f$ attached to M'.*

Proof. Let $f : \Delta \to \mathbb{C}^2$ be a k_0-stationary disc attached to $M = \{r = 0\}$. and let $c : b\Delta \to \mathbb{R}^*$ be such that $\zeta^{k_0} c(\zeta) \partial r(f(\zeta))$ extends as a map in $(\mathcal{A}^{k,\alpha})^2$. Since H sends M to M', the function $r \circ H^{-1}$ is a (local) defining function for M'. It follows that $\zeta^{k_0} c(\zeta) \partial(r \circ H^{-1})(H \circ f(\zeta)) = \zeta^{k_0} c(\zeta) \partial r(f(\zeta))(\partial H(f(\zeta))^{-1}$ extends as well as a $(\mathcal{A}^{k,\alpha})^2$ map. □

Remark 2.5. More generally, small enough k_0-stationary discs are invariant under CR-diffeomorphisms. Indeed, a CR-diffeomorphism H of class $\mathcal{C}^{k+1}$ admits a local holomorphic extension $\widetilde{H}$ to a one-sided neighborhood U of 0, which is of class $\mathcal{C}^{k+1}$ up to $\overline{U}$ (see Theorem 7.5.1 in [1]), and any small stationary disc $f \in (\mathcal{A}^{k,\alpha})^2$ is contained in $\overline{U}$. This allows to define the composition $\widetilde{H} \circ f \in (\mathcal{A}^{k,\alpha})^2$, and the previous proof applies.

We denote by $\mathcal{S}^{k_0,r}$ the set of lifts of k_0-stationary discs attached to $M = \{r = 0\}$. Moreover, assume that the local definition function of $M = \{r = 0\}$. We denote by $\mathcal{S}_0^{k_0,r}$, or simply $\mathcal{S}_0^r$, the elements $(c, f) \in \mathcal{S}^{k_0,r}$ satisfying $f(1) = 0$. In such a case, we say that the lift (c, f) is *tied to the origin*.

Example 2.6. *Consider a model hypersurface $S = \{\rho = 0\}$ with*

$$\rho(z, w) = -\operatorname{Re} w + P(z, \overline{z}) = -\operatorname{Re} w + \sum_{j=d-k_0}^{k_0} \alpha_j z^j \overline{z}^{d-j}.$$

We have

$$\begin{cases} \frac{\partial \rho}{\partial w}(z, w) = -\frac{1}{2} \\ \\ \frac{\partial \rho}{\partial z}(z, w) = P_z(z, \overline{z}) = \sum_{j=d-k_0}^{k_0} j\alpha_j z^{j-1} \overline{z}^{d-j}. \end{cases}$$

We restrict to coefficient functions of the form $c(\zeta) = \left(\overline{b}\zeta + 1 + b\zeta\right)^{k_0}$ where $b \in \mathbb{C}$ is such that $|b| < 1/2$. Set $c'(\zeta) = \overline{b}\zeta + 1 + b\zeta$. It follows that $\zeta c'$, and thus $-\frac{1}{2}\zeta^{k_0}c$, extend holomorphically to the unit disc. Following the arguments in [6], we set

$$h(\zeta) = \frac{1 - \zeta}{1 - a(b)\zeta}v,$$

with $a = \frac{-1+\sqrt{1-4|b|^2}}{2b}$ and $v \in \mathbb{C}^*$, which implies that $\zeta c'\overline{h} \in \mathcal{A}^{k,\alpha}$. Then

$$\zeta^{k_0}c(\zeta)P_z(h(\zeta), \overline{h(\zeta)}) = \sum_{j=d-k_0}^{k_0} j\alpha_j(\zeta c'(\zeta))^{k_0-d+j}h(\zeta)^{j-1}(\zeta c'(\zeta)\overline{h(\zeta)})^{d-j}.$$

Since $k_0 - d + j \geq 0$ and $d - j \geq 0$ for $d - k_0 \leq j \leq k_0$, every term in the sum on the right hand side belongs to $\mathcal{A}^{k,\alpha}$. Hence $\zeta^{k_0}cP_z(h, \overline{h}) \in \mathcal{A}^{k,\alpha}$. Imposing the further condition $g(1) = 0$, one can find, by standard results about the Hilbert transform, a map g such that $f = (h, g)$ is a k_0-stationary disc attached to S and tied to the origin.

In particular, if $b = 0$ and $v = 1$, it follows that c_0 is identically equal to 1 and that $h_0(\zeta) = 1 - \zeta$. Denote by $f_0 = (h_0, g_0)$ the corresponding stationary disc.

The disc f_0 we have given in Example 2.6 is essential in our approach in order to obtain a family of k_0-stationary discs by deformation of both f_0 and the model hypersurface.

2.4. **Regularity and Fredholmness of integral operators on $b\Delta$.** We denote by $H^2(b\Delta)$ the classical Hardy space on the unit circle and by $\mathcal{P}' : L^2(b\Delta) \to H^2(b\Delta)$ the Szegö projection. The Szegö projection defines a linear operator $\mathcal{P}' : \mathcal{C}_{\mathbb{C}}^{k,\alpha} \to \mathcal{A}^{k,\alpha}$. Denoting by $\mathcal{T}$ the Hilbert transform, one can deduce by [1] (6.1.37) the following relation

$$(2.1) \qquad\qquad -2i\mathcal{P}' = \mathcal{T} - i\mathrm{Id} - iC_0,$$

where $C_0u = \int_0^{2\pi} u(e^{i\theta})d\theta$. Due to Privalov's Theorem, the Hilbert transform is a well-behaved operator (see [1], Corollary 6.1.31):

Theorem 2.7. *Let $k \geq 0$ be an integer and let $0 < \alpha < 1$ be a real number. The Hilbert transform $\mathcal{T} : \mathcal{C}^{k,\alpha} \to \mathcal{C}^{k,\alpha}$ is a bounded linear operator.*

Hence, it follows from (2.1) that the Szegö projection $\mathcal{P}' : \mathcal{C}_{\mathbb{C}}^{k,\alpha} \to \mathcal{A}^{k,\alpha}$ is a bounded linear operator. Out of convenience, however, we will mainly consider the analogous bounded projection $\mathcal{P} : \mathcal{C}_{\mathbb{C}}^{k,\alpha} \to \overline{\zeta\mathcal{A}}^{k,\alpha}$ defined by

$$(2.2) \qquad\qquad \mathcal{P}(u) = \overline{\zeta\mathcal{P}'\left(\overline{\zeta u}\right)} = u - \mathcal{P}'(u).$$

The space $\overline{\zeta\mathcal{A}}^{k,\alpha}$ is endowed with the induced norm of $\mathcal{C}_{\mathbb{C}}^{k,\alpha}$. Notice that a function u extends as a function in $\mathcal{A}^{k,\alpha}$ if and only if $\mathcal{P}(u) = 0$. More precisely, if $u = u' + u''$ with $u' = \sum_{n\geq 0} u_n \zeta^n$ and $u'' = \mathcal{P}(u) = \sum_{n<0} u_n \zeta^n$ then $\mathcal{P}(u) = u''$.

Recall that when φ is a complex valued continuous function defined on $b\Delta$ that does not vanish, then the Toeplitz operator with symbol φ defined by $T_\varphi = \mathcal{P}'(\varphi.) :$ $H^2(b\Delta) \to H^2(b\Delta)$ is a Fredholm operator, that is with finite dimensional kernel and cokernel. Moreover, it index, namely $dim_{\mathbb{C}} \ker T_\varphi - dim_{\mathbb{C}} \mathrm{coker} T_\varphi$, is the opposite of the winding number $-\frac{1}{2\pi i}\int_{b\Delta} \frac{d\varphi}{\varphi}$ of φ. It follows then that for such a φ, the operator $\mathcal{P}(\varphi.) : \overline{H}^2(b\Delta) \to \overline{\zeta H}^2(b\Delta)$ is also Fredholm of index $-\frac{1}{2\pi i}\int_{b\Delta} \frac{d\overline{\zeta\varphi}}{\overline{\zeta\varphi}}$. Note that, although the operator $\mathcal{P}(\varphi.)$ is defined in a similar way than the Hankel operator $\mathcal{P}(\varphi.) : H^2(b\Delta) \to \overline{\zeta H}^2(b\Delta)$, they do not agree since their source spaces differ. In this paper, we need to consider the operator $\mathcal{P}(\varphi.)$ restricted to $\overline{\mathcal{A}}^{k,\alpha} \subset \overline{H}^2(b\Delta)$. Although it is not clear whether or not $\mathcal{P}(\varphi.)$ is Fredholm in that case, for a very special class of functions φ, we have the following:

Lemma 2.8. *Let $\varphi \in \mathcal{C}^{k,\alpha}$ be a holomorphic polynomial whose zeros are contained in the unit disc Δ. Then the operator $\mathcal{P}(\varphi.) : \overline{\mathcal{A}}^{k,\alpha} \to \overline{\zeta\mathcal{A}}^{k,\alpha}$ is Fredholm of index the opposite of the winding number of $\overline{\zeta\varphi}$. More precisely, $\mathcal{P}(\varphi.)$ is surjective and its kernel has complex dimension $-\frac{1}{2\pi i}\int_{b\Delta} \frac{d\overline{\zeta\varphi}}{\overline{\zeta\varphi}}$.*

Proof. First note that since $\overline{\mathcal{A}}^{k,\alpha} \subset \overline{H}^2(b\Delta)$, the kernel of $\mathcal{P}(\varphi.) : \overline{\mathcal{A}}^{k,\alpha} \to \overline{\zeta\mathcal{A}}^{k,\alpha}$ is included in the one of $\mathcal{P}(\varphi.) : \overline{H}^2(b\Delta) \to \overline{\zeta H}^2(b\Delta)$ and hence is finitely dimensional. Write

$$\varphi(\zeta) = C \prod_{1\leq j\leq \ell} (q_j - \zeta) = \frac{C}{\overline{\zeta}^\ell}\varphi_2(\zeta),$$

where $\varphi_2(\zeta) = \prod_{1\leq j\leq \ell}(\overline{\zeta}q_j - 1)$. Then φ_2 extends antiholomorphically to the unit disc and the extension is nowhere vanishing on $\overline{\Delta}$. It follows that both φ_2 and its inverse belong to $\overline{\mathcal{A}}^{k,\alpha}$. Now, for any $v \in \overline{\zeta\mathcal{A}}^{k,\alpha}$ define

$$u(\zeta) = \frac{\overline{\zeta}^\ell}{C}\frac{v(\zeta)}{\varphi_2(\zeta)} \in \overline{\mathcal{A}^{k,\alpha}}.$$

Since $v \in \overline{\zeta\mathcal{A}}^{k,\alpha}$, we have $\mathcal{P}(\varphi u) = v$, which proves the surjectiviy of $\mathcal{P}(\varphi.)$. Thus $\mathcal{P}(\varphi.)$ is a Fredholm operator and its index is equal to $dim_{\mathbb{C}} \ker \mathcal{P}(\varphi.)$.

In order to compute its Fredholm index, we need the following observation. The boundedness of the operator $\mathcal{P}$ implies that the map $\varphi \mapsto \mathcal{P}(\varphi.)$ is continuous. Due to the local constancy of the index of a Fredholm operator (see Theorem 5.2 p. 42 [7]), it follows that if φ and φ' are two homotopically equivalent holomorphic

polynomials whose zeros are contained in the unit disc Δ, then $\mathcal{P}(\varphi.)$ and $\mathcal{P}(\varphi.)$ have the same index. And since any such function $\varphi : b\Delta \to \mathbb{C}^*$ is homotopically equivalent to some ζ^m, for some integer $m \geq 0$, it is enough to compute the index of the Fredholm operator $\mathcal{P}(\zeta^m.)$. We write the Fourier expansion of $u \in \overline{\mathcal{A}}^{k,\alpha}$ as $u(\zeta) = \sum_{n<0} u_n \zeta^n$. Thus

$$\mathcal{P}(\zeta^m u) = \sum_{n<-m} u_n \zeta^{m+n} = 0$$

if and only if $u_n = 0$ whenever $n < -m$. This implies that $dim_{\mathbb{C}} \ker \mathcal{P}(\zeta^m.) = m + 1$. $\qquad\square$

Finally, we will need the following two lemmas.

Lemma 2.9. *Let $k_0 \geq 1$ be an integer and let $c : b\Delta \to \mathbb{R}^*$ be a continuous function. Then $\mathcal{P}(\zeta^{k_0} c) = 0$ if and only if*

$$(2.3) \qquad c(\zeta) = \bar{c}_{k_0} \zeta^{-k_0} + \bar{c}_{k_0-1} \zeta^{-(k_0-1)} + \ldots + c_0 + \ldots + c_{k_0-1} \zeta^{k_0-1} + c_{k_0} \zeta^{k_0},$$

where $c_0 \in \mathbb{R}$ and $c_1, \ldots, c_{k_0} \in \mathbb{C}$.

Proof. We write the Fourier expansion of $c : b\Delta \to \mathbb{R}^*$ as $c(\zeta) = \sum_{n\in\mathbb{Z}} c_n \zeta^n$, where $c_n \in \mathbb{C}$ and $c_n = \bar{c}_{-n}$ for all $n \in \mathbb{Z}$. Since the Fourier expansion of $\zeta^{k_0} c$ is given by $\sum_{n\in\mathbb{Z}} c_n \zeta^{n+k_0}$, in order for it to belong to $\mathcal{A}^{k,\alpha}$ we must have $c_n = 0$ for all $n < -k_0$, from which follows (2.3). Conversely, the fact that such a c satisfies $P(\zeta^{k_0} c) = 0$ is immediate. $\qquad\square$

Lemma 2.10. *Let $u \in \mathcal{C}_{\mathbb{C}}^{k,\alpha}$ and let $\ell \in \mathbb{Z}$. Then $\mathcal{P}\left((1 - \zeta)^\ell u\right) = 0$ if and only if $\mathcal{P}u = 0$.*

Proof. By induction it is sufficient to show that $\mathcal{P}((1-\zeta)u) = 0$ if and only if $\mathcal{P}u = 0$. Let $u(\zeta) = \sum_{n\in\mathbb{Z}} c_n \zeta^n$ be the Fourier expansion of u, and write $u = u' + u''$ with $u' = \sum_{n\geq 0} u_n \zeta^n$ and $u'' = \mathcal{P}(u) = \sum_{n<0} u_n \zeta^n$. It is clear that $(1 - \zeta)u'$ always extends holomorphically, hence we must show that $\mathcal{P}((1 - \zeta)u'') = 0$ if and only if $u'' = 0$. Now, we have $(1 - \zeta)u'' = \sum_{n\leq 0} v_n \zeta^n$, where

$$\begin{cases} v_0 = u_{-1} \\ \\ v_n = u_n - u_{n-1}, \ \ n \leq -1. \end{cases}$$

If $\mathcal{P}((1-\zeta)u'') = 0$, we must have $v_n = 0$ for all $n \leq -1$, which implies that $u_n = u_{n-1}$ for all $n \leq -1$. Since $u'' \in \mathcal{C}_{\mathbb{C}}^{k,\alpha}$, this is only possible if $u'' = 0$. $\qquad\square$

3. Construction of k_0-stationary discs

Consider a model hypersurface $S = \{\rho = 0\}$ of type d

$$\rho(z, w) = -\operatorname{Re} w + P(z, \overline{z}) = -\operatorname{Re} w + \sum_{j=d-k_0}^{k_0} \alpha_j z^j \overline{z}^{d-j}.$$

Recall that in view of Example 2.6, we consider the k_0-stationary disc attached to S given by $f_0 = (h_0, g_0) = (1 - \zeta, g_0)$, with the coefficient function $c_0 \equiv 1$. We fix $\delta > 0$ such that $f_0(\Delta) \subset (\delta\Delta)^2$. The aim of this section is to construct a finite dimensional manifold of lifts of k_0-stationary discs attached to small perturbations of the given model hypersurface.

Let $k > 0$ be an integer and let $0 < \alpha < 1$. We denote by X the (affine) Banach space parametrizing the deformations of the model S that we will consider, defined as the set of functions $r \in \mathcal{C}^{k+3}\left(\overline{\delta\Delta}^2\right)$ which can be written as

(3.1) $$r(z, w) = \rho(z, w) + \theta(z, \operatorname{Im} w),$$

with

$$\theta(z, \operatorname{Im} w) = \sum_{i+j=d+1} (z^i \overline{z}^j) \cdot r_{ij0}(z) + \sum_{l=1}^{d} \sum_{i+j=d-l} z^i \overline{z}^j (\operatorname{Im} w)^l \cdot r_{ijl}(z, \operatorname{Im} w)$$

where $r_{ij0} \in \mathcal{C}_{\mathbb{C}}^{k+3}\left(\overline{\delta\Delta}\right), r_{ijl} \in \mathcal{C}_{\mathbb{C}}^{k+3}\left(\overline{\delta\Delta} \times [-\delta, \delta]\right)$. Furthermore, we will consider the norm

$$\|r\|_X = \sup \|r_{ijl}\|_{\mathcal{C}^{k+3}},$$

so that X is isomorphic to a (real) closed subspace of $\mathcal{C}_{\mathbb{C}}^{k+3}\left(\overline{\delta\Delta} \times [-\delta, \delta]\right)$, hence it is a Banach space. The inclusion of X into $\mathcal{C}^{k+3}\left(\overline{\delta\Delta} \times [-\delta, \delta]\right)$ is an (affine) linear bounded map, and in particular it is of class $\mathcal{C}^1$. We define the set

$$Y = \mathcal{C}^{k,\alpha} \times (1 - \zeta)\mathcal{A}^{k,\alpha} \times (1 - \zeta)\mathcal{A}^{k,\alpha},$$

and we equip it with the following norm

$$\|(c, h, g)\|_Y = \|c\|_{\mathcal{C}^{k,\alpha}} + \|h\|_{(1-\zeta)\mathcal{A}^{k,\alpha}} + \|g\|_{(1-\zeta)\mathcal{A}^{k,\alpha}}.$$

Theorem 1.1 is a consequence of the following more precise result:

Theorem 3.1. *Let $S = \{\rho = 0\}$ be a model hypersurface of finite type d. Then for any integer $k \geq 0$ and $0 < \alpha < 1$, there exist some open neighborhoods V of ρ in X and U of 0 in $\mathbb{R}^{4k_0-d+3}$, $\eta > 0$, and a map $\mathcal{F} : V \times U \to Y$ of class $\mathcal{C}^1$, such that:*

i) $\mathcal{F}(\rho, 0) = (1, f_0)$,

ii) for all $r \in V$, the map $\mathcal{F}(r, \cdot) : U \to \{(c, f) \in Y \mid (c, f) \in \mathcal{S}_0^r, \|(c, f) - (1, f_0)\|_Y < \eta\}$ is one-to-one and onto.

In order to prove Theorem 3.1, we need to consider the zero set of the map $T = (T_1, T_2, T_3)$

$$T : X \times Y \to (\overline{\zeta \mathcal{A}}^{k,\alpha})^2 \times \mathcal{C}_0^{k,\alpha},$$

defined by

$$\begin{cases} T_1(r, c, h, g) = \mathcal{P}\left(\zeta^{k_0} c \frac{\partial r}{\partial z}(h, g)\right) \\[2mm] T_2(r, c, h, g) = \mathcal{P}\left(\zeta^{k_0} c \frac{\partial r}{\partial w}(h, g)\right) \\[2mm] T_3(r, c, h, g) = r(h, g) \end{cases}$$

in a neighborhood of (ρ, c_0, h_0, g_0). For any fixed $r \in X$, the zero set of $T(r, \cdot)$ is the set $\mathcal{S}_0^r$ of k_0-stationary discs attached to $\{r = 0\}$ and tied to the origin.

Although the map T is of class $\mathcal{C}^1$, its derivative with respect to Y is, in general, not surjective. Our aim is then to replace it with another map T' having the same zero set as T and whose derivative with respect to Y is surjective, hence allowing the application of the implicit function theorem.

We first define a polynomial Q in such a way that

$$\zeta^{k_0} P_{z\bar{z}}(1 - \zeta, 1 - \bar{\zeta}) = (\zeta - 1)^{d-2} Q(\zeta)$$

for all $\zeta \in b\Delta$. The polynomial Q can be explicitly computed:

$$\begin{aligned} \zeta^{k_0} P_{z\bar{z}}(1 - \zeta, 1 - \bar{\zeta}) &= \sum_{j=(d-k_0)}^{k_0} \gamma_j (1 - \zeta)^{j-1} \zeta^{k_0} (1 - \bar{\zeta})^{d-j-1} \\ &= (\zeta - 1)^{d-2} \sum_{j=d-k_0}^{k_0} (-1)^{j-1} \gamma_j \zeta^{k_0+j+1-d} \\ &= (\zeta - 1)^{d-2} Q(\zeta) \end{aligned}$$

where $\gamma_j = j(d - j)\alpha_j$. The assumption made on P implies that Q does not have any roots on the unit circle $b\Delta$; also note that Q has always one root of multiplicity one at the origin. Denote by q_j, $1 \leq j \leq 2k_0 + 1 - d$, the roots of Q, and suppose that $q_1, \ldots, q_{i_0}$ are the ones lying outside of Δ and $q_{2k_0+1-d} = 0$. We set $s(\zeta) = \prod_{1 \leq j \leq i_0} (q_j - \zeta)$, $t(\zeta) = \prod_{i_0+1 \leq j \leq 2k_0-d} (q_j - \zeta)$, and write

$$Q(\zeta) = C\zeta s(\zeta) t(\zeta)$$

for some constant $C \in \mathbb{C}$. Put $\ell_0 = 2k_0 - i_0 - d$, so that $Q(\zeta)$ admits exactly $\ell_0 + 1$ roots inside the unit disc and i_0 roots outside.

Lemma 3.2. *We have $\ell_0 = i_0 = k_0 - d/2$.*

Proof. For the given $k_0 \in \mathbb{N}$, consider the set $P_{k_0,d}$ of the real polynomials $P'(z, \overline{z})$ such that

$$P'(z, \overline{z}) = \sum_{j=d-k_0}^{k_0} \alpha'_j z^j \overline{z}^{d-j}, \quad \alpha'_j = \overline{\alpha}'_{d-j}$$

and such that $P'_{z\overline{z}}$ vanishes exactly at 0 and is positive elsewhere. For each $P' \in P_{k_0,d}$, denote by $Q_{P'}$ the polynomial defined above. The computation performed before the lemma shows that the coefficients of $Q_{P'}$ depend continuously on those of P', and that $Q_{P'}$ never vanishes on the unit circle. Thus, the argument principle implies that the number of zeroes of $Q_{P'}$ lying inside the unit disc is constant on any connected component of $P_{k_0,d}$. On the other hand, it is easy to check that $P_{k_0,d}$ is a convex cone in the space of the real polynomials of degree d, hence it is connected.

Choosing now $P'(z, \overline{z}) = |z|^d$, we obtain $Q_{P'}(\zeta) = (-1)^{\frac{d}{2}-1} \frac{d^2}{4} \zeta^{k_0+1-\frac{d}{2}}$, which vanishes at $z = 0$ with multiplicity $k_0 + 1 - d/2$. It follows that the polynomial P also admits exactly $k_0 + 1 - d/2$ zeroes inside the unit disc, hence $\ell_0 = k_0 - d/2$ as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now define the map $T' = (T'_1, T'_2, T'_3)$ by putting $T'_2 = T_2$, $T'_3 = T_3$ and

$$T'_1(r, c, h, g) = \mathcal{P}\left(\frac{\zeta^{k_0}}{(1-\zeta)^{d-1}s(\zeta)} \cdot c\frac{\partial r}{\partial z}(h, g)\right).$$

The map T'_1 is well-defined because the choice of the space X implies $\frac{\partial r}{\partial z}(h, g) \in (1-\zeta)^{d-1}\mathcal{C}^{k,\alpha}_{\mathbb{C}}$ for all $r \in X$ and $h, g \in (1-\zeta)\mathcal{A}^{k,\alpha}$. The fact that the zero set of T'_1 is the same as the one of T_1 follows from Lemma 2.10 and from the fact that both $s(\zeta) = \prod_{1 \le j \le i_0}(q_j - \zeta)$ and its inverse extend holomorphically to the unit disc, which implies that $s \cdot u \in \mathcal{A}^{k,\alpha}$ if and only if $u \in \mathcal{A}^{k,\alpha}$.

Lemma 3.3. *The map T' is of class $\mathcal{C}^1$.*

Proof. Since $\mathcal{P} : \mathcal{C}^{k,\alpha}_{\mathbb{C}} \to \overline{\zeta\mathcal{A}}^{k,\alpha}$ is a bounded linear operator, it is enough to show that the following map

$$\tilde{T}' = (\tilde{T}'_1, \tilde{T}'_2, T'_3) : X \times Y \to (\mathcal{C}^{k,\alpha}_{\mathbb{C}})^2 \times \mathcal{C}^{k,\alpha}_0,$$

defined by

$$\begin{cases} \tilde{T}'_1(r, c, h, g) = \frac{\zeta^{k_0}}{(1-\zeta)^{d-1}s(\zeta)} \cdot c\frac{\partial r}{\partial z}(h, g) \\[2ex] \tilde{T}'_2(r, c, h, g) = \zeta^{k_0} c\frac{\partial r}{\partial w}(h, g) \\[2ex] T'_3(r, c, h, g) = r(h, g) \end{cases}$$

is of class $\mathcal{C}^1$.

According to Lemma 5.1 in [26], the map $\tilde{T}'_2$ is of class $\mathcal{C}^1$ when considered as a map from $X \times \mathcal{C}^{k,\alpha} \times \mathcal{A}^{k,\alpha} \times \mathcal{A}^{k,\alpha}$. Then it follows from the boundedness of the inclusion of $(1-\zeta)\mathcal{A}^{k,\alpha}$ into $\mathcal{A}^{k,\alpha}$ noticed in Subsection 2.1 that $\tilde{T}'_2 : X \times Y \to \mathcal{C}^{k,\alpha}_{\mathbb{C}}$ is of class $\mathcal{C}^1$.

To treat the smoothness of $\tilde{T}'_1$, we define, for every i,j such that $i+j = d+1$, maps $S_{ij0}, U_{ij0} : X \to \mathcal{C}^{k+2}_{\mathbb{C}}(\overline{\delta\Delta}^2)$ of class $\mathcal{C}^1$ in the following way:

$$(S_{ij0}(r))(z, z') = iz^{i-1}\overline{z}^j \cdot r_{ij0}(z'),$$

$$(U_{ij0}(r))(z, z') = z^i\overline{z}^j \cdot \frac{\partial r_{ij0}}{\partial z}(z').$$

We also define, for every i,j and $1 \le l \le d$ satisfying $i+j+l = d$, maps $S_{ijl}, U_{ijl} : X \to \mathcal{C}^{k+2}_{\mathbb{C}}(\overline{\delta\Delta}^3 \times [-\delta, \delta])$ of class $\mathcal{C}^1$ by:

$$(S_{ijl}(r))(z, w, z', \operatorname{Im} w') = iz^{i-1}\overline{z}^j \left(w + \overline{\zeta w}\right)^l \cdot r_{ijl}(z', \operatorname{Im} w'),$$

$$(V_{ijl}(r))(z, w, z', \operatorname{Im} w') = z^i\overline{z}^j \left(w + \overline{\zeta w}\right)^l \cdot \frac{\partial r_{ijl}}{\partial z}(z', \operatorname{Im} w').$$

Posing $h = (1-\zeta)\tilde{h}$, $g = (1-\zeta)\tilde{g}$, a straightforward computation gives

$$\tilde{T}'_1(r, c, h, g) = \frac{\zeta^{k_0}}{s(\zeta)}cP_z\left(\tilde{h}, -\overline{\zeta\tilde{h}}\right) + \frac{\zeta^{k_0}(1-\zeta)}{s(\zeta)}c\sum_{i+j=d+1}\frac{1}{(-\zeta)^j}(S_{ij0}(r))(\tilde{h}, h)$$

$$+\frac{\zeta^{k_0}(1-\zeta)^2}{s(\zeta)}c\sum_{i+j=d+1}\frac{1}{(-\zeta)^j}(U_{ij0}(r))(\tilde{h}, h)$$

$$+\frac{\zeta^{k_0}}{s(\zeta)}c\sum_{l=1}^{d-1}\sum_{i+j=d-l}\frac{1}{(-\zeta)^j}(S_{ijl}(r))(\tilde{h}, \tilde{g}, h, \operatorname{Im} g)$$

$$+\frac{\zeta^{k_0}(1-\zeta)}{s(\zeta)}c\sum_{l=1}^{d-1}\sum_{i+j=d-l}\frac{1}{(-\zeta)^j}(U_{ijl}(r))(\tilde{h}, \tilde{g}, h, \operatorname{Im} g)$$

Since, again by Lemma 5.1 in [26] and the discussions made in Subsection 2.1, the maps $X \times (1-\zeta)\mathcal{A}^{k,\alpha} \times (1-\zeta)\mathcal{A}^{k,\alpha} \to \mathcal{C}^{k,\alpha}_{\mathbb{C}}$ defined as $(r, h, g) \mapsto (S_{ij0}(r))(\tilde{h}, h)$, $(r, h, g) \mapsto (U_{ij0}(r))(\tilde{h}, h)$, $(r, h, g) \mapsto (S_{ijl}(r))(\tilde{h}, \tilde{g}, h, \operatorname{Im} g)$, $(r, h, g) \mapsto (U_{ijl}(r))(\tilde{h}, \tilde{g}, h, \operatorname{Im} g)$ are of class $\mathcal{C}^1$, it follows that T'_1 is in turn of class $\mathcal{C}^1$.

The proof that $T'_3 : X \times Y \to \mathcal{C}^{k,\alpha}_0$ is of class $\mathcal{C}^1$ is analogous to (but simpler than) the proof for $\tilde{T}'_1$, hence we shall omit it. $\square$

We then show the following:

Lemma 3.4. *The Banach space derivative T'_Y is surjective at the point $p_0 = (\rho, c_0, h_0, g_0)$.*

Proof. Choosing any $(c', h', g') \in Y$, for any $p = (r, c, h, g)$ we can write

$$T'_Y(p)[c', h', g'] = \left((T'_1)_Y(p)[c', h', g'], (T'_2)_Y(p)[c', h', g'], (T'_3)_Y(p)[c', h', g']\right),$$

where

$$\begin{cases} (T_1')_Y(p)[c',h',g'] &= \mathcal{P}\left(\frac{\zeta^{k_0}}{s(\zeta)(1-\zeta)^{d-1}} \cdot \left(c'\frac{\partial r}{\partial z}(h,g) + ch'\frac{\partial^2 r}{\partial z^2}(h,g) + c\overline{h}'\frac{\partial^2 r}{\partial z \partial \overline{z}}(h,g)\right)\right) \\[2mm] &\quad + \mathcal{P}\left(\frac{\zeta^{k_0}}{s(\zeta)(1-\zeta)^{d-1}} \cdot \left(cg'\frac{\partial^2 r}{\partial z \partial w}(h,g) + c\overline{g}'\frac{\partial^2 r}{\partial z \partial \overline{w}}(h,g)\right)\right), \\[2mm] (T_2')_Y(p)[c',h',g'] &= \mathcal{P}\left(\zeta^{k_0} \cdot \left(c'\frac{\partial r}{\partial w}(h,g) + ch'\frac{\partial^2 r}{\partial w \partial z}(h,g) + c\overline{h}'\frac{\partial^2 r}{\partial w \partial \overline{z}}(h,g)\right)\right) \\[2mm] &\quad + \mathcal{P}\left(\zeta^{k_0} \cdot \left(cg'\frac{\partial^2 r}{\partial w^2}(h,g) + c\overline{g}'\frac{\partial^2 r}{\partial w \partial \overline{w}}(h,g)\right)\right), \\[2mm] (T_3')_Y(p)[c',h',g'] &= g'\frac{\partial r}{\partial w}(h,g) + \overline{g}'\frac{\partial r}{\partial \overline{w}}(h,g) + h'\frac{\partial r}{\partial z}(h,g) + \overline{h}'\frac{\partial r}{\partial \overline{z}}(h,g). \end{cases}$$

At $p_0 = (\rho, c_0, h_0, g_0)$ this becomes

$$\begin{cases} (T_1')_Y(p_0)[c',h',g'] &= \mathcal{P}\left(\frac{\zeta^{k_0}}{s(\zeta)(1-\zeta)^{d-1}} \cdot \left(P_{z^2}(1-\zeta, 1-\overline{\zeta})h' + P_{z\overline{z}}(1-\zeta, 1-\overline{\zeta})\overline{h}'\right)\right) \\[2mm] &\quad + \mathcal{P}\left(\frac{\zeta^{k_0}}{s(\zeta)(1-\zeta)^{d-1}} P_z(1-\zeta, 1-\overline{\zeta})c'\right), \\[2mm] (T_2')_Y(p_0)[c',h',g'] &= \mathcal{P}\left(-\frac{\zeta^{k_0}}{2}c'\right), \\[2mm] (T_3')_Y(p_0)[c',h',g'] &= -\frac{g'}{2} - \frac{\overline{g}'}{2} + P_z(1-\zeta, 1-\overline{\zeta})h' + P_{\overline{z}}(1-\zeta, 1-\overline{\zeta})\overline{h}'. \end{cases}$$

Because of the triangular form of the previous expressions we see that it is sufficient to show that the derivative of T_2' in the direction of c' is surjective (onto $\overline{\zeta \mathcal{A}}^{k,\alpha}$) and that the same is true for the derivatives of T_1' in the h' direction and of T_3' in the g' direction.

We first focus on $T_2' = T_2$, i.e. we consider the map $\mathcal{C}^{k,\alpha} \ni c' \mapsto \mathcal{P}(-\zeta^{k_0}c'/2) \in \overline{\zeta \mathcal{A}}^{k,\alpha}$. If $c'(\zeta) = \sum_{n \in \mathbb{Z}} c_n' \zeta^n$ with $c_n' = \overline{c'_{-n}}$, we have

$$\mathcal{P}(-\zeta^{k_0}c'/2) = -\frac{1}{2}\sum_{n<0} c'_{n-k_0}\zeta^n.$$

Since the indices $n - k_0$ are all negative, this map is clearly onto.

We turn now to T_1'. The derivative of T_1' in the h' direction consists of two summands: computing the first one we obtain, setting $h' = (1-\zeta)h''$ with $h'' \in \mathcal{A}^{k,\alpha}$ and

$\beta_j = j(j-1)\alpha_j$, and taking in account that $\overline{\zeta} = 1/\zeta$,

$$\frac{\zeta^{k_0}h'(\zeta)}{s(\zeta)(1-\zeta)^{d-1}}P_{z^2}(1-\zeta, 1-\overline{\zeta}) = \frac{h''(\zeta)}{s(\zeta)(1-\zeta)^{d-2}}\sum_{j=d-k_0}^{k_0}\beta_j(1-\zeta)^{j-2}\zeta^{k_0}(1-\overline{\zeta})^{d-j}$$

$$= \frac{h''(\zeta)}{s(\zeta)(1-\zeta)^{d-2}}\sum_{j=d-k_0}^{k_0}\beta_j(1-\zeta)^{j-2}\zeta^{k_0+j-d}(\zeta-1)^{d-j}$$

$$= \frac{h''(\zeta)}{s(\zeta)}\sum_{j=d-k_0}^{k_0}(-1)^{d-j}\beta_j\zeta^{k_0+j-d}.$$

Since in the sum the exponents k_0+j-d are always non-negative, this term belongs to $\mathcal{A}^{k,\alpha}$ for any $h'' \in \mathcal{A}^{k,\alpha}$, hence its projection vanishes. On the other hand, computing the second summand we get

$$\frac{\overline{h'(\zeta)}}{s(\zeta)(1-\zeta)^{d-1}}\zeta^{k_0}P_{z\overline{z}}(1-\zeta, 1-\overline{\zeta}) = -\frac{\overline{h''(\zeta)}}{\zeta s(\zeta)(1-\zeta)^{d-2}}(\zeta-1)^{d-2}Q(\zeta)$$

$$= (-1)^d C\overline{h''(\zeta)}t(\zeta)$$

Since $t(\zeta)$ is a holomorphic polynomial whose zeros are contained in the unit disc Δ, Lemma 2.8 implies that the map $(T_1')_Y$ is surjective at p_0 in the direction of h'.

Finally we have to consider the derivative of T_3' in the g' direction. Proving its surjectivity amounts to show that for any $v \in \mathcal{C}_0^{k,\alpha}$ there exists $g' \in (1-\zeta)\mathcal{A}^{k,\alpha}$ such that $2\text{Re}\,g' = v$. Thus $\text{Re}\,g'$ is the harmonic extension of $v/2$ to the unit disc and $\text{Im}\,g'$ is uniquely determined as the harmonic conjugate of $v/2$ such that $\text{Im}\,g'(1) = 0$. It follows that $g'(1) = 0$, i.e. $g' = (1-\zeta)g''$ for some holomorphic function g'' continuous up to $\overline{\Delta}$ (in fact g'' is at least in $\mathcal{A}^{k-1}$ since $g' \in \mathcal{A}^{k,\alpha}$). We want to show that $g'' \in \mathcal{A}^{k,\alpha}$. By definition of $\mathcal{C}_0^{k,\alpha}$, we have $v = (1-\zeta)v'$ for some $v' \in \mathcal{C}_{\mathbb{C}}^{k,\alpha}$. We can write for $\zeta \in b\Delta$

$$(1-\zeta)g''(\zeta) + (1-\overline{\zeta})\overline{g''(\zeta)} = (1-\zeta)v'(\zeta)$$

and since $(1-\overline{\zeta}) = -\overline{\zeta}(1-\zeta)$ for $\zeta \in b\Delta$,

$$g''(\zeta) - \overline{\zeta g''(\zeta)} = v'(\zeta).$$

Applying the Szego projection $\mathcal{P}' : \mathcal{C}_{\mathbb{C}}^{k,\alpha} \to \mathcal{A}^{k,\alpha}$ to both sides, we conclude that $g'' = \mathcal{P}'(v') \in \mathcal{A}^{k,\alpha}$. By the discussion above, this finishes the proof of the surjectivity of T_Y' at $p_0 = (\rho, c_0, h_0, g_0)$. $\qquad\square$

Now, we show that the kernel of T_Y' at $p_0 = (\rho, c_0, h_0, g_0)$ is finite dimensional:

Lemma 3.5. *The kernel of $T_Y'(p_0)$ has real dimension $2(k_0 + \ell_0) + 3 = 4k_0 - d + 3$.*

Proof. We will revisit the computations performed in Lemma 3.4, and start solving $(T_2')_Y(p_0)[c', h', g'] = \mathcal{P}\left(-\frac{\zeta^{k_0}}{2}c'\right) = 0$. According to Lemma 2.9, c' must be of the

form (2.3). Thus the projection of $\ker T'_Y(p_0)$ to the factor $\mathcal{C}^{k,\alpha}$ has real dimension $2k_0 + 1$.

Next, looking at the expression of $(T'_1)_Y(p_0)$, and taking in account that, as proved in Lemma 3.4, $\frac{\zeta^{k_0} h'(\zeta)}{s(\zeta)(1-\zeta)^{d-1}} P_{z^2}(1 - \zeta, 1 - \overline{\zeta}) \in \mathcal{A}^{k,\alpha}$ for any $h' \in (1 - \zeta)\mathcal{A}^{k,\alpha}$, we see that for any given c' of the form (2.3), we need to solve for h' the equation

$$(3.2) \qquad \mathcal{P}\left(\frac{\overline{h'(\zeta)}}{s(\zeta)(1-\zeta)^{d-1}} \zeta^{k_0} P_{z\overline{z}}(1 - \zeta, 1 - \overline{\zeta}) \right) = v$$

where

$$v = -\mathcal{P}\left(\frac{\zeta^{k_0}}{s(\zeta)(1-\zeta)^{d-1}} P_z(1 - \zeta, 1 - \overline{\zeta})c' \right) \in \overline{\mathcal{A}}^{k,\alpha}.$$

Since any two solutions of (3.2) differ by a solution of its homogenized version (i.e. with $v = 0$), it is of course enough to consider the latter. In view of the computations in Lemma 3.4, this amounts to solving

$$(3.3) \qquad \mathcal{P}\left(t(\zeta)\overline{h''(\zeta)} \right) = 0$$

for $h'' \in \mathcal{A}^{k,\alpha}$. According to Lemma 2.8, the kernel of $\mathcal{P}(t.)$ has complex dimension $\ell_0 + 1$. Thus the (affine) space of solutions of (3.2) has real dimension $2(\ell_0 + 1)$, and in turn that the projection of $\ker T'_Y(p_0)$ to the factor $\mathcal{C}^{k,\alpha} \times (1 - \zeta)\mathcal{A}^{k,\alpha}$ has real dimension $2k_0 + 2\ell_0 + 3$.

Finally, considering $(T'_3)_Y$, given any h' which satisfies (3.2) we must solve for $g' \in (1 - \zeta)\mathcal{A}^{k,\alpha}$ the equation

$$\frac{g'}{2} + \frac{\overline{g'}}{2} = P_z(1 - \zeta, 1 - \overline{\zeta})h' + P_{\overline{z}}(1 - \zeta, 1 - \overline{\zeta})\overline{h}'.$$

The same proof as in Lemma 3.4, however, shows that the previous equation admits a unique solution g'. By the previous arguments, we conclude that the kernel of $T'_Y(p_0)$ has real dimension $2(k_0 + \ell_0) + 3$. $\qquad \square$

We have proved that the derivative at the point $p_0 = (\rho, c_0, h_0, g_0)$ of $T'_Y(p_0)$: $Y \to (\mathcal{C}_{\mathbb{C}}^{k,\alpha})^2 \times \mathcal{C}_0^{k,\alpha}$ is surjective and that its kernel $\ker T'_Y(p_0)$ has real dimension $4k_0 - d + 3$. In particular, the Banach space Y can be decomposed as a direct sum $Y = \ker T'_Y(p_0) \oplus W$, where W is isomorphic to $(\mathcal{C}_{\mathbb{C}}^{k,\alpha})^2 \times \mathcal{C}_0^{k,\alpha}$. According to the implicit function theorem, there exist open neighborhoods V of ρ in X, U of 0 in $\ker T'_Y(p_0) \simeq \mathbb{R}^{4k_0 - d + 3}$, and U' of (c_0, f_0) in W, and a map $v : V \times U \to U'$, such that $T'(r, t \oplus w) = 0$ if and only if $w = v(r, t)$. It follows that the $\mathcal{C}^1$ map $\mathcal{F} : V \times U \to Y$ defined by $\mathcal{F}(r, t) = t \oplus v(r, t)$ satisfies the properties of Theorem 3.1.

4. Properties of k_0-stationary discs

Let $S = \{\rho = 0\}$ be a model hypersurface of type d

$$\rho(z, w) = -\operatorname{Re} w + P(z, \overline{z}) = -\operatorname{Re} w + \sum_{j=d-k_0}^{k_0} \alpha_j z^j \overline{z}^{d-j}.$$

Let k be an integer and let $0 < \alpha < 1$. Fix a k_0-stationary disc $f_0(\zeta) = (1 - \zeta, g_0)$ with $c_0 \equiv 1$. For $\eta > 0$, we denote by $\mathcal{S}_{0,\eta}^r$ the set

$$\mathcal{S}_{0,\eta}^r = \{(c, f) = (c, h, g) \in Y \mid (c, f) \in \mathcal{S}_0^r, \ \|(c, f) - (1, f_0)\|_Y < \eta\}.$$

Let U, V, $\eta > 0$, and $\mathcal{F} : U \times V \to \mathcal{S}_{0,\eta}^r$ given by Theorem 3.1. We write

$$\mathcal{F}(r, t) = (c_{r,t}, h_{r,t}, g_{r,t}).$$

Define a map $\Phi : U \times V \to \mathbb{R} \times \mathbb{C}^{k_0} \simeq \mathbb{R}^{2k_0+1}$ by

$$\Phi(r, t) = (c_{r,t,0}, c_{r,t,1}, \cdots, c_{r,t,k_0})$$

where $c_{r,t}(\zeta) = \sum_{n<0} c_{r,t,n} \zeta^n + c_{r,t,0} + \sum_{n>0} c_{r,t,n} \zeta^n$ with $c_{r,t,n} = \overline{c_{r,t,-n}}$. With a slight abuse of notations, we will also write $\Phi(r, c, f) = \Phi(r, \mathcal{F}(r, .)^{-1}(c, f))$; this notation will be used in the proof of Theorem 1.2.

Lemma 4.1. *Shrinking the neighborhoods U and V if necessary, the map Φ is a submersion.*

Proof. According to the proof of Lemma 3.5 and Lemma 2.9, the derivative $\frac{\partial \Phi}{\partial t}(\rho, 0)$ is of rank $2k_0 + 1$. Thus for r in a neighborhood of ρ and t sufficiently small, the rank of $\frac{\partial \Phi}{\partial t}(r, t)$ is also $2k_0 + 1$. This proves that for a fixed ρ, the map $\Phi(\rho, .) : V \to \mathbb{R}^{2k_0+1}$ is a submersion. $\square$

We define a real 2-dimensional submanifold Γ of $\mathbb{R} \times \mathbb{C}^{k_0}$ in the following way:

$$\Gamma = \{(c_0, \ldots, c_{k_0}) \in \mathbb{R} \times \mathbb{C}^{k_0} \mid c(b) = (\overline{b}\overline{\zeta} + 1 + b\zeta)^{k_0} = \sum_{n=-k_0}^{-1} \overline{c_n} \zeta^n + c_0 + \sum_{n=1}^{k_0} c_n \zeta^n,$$

$$b \in \mathbb{C}, |b| < 1/2\},$$

An easy computation shows that

$$\begin{cases} c_0(b) &= 1 + O(|b|^2), \\ c_1(b) &= k_0 b + O(|b|^2), \\ c_n(b) &= O(|b|^2) \text{ for } 2 \le n \le k_0. \end{cases}$$

It follows that the tangent space of Γ at the point $c_0 = 1$ is generated by the vectors $(0, 1, 0, \ldots, 0)$ and $(0, i, 0, \ldots, 0)$. Using Lemma 4.1, the set $\Phi^{-1}(\Gamma)$ is a smooth submanifold of $U \times V$, and, for $r \in U$, the submanifold

$$\mathcal{M}_\eta^r = \mathcal{S}_{0,\eta}^r \cap \mathcal{F}(\Phi^{-1}(\Gamma))$$

has real dimension $2\ell_0 + 4 = 2k_0 - d + 4 = 4k_0 - d + 3 - (2k_0 - 1)$. Henceforth, we will restrict our attention to the submanifold $\mathcal{M}_\eta^r$ and we will study the properties of discs in that submanifold.

Remark 4.2. Notice that the point $p_0 = (\rho, c_0, h_0, g_0)$ belongs to $\mathcal{M}_\eta^\rho$. More in general, for $b \in \mathbb{C}$ such that $|b| < 1/2$, define $p(b) = (\rho, c(b), h(b), g(b))$ with $c(b) = (\overline{b}\zeta + 1 + b\zeta)^{k_0}$, $h(b) = \frac{1-\zeta}{1-a(b)\zeta}$ where $a(b) = \frac{-1 + \sqrt{1 - 4|b|^2}}{2b}$ and $g(b)$ is uniquely determined by $h(b)$. Then $p(b)$ belongs to $\mathcal{M}_\eta^\rho$ for all $b \in \mathbb{C}$ with $|b| < 1/2$. We will use this special family of k_0-stationary discs alongside with Lemma 3.5 to compute the tangent space of $\mathcal{M}_\eta^\rho$ at p_0.

Define $\pi : Y \to (1 - \zeta)\mathcal{A}^{k,\alpha} \times (1 - \zeta)\mathcal{A}^{k,\alpha}$ to be the projection on the (h, g) factor and $\pi' : Y \to (1 - \zeta)\mathcal{A}^{k,\alpha}$ to be the projection on the h factor.

Lemma 4.3. *The restrictions of π' and of π to $T_{p_0}\mathcal{M}_\eta^\rho$ are injective.*

Proof. We first note that, as follows from the proofs of Lemmas 3.4 and 3.5), for any $(c', h', g') \in T_{p_0}\mathcal{M}_\eta^\rho$ we have that g' is uniquely determined by h'. It immediately follows that π is injective if and only if π' is injective, and in the rest of the proof we can ignore the factor relative to g'.

We start by computing the tangent space of the real 2-dimensional submanifold of $\mathcal{M}_\eta^\rho$ given by the parametrization $p(b)$ defined above, at $p_0 = p(0)$. Developing $c(b)$ and $h(b)$ up to first order in b and taking in account that $a(b) = -\overline{b} + O(|b|^2)$, we get

$$\begin{cases} c(b) &= \overline{b}\zeta + 1 + b\zeta + O(|b|^2), \\\\ h(b) &= (1 - \zeta)(1 - \overline{b}\zeta) + O(|b|^2). \end{cases}$$

It follows that the (projection to the (c, h) factor of the) tangent space of the submanifold parametrized by $p(b)$ at p_0 is generated over $\mathbb{R}$ by $(\overline{\zeta} + \zeta, -(1 - \zeta)\zeta)$ and $(-i\overline{\zeta} + i\zeta, i(1 - \zeta)\zeta)$. In view of the proof of Lemma 3.5, the elements of (the projection to the (c, h) space of) $T_{p_0}\mathcal{M}_\eta^\rho$ are the vectors given by

$$(4.1) \qquad (2t_1 \operatorname{Re}\zeta - 2t_2 \operatorname{Im}\zeta, (1 - \zeta) \cdot ((-t_1 + it_2)\zeta + h''(\zeta)))$$

for $t_1, t_2 \in \mathbb{R}$ and where h'' is a solution of (3.3) for $v = 0$.

Define now the linear subspace $\mathcal{H}'' \subset \mathcal{A}^{k,\alpha}$ as the span over $\mathbb{C}$ of ζ and of homogeneous solutions of (3.3). Moreover, set $\mathcal{H}' = (1 - \zeta)\mathcal{H}'' \subset (1 - \zeta)\mathcal{A}^{k,\alpha}$. By

(4.1), it follows that $\mathcal{H}'$ is the image of the restriction of π' to $T_{p_0}\mathcal{M}_\eta^\rho$, and π' is injective if and only if the function ζ is not a solution of (3.3). Recal that $t(\zeta) = \prod_{i_0+1\leq j\leq 2k_0-d}(q_j - \zeta)$. We have

$$\mathcal{P}(t(\zeta)\zeta) = \frac{\prod_{i_0+1\leq j\leq 2k_0-d} q_j}{\zeta}.$$

Since none of the roots of t is equal to zero we have $\mathcal{P}(t(\zeta)\zeta) \neq 0$.

$\square$

In particular, notice that $\dim_\mathbb{R} \mathcal{H}' = 2\ell_0 + 4$.

Define now, for any $n \in \mathbb{N}$, $n \leq k$, the *n-jet map at 1* as the complex linear map $j_n : (1 - \zeta)\mathcal{A}^{k,\alpha} \to \mathbb{C}^n$ given by

$$j_n(h) = (h'(1), h''(1), \ldots, h^{(n)}(1))$$

for any $h \in (1 - \zeta)\mathcal{A}^{k,\alpha}$.

Proposition 4.4. *Let $S = \{\rho = 0\}$ be a model hypersurface of finite type d. Fix a k_0-stationary disc $f_0(\zeta) = (1 - \zeta, g_0(\zeta))$ with $c_0 \equiv 1$, attached to S. Then*

  i) *There exists a sufficiently small $\theta \in \mathbb{R}$ such that the derivative of the evaluation map $\varphi : \pi'(\mathcal{M}_\eta^\rho) \to \mathbb{C}^2$ defined by $\varphi(h) = (h(0), g(0))$, at the point $h_\theta(\theta) = (1 - \zeta)e^{i\theta}$, is surjective.*
  ii) *Shrinking the neighborhood U of ρ, and $\eta > 0$, given in Theorem 3.1, for $r \in U$, for $\ell_0 = k_0 - d/2$, the restriction of j_{ℓ_0+2} to $\pi'(\mathcal{M}_\eta^r)$ is a diffeomorphism onto its image.*

In particular, point $i)$ of Proposition 4.4 implies that the set $\{(h(0), g(0)) \in \mathbb{C}^2 | (h, g) \in \pi(\mathcal{M}_\eta^\rho)\} \subseteq \{(h(0), g(0)) \in \mathbb{C}^2 | (c, h, g) \in \mathcal{S}_0^\rho\}$ contains an open set.

4.1. **Proof of $i)$ of Proposition 4.4.** Define the composed map

$$\psi : \pi'(\mathcal{M}_\eta^\rho) \to \pi(\mathcal{M}_\eta^\rho) \to \mathbb{C}^2$$

defined by $\psi : h \mapsto (h, g) \mapsto (h(0), g(0))$. Recall that $(h, g) \in \mathcal{S}_0^\rho$ satisfy for $\zeta \in b\Delta$

$$\mathrm{Re}\, g(\zeta) = P(h(\zeta), \overline{h(\zeta)}).$$

Thus, from $g(1) = 0$ and from classical facts on Cauchy transform (see Lemma 3 in [36]), we have

$$g(0) = \frac{1}{i\pi}\int_{b\Delta}\frac{\mathrm{Re}\, g(\zeta)}{1 - \zeta}\frac{d\zeta}{\zeta} = \frac{1}{i\pi}\int_{b\Delta}\frac{P(h(\zeta), \overline{h(\zeta)})}{1 - \zeta}\frac{d\zeta}{\zeta}.$$

We consider the subset $\mathcal{M}' := \left\{ h \in \pi'(\mathcal{M}_\eta^\rho) \mid h(\zeta) = \frac{1-\zeta}{1-a\zeta} v, a \in \Delta, v \in \mathbb{C}^* \right\}$. Notice that for such a disc h, the tangent disc $h' \in T_h \mathcal{M}'$ can be written

$$h'(\zeta) = \frac{(1-\zeta)\zeta}{(1-a\zeta)^2} v a' + \frac{1-\zeta}{1-a\zeta} v'$$

where $(a', v') \in \mathbb{C}^2$. Consider the disc $h_\theta \in \mathcal{M}'$ given by $h_\theta(\zeta) = (1-\zeta)e^{i\theta} \in \mathcal{M}'$ for θ sufficiently small. The derivative of ψ at $h_\theta \in \mathcal{M}'$ is given by

$$d_{h_\theta}\psi(h') \quad = \quad (h'(0), d_{h_\theta}g(0)(h'))$$

$$= \quad \left(h'(0), \frac{1}{i\pi} \int_{b\Delta} \frac{P_z(h_\theta(\zeta), \overline{h_\theta(\zeta)})h'(\zeta)}{1-\zeta} \frac{d\zeta}{\zeta} + \frac{1}{i\pi} \int_{b\Delta} \frac{P_{\bar{z}}(h_\theta(\zeta), \overline{h_\theta(\zeta)})\overline{h'(\zeta)}}{1-\zeta} \frac{d\zeta}{\zeta} \right).$$

where $h' \in T_{h_\theta}\mathcal{M}'$. Now we fix $(Z, W) \in \mathbb{C}^2$ and we solve $d_{h_\theta}\psi(h') = (Z, W)$. The first component gives $v' = Z$ and the second component leads to

$$i\pi W \quad = \quad \int_{b\Delta} P_z\left(h_\theta(\zeta), \overline{h_\theta(\zeta)}\right) Z \frac{d\zeta}{\zeta} + \int_{b\Delta} P_z\left(h_\theta(\zeta), \overline{h_\theta(\zeta)}\right) e^{i\theta} a' d\zeta$$

$$- \int_{b\Delta} P_{\bar{z}}\left(h_\theta(\zeta), \overline{h_\theta(\zeta)}\right) \overline{Z} \frac{d\zeta}{\zeta^2} - \int_{b\Delta} P_{\bar{z}}\left(h_\theta(\zeta), \overline{h_\theta(\zeta)}\right) e^{-i\theta} \overline{a'} \frac{d\zeta}{\zeta^3}.$$

Set $I_1(\theta) = \int_{b\Delta} P_z\left(h_\theta(\zeta), \overline{h_\theta(\zeta)}\right) d\zeta$ and $I_2(\theta) = -\int_{b\Delta} P_{\bar{z}}\left(h_\theta(\zeta), \overline{h_\theta(\zeta)}\right) \frac{d\zeta}{\zeta^3}$. It follows that $d_{h_\theta}\psi(h')$ is surjective if and only if $|I_1(\theta)|^2 \neq |I_2(\theta)|^2$. A straightforward computation leads to

$$I_1(\theta) = - \sum_{j=d-k_0}^{k_0} \binom{d-1}{d-1-j} j\alpha_j e^{i(2j-d-1)\theta}.$$

$$I_2(\theta) = \sum_{j=d-k_0}^{k_0-2} \binom{d-1}{d-3-j} (d-j)\alpha_j e^{i(2j-d+1)\theta}$$

if $d - 3 < k_0$, and

$$I_2(\theta) = \sum_{j=d-k_0}^{k_0} \binom{d-1}{d-3-j} (d-j)\alpha_j e^{i(2j-d+1)\theta}.$$

otherwise. The highest degree term of the trigonometric polynomial $|I_1(\theta)|^2$ is given by

$$I_1(\theta)' = k_0(d-k_0)\alpha_{k_0}\overline{\alpha_{d-k_0}} \binom{d-1}{k_0-1}\binom{d-1}{d-1-k_0} j\alpha_j e^{i(4k_0-2d)\theta}.$$

In case $k_0 \leq d - 3$, the highest degree term of the trigonometric polynomial $|I_2(\theta)|^2$ is given by

$$I_2(\theta)' = k_0(d - k_0)\alpha_{k_0}\overline{\alpha_{d-k_0}}\binom{d-1}{k_0-3}\binom{d-1}{d-3-k_0}j\alpha_j e^{i(4k_0-2d)\theta}.$$

Since $\binom{d-1}{k_0-1}\binom{d-1}{d-1-k_0} < \binom{d-1}{k_0-3}\binom{d-1}{d-3-k_0}$, there exist a sufficiently small θ such that $|I_1(\theta)|^2 \neq |I_2(\theta)|^2$. Finally if $d-3 < k_0$, the degrees of the trigonometric polynomial of $|I_2(\theta)|^2$ and $|I_1(\theta)|^2$ are different and therefore $|I_1(\theta)|^2 \neq |I_2(\theta)|^2$ for some sufficiently small θ. $\qquad\square$

4.2. **Proof of** $ii)$ **of Proposition 4.4.** In order to prove the second part of Proposition 4.4. it is sufficient to show that the restriction of $\mathbf{j}_{\ell_0+2}$ to $\pi'(T_{p_0}\mathcal{M}_\eta^\rho)$ is injective.

We first compute explicitly the tangent space of $\mathcal{M}_\eta^\rho$ at p_0. Recall that the elements of (the projection to the (c,h) space of) $T_{p_0}\mathcal{M}_\eta^\rho$ are given by

$$(2t_1\mathrm{Re}\,\zeta - 2t_2\mathrm{Im}\,\zeta, (1 - \zeta) \cdot ((-t_1 + it_2)\zeta + h''(\zeta)))$$

for $t_1, t_2 \in \mathbb{R}$ and where h'' is a solution of (3.3) for $v = 0$. We need to describe explicitly the solutions h'' of (3.3) for $v = 0$. If $t(\zeta) \equiv 1$ put $\ell_0 = 0$, otherwise write $t(\zeta) = \sum_{i=0}^{\ell_0} t_i\zeta^i$ where ℓ_0 is defined in Lemma 3.4 and $t_i \in \mathbb{C}$. We need to find $h''(\zeta) = \sum_{n\geq 0} h_n''\zeta^n$ such that $t(\zeta)\overline{h''(\zeta)} \in \mathcal{A}^{k,\alpha}$. This translates into the following recursion:

$$(4.2) \qquad t_{\ell_0}\overline{h}''_{n+\ell_0} + \ldots + t_1\overline{h}''_{n+1} + t_0\overline{h}''_n = 0$$

for all $n \geq 1$. Therefore, we must examine the solutions of (4.2). Let $r_1, \ldots, r_{\ell_1}$ be the *distinct* roots of the polynomial $t(\zeta)$, with, respectively, multiplicity $m_1, \ldots, m_{\ell_1}$, so that $\sum_{j=1}^{\ell_1} m_j = \ell_0$. Moreover, put $r_0 = 0$, $m_0 = 1$. The general solution of (4.2) is then given by $h_n'' = \sum_{j=0}^{\ell_1} Q_j(n)\overline{r}_j^n$, $n \geq 0$, where with a slight abuse of notation we set $(r_0)^0 = 1$ and where each Q_j is a polynomial of degree at most $m_j - 1$ with complex coefficients. Put $R_j(\zeta) = \sum_{n\geq 0} \overline{r}_j^n\zeta^n = \frac{1}{1-\overline{r}_j\zeta}$. It is easy to see that, for any fixed polynomial Q_j of degree $m_j - 1$, there exist $q_{j,0}, \ldots, q_{j,m_j-1} \in \mathbb{C}$ such that $\sum_{n\geq 0} Q_j(n)\overline{r}_j^n\zeta^n = \sum_{i=0}^{m_j-1} q_{j,i}\zeta^i R_j^{(i)}(\zeta)$. Hence for the general solution $h''(\zeta)$ of the homogeneous version of (3.3) we get

$$h''(\zeta) = \sum_{j=0}^{\ell_1}\sum_{n\geq 0} Q_j(n)\overline{r}_j^n\zeta^n = \sum_{j=0}^{\ell_1}\sum_{i=0}^{m_j-1} q_{j,i}\zeta^i R_j^{(i)}(\zeta) = \sum_{j=0}^{\ell_1}\sum_{i=0}^{m_j-1} i! \cdot q_{j,i}\frac{\overline{r}_j^i\zeta^i}{(1 - \overline{r}_j\zeta)^{i+1}}$$

for $q_{j,i} \in \mathbb{C}$. Note that, for all $1 \leq j \leq \ell_1$, $0 \leq i \leq m_j - 1$

$$\frac{\overline{r}_j^i\zeta^i}{(1 - \overline{r}_j\zeta)^{i+1}} = \frac{(1 - (1 - \overline{r}_j\zeta))^i}{(1 - \overline{r}_j\zeta)^{i+1}} = \sum_{\kappa=0}^{i}(-1)^\kappa\binom{i}{\kappa}\frac{1}{(1 - \overline{r}_j\zeta)^{\kappa+1}},$$

hence we can also write the general solution as

$$(4.3) \qquad h''(\zeta) = s_{0,0} + \sum_{j=1}^{\ell_1} \sum_{i=0}^{m_j - 1} s_{j,i} \frac{1}{(1 - \overline{r}_j \zeta)^{i+1}}$$

for $s_{j,i} \in \mathbb{C}$. It follows that the elements of (the projection to the (c,h) space of) $T_{p_0} \mathcal{M}_\eta^\rho$ are the vectors given by

$$(2t_1 \mathrm{Re}\,\zeta - 2t_2 \mathrm{Im}\,\zeta, (1-\zeta) \cdot ((-t_1 + it_2)\zeta + h''(\zeta)))$$

for $t_1, t_2 \in \mathbb{R}$ and $h''(\zeta)$ as in (4.3).

Let $\mathcal{H}' = \pi'(T_{p_0}\mathcal{M}_\eta^\rho)$ be the $(2\ell_0 + 4)$-dimensional space introduced in Lemma 4.3, and set

$$u_0 = 1, \ u_1 = \zeta, \ \left\{ u_{j,i} = \frac{1}{(1 - r_j\zeta)^{i+1}} \right\}_{\substack{1 \leq j \leq j_1, \\ 0 \leq i \leq m_j - 1}}.$$

By definition, a basis of $\mathcal{H}'$ is then given by $(1-\zeta)u_0$, $(1-\zeta)u_1$ and $(1-\zeta)u_{j,i}$. By the Leibniz rule, for every $v(\zeta) = (1-\zeta)u(\zeta)$ and $n \geq 1$ we have $v^{(n)}(\zeta) = (1-\zeta)u^{(n)} - nu^{(n-1)}(\zeta)$, so that $v^{(n)}(1) = -nu^{(n-1)}(1)$. Since

$$\frac{d^n}{d\zeta^n} \left(\frac{1}{(1 - r_j\zeta)^{i+1}} \right) = \frac{(i+n)!}{i!} \cdot \frac{r_j^n}{(1 - r_j\zeta)^{i+n+1}},$$

it follows that, for every $n > 2$, the expression of $\mathfrak{j}_n$ in the given basis is

$$(-1) \cdot \begin{pmatrix} 1 & 1 & \frac{1}{1-r_1} & \frac{1}{(1-r_1)^2} & \cdots & \frac{1}{(1-r_{\ell_1})^{m_{\ell_1}}} \\ 0 & 2 & 2\frac{r_1}{(1-r_1)^2} & 2\frac{2r_1}{(1-r_1)^3} & \cdots & 2\frac{m_{\ell_1} r_{\ell_1}}{(1-r_{\ell_1})^{m_{\ell_1}+1}} \\ 0 & 0 & 3\frac{2r_1^2}{(1-r_1)^3} & 3\frac{6r_1^2}{(1-r_1)^4} & \cdots & 3\frac{m_{\ell_1}(m_{\ell_1}+1)r_{\ell_1}^2}{(1-r_{\ell_1})^{m_{\ell_1}+2}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & n\frac{(n-1)!r_1^{n-1}}{(1-r_1)^n} & n\frac{n!r_1^{n-1}}{(1-r_1)^{n+1}} & \cdots & n\frac{(m_{\ell_1}+n-2)!r_{\ell_1}^{n-1}}{(m_{\ell_1}-1)!(1-r_{\ell_1})^{m_{\ell_1}+n-1}} \end{pmatrix}.$$

Taking $n = \ell_0 + 2$, the one above is a square matrix of size $\ell_0 + 2$. Put now $\chi_j = \frac{r_j}{1-r_j}$. After a suitable sequence of scalar multiplications of the row/columns, we can transform the previous matrix into the following one:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & \chi_1 & 2\chi_1 & \cdots & m_{\ell_1}\chi_{\ell_1} \\ 0 & 0 & \chi_1^2 & 3\chi_1^2 & \cdots & \binom{m_{\ell_1}+1}{2}\chi_{\ell_1}^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \chi_1^{\ell_0+1} & (\ell_0+2)\chi_1^{\ell_0+1} & \cdots & \binom{m_{\ell_1}+\ell_0}{\ell_0+1}\chi_{\ell_1}^{\ell_0+1} \end{pmatrix}.$$

Next, performing a sequence of subtractions of columns and keeping in account the properties of the binomial coefficients, we get

$$
\begin{pmatrix}
1 & 1 & 1 & -2 & \cdots & 1 & -2 & \cdots & (-1)^{m_{\ell_1}+1}(m_{\ell_1}+1) \\
0 & 1 & \chi_1 & -\chi_1 & \cdots & \chi_{\ell_1} & -\chi_{\ell_1} & \cdots & (-1)^{m_{\ell_1}+1}\chi_{\ell_1} \\
0 & 0 & \chi_1^2 & 0 & \cdots & \chi_{\ell_1}^2 & 0 & \cdots & 0 \\
0 & 0 & \chi_1^3 & \chi_1^3 & \cdots & \chi_{\ell_1}^3 & \chi_{\ell_1}^3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \chi_1^{\ell_0+1} & (\ell_0-1)\chi_1^{\ell_0+1} & \cdots & \chi_{\ell_1}^{\ell_0+1} & (\ell_0-1)\chi_1^{\ell_0+1} & \cdots & \binom{\ell_0-1}{\ell_0-m_{\ell_1}}\chi_{\ell_1}^{\ell_0+1}
\end{pmatrix}.
$$

Finally, a further sequence of column multiplications gives

$$
\begin{pmatrix}
1 & 1 & \frac{1}{\chi_1^2} & \frac{-2}{\chi_1^3} & \cdots & \frac{1}{\chi_{\ell_1}^2} & \frac{-2}{\chi_{\ell_1}^3} & \cdots & \frac{(-1)^{m_{\ell_1}+1}(m_{\ell_1}+1)(m_{\ell_1}-1)!}{\chi_{\ell_1}^{m_{\ell_1}+1}} \\
0 & 1 & \frac{1}{\chi_1} & \frac{-1}{\chi_1^2} & \cdots & \frac{1}{\chi_{\ell_1}} & \frac{-1}{\chi_{\ell_1}^2} & \cdots & \frac{(-1)^{m_{\ell_1}+1}(m_{\ell_1}-1)!}{\chi_{\ell_1}^{m_{\ell_1}}} \\
0 & 0 & 1 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 0 & \chi_1 & 1 & \cdots & \chi_{\ell_1} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \chi_1^{\ell_0-1} & (\ell_0-1)\chi_1^{\ell_0-2} & \cdots & \chi_{\ell_1}^{\ell_0-1} & (\ell_0-1)\chi_1^{\ell_0-2} & \cdots & \frac{(\ell_0-1)!}{(\ell_0-m_{\ell_1})!}\chi_{\ell_1}^{\ell_0-m_{\ell_1}}
\end{pmatrix}.
$$

The $\ell_0 \times \ell_0$ minor obtained by erasing the first two rows and columns of the previous matrix is a confluent Vandermonde matrix: it is a well-known fact that its determinant does not vanish. It follows that the determinant of the whole matrix is in turn non-vanishing, hence j_{ℓ_0+2} is injective as claimed.

$\square$

5. APPLICATION TO FINITE JET DETERMINATION

5.1. **Dilation.** Let $\mathcal{U}$ be a neighborhood of 0 in $\mathbb{C}^2$, and let $r : \mathcal{U} \to \mathbb{R}$ be a real valued function of class $\mathcal{C}^{d+k+3}$ that can be written as (3.1), namely

$$
r(z,w) = \rho(z,w) + \sum_{i+j=d+1} (z^i\overline{z}^j) \cdot r_{ij0}(z) + \sum_{l=1}^{d} \sum_{i+j=d-l} z^i\overline{z}^j (\operatorname{Im}w)^l \cdot r_{ijl}(z, \operatorname{Im}w)
$$

for suitable $r_{ij0} \in \mathcal{C}_{\mathbb{C}}^{k+3}$, $r_{ijl} \in \mathcal{C}_{\mathbb{C}}^{k+3}$. Moreover, let $H : \mathcal{U} \to \mathbb{C}^2$ be a local CR-diffeomorphism of class $\mathcal{C}^{k+d+2}$, tangent to the identity, and preserving $\{r=0\}$. We still denote by H its local one-sided extension which is of class $\mathcal{C}^{k+d+2}$ up to $\{r=0\}$.

We can write $H = (H_1, H_2)$, where

$$\begin{cases} H_1(z,w) & = & z + \sum_{j+l=2} z^j w^l H_1^{jl}(z,w), \\[2mm] H_2(z,w) & = & w + \sum_{j+l=2} z^j w^l H_2^{jl}(z,w), \end{cases}$$

where H_1^{jl} and H_2^{jl} are of class $\mathcal{C}^{k+d}$.

Lemma 5.1. *We have $H_2(z,0) = O(|z|^{d+1})$.*

Proof. Setting $s(z, \operatorname{Im} w) = r(z,w) - \operatorname{Re} w$, we can locally write $\{r = 0\}$ as $\{\operatorname{Re} w = s(z, \operatorname{Im} w)\}$. The fact the hypersurface is invariant under H translates into the mapping equation

$$\operatorname{Re} H_2(z, s(z, \operatorname{Im} w) + i\operatorname{Im} w) = s(H_1(z, s(z, \operatorname{Im} w) + i\operatorname{Im} w), \operatorname{Im} H_2(z, s(z, \operatorname{Im} w) + i\operatorname{Im} w))$$

which, computed along $\operatorname{Im} w = 0$, gives

(5.1) $$\operatorname{Re} H_2(z, s(z,0)) = s(H_1(z, s(z,0)), \operatorname{Im} H_2(z, s(z,0))).$$

Suppose, by contradiction, that for some $\kappa \leq d$ the monomial of smallest order appearing in the Taylor expansion of $H_2(z,0)$ is z^κ. Then since $\operatorname{Re} H_2(z,w) = \operatorname{Re} w + O(2)$, the left-hand-side of (5.1) can be expressed as

$$\operatorname{Re} H_2(z, s(z,0)) = \operatorname{Re}(az^\kappa) + P(z, \overline{z}) + O(|z|^{\min(\kappa, d)+1})$$

for some $a \neq 0$. Taking in account that $\operatorname{Im} H_2(z, s(z,0)) = \operatorname{Im}(az^\kappa) + O(|z|^{\kappa+1})$ and the expression of r, we can expand the right-hand-side as

$$s(H_1(z, s(z,0)), \operatorname{Im} H_2(z, s(z,0))) = P(z, \overline{z}) + O(|z|^{\min(\kappa, d)+1})$$

which immediately gives a contradiction if $\kappa \leq d$. $\qquad\square$

For $t > 0$, denote by ϕ_t the linear map

$$\phi_t(z, w) = (tz, t^d w),$$

and let $r_t = \frac{1}{t^d} r \circ \phi_t$ and $H_t = \phi_t^{-1} \circ H \circ \phi_t$ be the direct images of, respectively, r and H, under ϕ_t. The purpose of using a dilation method is to ensure that the scaled hypersurface $\{r_t = 0\}$ is close enough to the model hypersurface and that H_t is a small perturbation of the identity as $t \to 0$. This allows to use the results of the previous sections to obtain a jet determination for H_t, and thus for H.

For any $\delta > 0$, if $t > 0$ is small enough, we have $\phi_t\left(\overline{\delta\Delta}^2\right) \subset \mathcal{U}$, hence H_t is defined on $\overline{\delta\Delta}^2$ and $r_t \in X$ (see Section 3 for the definition of the space X). Although the following lemma is a classical fact, we include the proof because of the nonstandard choice of the norm $\|.\|_X$.

Lemma 5.2. *We have $\|r_t - \rho\|_X \to 0$ as $t \to 0$.*

Proof. Indeed,

$$r_t(z,w) = \rho(z,w) + \sum_{i+j=d+1} (z^i \bar{z}^j) \cdot r^t_{ij0}(z) + \sum_{l=1}^{d} \sum_{i+j=d-l} z^i \bar{z}^j (\operatorname{Im} w)^l \cdot r^t_{ijl}(z, \operatorname{Im} w)$$

with

$$\begin{cases} r^t_{ij0}(z) & = \ t \ r_{ij0}(tz), \\[2mm] r^t_{ijl}(z,w) & = \ t^{l(d-1)} r_{ijl}(tz, t^d \operatorname{Im} w) \ \text{ for } l \geq 1, \end{cases}$$

It is clear that $\|r^t_{ijl}\|_{\mathcal{C}^{k+3}(\overline{\delta\Delta^2})} \to 0$ as $t \to 0$, and the claim of the lemma follows from the definition of the norm in the space X. $\qquad\square$

We will employ now the notation $Z = (1-\zeta)\mathcal{A}^{k,\alpha} \times (1-\zeta)\mathcal{A}^{k,\alpha}$ and the norm

$$\|(h,g)\|_Z = \|h\|_{(1-\zeta)\mathcal{A}^{k,\alpha}} + \|g\|_{(1-\zeta)\mathcal{A}^{k,\alpha}}.$$

We also need to show that

Lemma 5.3. *For all $f = (h,g) \in Z$ and $t > 0$ small enough we have $H_t \circ f \in Z$ and*

$$\|H_t \circ f - f\|_Z \leq tK \max(1, \|f\|_Z^{d+k+2})$$

for some constant $K > 0$.

Proof. Indeed we have $H_t(z,w) = (H^t_1(z,w), H^t_2(z,w))$, with

$$\begin{cases} H^t_1(z,w) = z + \sum_{j+l=2} t^{j+dl-1} z^j w^l H^{jl}_1(tz, t^d w), \\[2mm] H^t_2(z,w) = w + \sum_{j+l=2} t^{j+dl-d} z^j w^l H^{jl}_2(tz, t^d w). \end{cases}$$

Posing $h = (1-\zeta)\tilde{h}, g = (1-\zeta)\tilde{g}$, with $\tilde{h}, \tilde{g} \in \mathcal{A}^{k,\alpha}$, we get

$$\begin{cases} H^t_1(h,g) & = \ h + (1-\zeta)^2 \sum_{j+l=2} t^{j+dl-1} \tilde{h}^j \tilde{g}^l H^{jl}_1(t(1-\zeta)\tilde{h}, t^d(1-\zeta)\tilde{g}), \\[2mm] H^t_2(h,g) & = \ g + (1-\zeta)^2 \sum_{j+l=2} t^{j+dl-d} \tilde{h}^j \tilde{g}^l H^{jl}_2(t(1-\zeta)\tilde{h}, t^d(1-\zeta)\tilde{g}). \end{cases}$$

It follows that

$$\begin{cases} \|H^t_1(h,g) - h\|_{(1-\zeta)\mathcal{A}^{k,\alpha}} & = \ \left\| (1-\zeta) \sum_{j+l=2} t^{j+dl-1} \tilde{h}^j \tilde{g}^l H^{jl}_1(t(1-\zeta)\tilde{h}, t^d(1-\zeta)\tilde{g}) \right\|_{\mathcal{C}^{k,\alpha}_{\mathbb{C}}}, \\[3mm] \|H^t_2(h,g) - g\|_{(1-\zeta)\mathcal{A}^{k,\alpha}} & = \ \left\| (1-\zeta) \sum_{j+l=2} t^{j+dl-d} \tilde{h}^j \tilde{g}^l H^{jl}_2(t(1-\zeta)\tilde{h}, t^d(1-\zeta)\tilde{g}) \right\|_{\mathcal{C}^{k,\alpha}_{\mathbb{C}}}. \end{cases}$$

Note that in the previous sum, if $l \neq 0$ then $j + ld - d \geq 1$. If $l = 0$, we have to consider the summand

$$t^{2-d}\tilde{h}^2 H^{2,0}_2(t(1-\zeta)\tilde{h}, t^d(1-\zeta)\tilde{g}).$$

since by Lemma 5.1 we can express $H_2^{2,0}$ as

$$H_2^{2,0}(z,w) = z^{d-1}\phi_1(z) + w\phi_2(z,w)$$

where ϕ_1 and ϕ_2 are of class $\mathcal{C}^{k+1}$, we can write the summand as

$$t(1-\zeta)^{d-1}\tilde{h}^{d+1}\phi_1(t(1-\zeta)\tilde{h}) + t^2(1-\zeta)\tilde{h}^2\tilde{g}\phi_2(t(1-\zeta)\tilde{h}, t^d(1-\zeta)\tilde{g})$$

The estimate of the lemma then follows from the previous expressions, using the facts that the norm of $(1-\zeta)$ in $\mathcal{A}_{k,\alpha}$ is finite, and that the $\mathcal{C}^{k,\alpha}$ norm of a composition is estimated by a constant times suitable powers of the $\mathcal{C}^{k,\alpha}$ norms of the maps which are composed. □

5.2. **Proof of Theorem 1.2.** Let M be a real hypersurface of class $\mathcal{C}^{d/2+k_0+5}$ whose defining function r is locally written as in Theorem 1.1 and let H be a germ at 0 of CR-diffeomorphism of class $\mathcal{C}^{d/2+k_0+3}$ such that $H(0) = 0$ and $H(M) = M$. We assume that H is tangent to the identity up to order $\ell_0 + 2 = k_0 - d/2 + 2$. We denote by H its local one-sided extension, of class $\mathcal{C}^{d/2+k_0+3}$ up to M.

Let $0 < \alpha < 1$. According to the first point of Proposition 4.4, for every sufficiently small $\epsilon > 0$ there exists a k_0-stationary disc $f \in Z$ attached to the model hypersurface $S = \{\rho = 0\}$ with coefficient $c \in \mathcal{C}^{\ell_0+2,\alpha}$ such that $\|(c,f) - (1,f_0)\|_Y < \epsilon$ and the restriction of the evaluation map φ, as a map from $\mathcal{M}_\eta^\rho$ to $\mathbb{C}^2$, to the tangent space of $\mathcal{M}_\eta^\rho$ at (c,f) is onto. As a consequence we have the following: for every sufficiently small $\epsilon > 0$, there exists $\epsilon' > 0$ such that for all $r \in X$ with $\|r - \rho\|_X < \epsilon'$ there exists a k_0-stationary disc $f \in Z$ attached to $\{r = 0\}$ with coefficient function $c \in \mathcal{C}^{\ell_0+2,\alpha}$ such that $\|(c,f) - (1,f_0)\|_Y < \epsilon$ and the restriction of the evaluation map φ to the tangent space of $\mathcal{M}_\epsilon^r$ at (c,f) is onto.

Fix now $\epsilon > 0$ such that $\epsilon < \eta/2$, where η is defined in Theorem 3.1. Using Lemma 5.2, we can find $t > 0$ such that $r_t \in V$ and $\|r_t - \rho\| < \epsilon'$, where V is neighborhood of ρ in X identified in Theorem 3.1. We can further require that $t < \epsilon/2^{d+\ell_0+4}K\|f_0\|_Z^{d+\ell_0+4}$ with K as in the statement of Lemma 5.3. In view of the previous paragraph, we can find a k_0-stationary disc $(c_1, f_1) \in \mathcal{M}_\epsilon^{r_t}$, such that the evaluation map φ is locally surjective in a neighborhood (c_1, f_1).

Hence, we can find a neighborhood O' of (c_1, f_1) in $\mathcal{M}_\epsilon^{r_t}$ such that $\varphi(O') = O$ is an open subset of the bidisc $(\delta\Delta)^2$. Choose any $q \in O$, and let $(c_q, f_q) \in O' \subset \mathcal{M}_\epsilon^{r_t}$ be such that $f(0) = q$. Note that $\|f_q\|_Z \leq \|f_0\|_Z + \epsilon \leq 2\|f_0\|_Z$ if ϵ is small enough. Since H_t is tangent to the identity, we can apply Lemma 5.3 to get

$$(5.2) \qquad \|H_t \circ f_q - f_q\|_Z \leq tK\|f_q\|_Z^{d+\ell_0+4} \leq tK2^{d+\ell_0+4}\|f_0\|^{d+\ell_0+4} \leq \epsilon$$

by the choice of t. On the other hand, by Proposition 2.4 the disc $H_t \circ f_q$ is again a stationary disc attached to $\{r_t = 0\}$, with coefficient function c_q, and tied to the

origin. Moreover, from (5.2) follows that

$$\|(c_q, H_t \circ f_q) - (c_0, f_0)\|_Y \leq 2\epsilon < \eta.$$

Thus $(c_q, H_t \circ f_q) \in \mathcal{S}^{r_t}_{0,\eta}$, and since $\Phi(r_t, c_q, H_t \circ f_q) = \Phi(r_t, c_q, f_q) \in \Gamma$ we deduce that $(c_q, H_t \circ f_q) \in \mathcal{M}^{r_t}_\eta$.

Since H_t is tangent to the identity up to order $\ell_0 + 2$, we also have that the holomorphic disc $H_t \circ f_q(\zeta)$ is tangent to $f_q(\zeta)$ at $\zeta = 1$ up to order $\ell_0 + 2$. The second point of Proposition 4.4 then implies that $H_t \circ f_q \equiv f_q$, and computing at $\zeta = 0$, we get in particular $H_t(q) = q$. Since this holds for any q belonging to the open set O, we conclude that H_t, and therefore H, coincides with the identity map. $\square$

5.3. **A remark on Theorem 1.3.** We note that Theorem 1.3 is in the vein of some previous boundary rigidity results involving smooth hypersurfaces. D. Burns and S. Krantz [8] proved that if $D \subset \mathbb{C}^n$ is a smooth bounded and strongly pseudoconvex domain, then if $H : D \to D$ is holomorphic and satisfies $H(Z) = Z + o(|Z - p|^3)$ as $Z \to p \in bD$, then H coincides with the identity. As shown by X. Huang [29], in case D is strongly convex, the assumption reduces to $H(Z) = Z + o(|Z - p|^2)$. Moreover, if $D \subset \mathbb{C}^n$ is a smooth bounded convex domain of finite type, then there exists a number ℓ, depending only the geometric properties of bD near p, such that if $H(Z) = Z + o(|Z - p|^m)$ as $Z \to p \in bD$, then H coincides with the identity; for instance in $\mathbb{C}^2$, the number ℓ can be taken to be $5d + \varepsilon$ for some $\epsilon > 0$, where d is the type of bD at p. As a matter of fact, we point out that the methods developed in [8, 29] were based on Lempert's theory [34] of extremal discs for the Kobayashi metric in convex bounded domains.

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