GROMOV HYPERBOLICITY FOR THE KOBAYASHI METRIC IN ALMOST COMPLEX MANIFOLDS

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ABSTRACT. Let $D = \{\rho < 0\}$ be a smooth relatively compact domain in an almost complex manifold (M, J), where ρ is a smooth defining function of D, strictly J-plurisubharmonic in a neighborhood of the closure $\overline{D}$ of D. We prove that D has a connected boundary and is Gromov hyperbolic.

INTRODUCTION

Complex Finsler geometry is an important branch of differential geometry, generalizing Hermitian geometry and carrying precious information on the geometry of the ambient complex manifold; its development goes back to the introduction of the Carathéodory pseudometric. The interest in complex Finsler geometry increased with the works of S.Kobayashi and his characterization of ample vector bundles, see [19]. We will focus in our paper on the Kobayashi metric, another well-known example of a (not necessarily smooth) complex Finsler pseudometric. One may refer to [20] for a presentation of complex hyperbolic spaces. When the integrated pseudodistance associated to that metric is a distance, the corresponding metric space is called Kobayashi hyperbolic and the complex manifold carrying the distance inherits complex dynamical properties particularly adapted to study spaces of holomorphic maps. In the case of bounded domains in $\mathbb{C}^n$, curvature conditions on the boundary furnish information on the global behaviour of geodesics in the associated metric space, as in the celebrated paper of L.Lempert [22] for smooth strongly convex domains, or on their local boundary behavior, as in [12, 15, 23] for strongly pseudoconvex domains.

Gromov hyperbolic spaces were introduced by M.Gromov [16] in the eighties as geodesic metric spaces in which geodesic triangles are thin. See for instance [6, 7, 9, 14] for different extensions of the theory. Trying to find examples of Gromov hyperbolic metric spaces or to characterize them received much attention. One could quote for instance the papers by Z.Balogh-S.Buckley [2] in which the authors characterize Gromov hyperbolic domains in the Euclidean space $\mathbb{R}^n$ endowed with their quasi-hyperbolic distance, or by Y.Benoist in which the author characterizes Gromov hyperbolic convex domains for the Hilbert metric [4]. The Hilbert metric may be defined in the general setting of projective maps in real projective spaces, using the interval $]-1, 1[$ as a gauje. The exact complex analogue provides the Kobayashi distance, where projective maps are replaced with holomorphic maps, the gauje being the unit disc endowed with the Poincaré metric. That similarity between the Hilbert metric and the Kobayashi metric was first observed by S.Kobayashi [21]. The question of finding examples of Gromov hyperbolic domains, in $\mathbb{C}^n$, for their Kobayashi distance, is therfore natural. The Poincaré disc and as a generalization the unit ball in $\mathbb{C}^n$ endowed with its Kobayashi metric are well-known such examples.

The aim of the paper is to prove that every relatively compact, strictly pseudoconvex region in an almost complex manifold, endowed with the Kobayashi distance, is Gromov hyperbolic. In a first step we prove that such regions have a connected boundary using the Morse theory (Theorem 1). In a second step we prove the Gromov hyperbolicity of the region by studying the large scale behaviour of geodesics (Theorem 2). That

second part is in the vein of the results due to Z.Balogh-M.Bonk [1] and L.Blanc-Centi [5]. For clarity and for the convenience of the reader we include a detailed proof, with the necessary adaptation in the (almost complex) manifold setting. We point out that a product of any two Kobayashi hyperbolic manifolds is never Gromov hyperbolic [1], explaining the strong pseudoconvexity condition on the domain. We also point out that a relatively compact strongly pseudoconvex domain in an almost complex manifold is not necessarily Kobayashi hyperbolic. One can prove in fact that a relatively compact strongly pseudoconvex domain in an almost complex manifold is complete hyperbolic if and only if it does not contain a Brody curve. However it is easy to see that a relatively compact strongly pseudoconvex region in an almost complex manifolds does not contain any Brody curve.

The two main results of the paper are the following:

Theorem 1. *Let $D = \{\rho < 0\}$ be a smooth relatively compact domain in an almost complex manifold (M, J), where ρ is a smooth defining function of D, strictly J-plurisubharmonic in a neighborhood of the closure $\overline{D}$ of D. Then the boundary ∂D of D is connected.*

Theorem 2. *Under the same assumptions as in Theorem 1, the domain D endowed with its Kobayashi distance $d_{(D,J)})$ is Gromov hyperbolic.*

In Section 1 we present the necessary preliminaries. In Section 2 we prove Theorem 1. Finally in Section 3 we prove Theorem 2.

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1. PRELIMINARIES

1.1. **Almost complex manifolds and pseudoholomorphic discs.** An *almost complex structure J* on a real smooth manifold M is a $(1, 1)$ tensor field which satisfies $J^2 = -Id$. We suppose that J is smooth. The pair (M, J) is called an *almost complex manifold*. We denote by J_{st} the standard integrable structure on $\mathbb{C}^n$ for every n. A differentiable map $f : (M', J') \longrightarrow (M, J)$ between two almost complex manifolds is said to be (J', J)-*holomorphic* if $J(f(p)) \circ d_p f = d_p f \circ J'(p)$, for every $p \in M'$. In case $M' = \Delta$ is the unit disc in $\mathbb{C}$, such a map is called a *pseudoholomorphic disc*.

1.2. **Strongly J-pseudoconvex domains.** Let ρ be a smooth real valued function on a smooth almost complex manifold (M, J). We denote by $d_J^c \rho$ the differential form defined by $d_J^c \rho(v) := -d\rho(Jv)$, where v is a section of TM. The *Levi form* of ρ at a point $p \in M$ and a vector $v \in T_p M$ is defined by $\mathcal{L}_J \rho(p, v) := dd_J^c \rho(p)(v, J(p)v)$. We say that ρ is *strictly J-plurisubharmonic* if $\mathcal{L}_J \rho(p, v) > 0$ for any $p \in M$ and $v \neq 0 \in T_p M$. The boundary of a domain D is *strongly J-pseudoconvex* if at any point $p \in \partial D$ there exists a smooth strictly J-plurisubharmonic function ρ defined in a neighborhood U of p in M satisfying $\nabla \rho \neq 0$ on $\partial D \cap U$ and such that $D \cap U = \{\rho < 0\}$. We say that a domain $D = \{\rho < 0\}$ is a *strongly J-pseudoconvex region* in (M, J) if ρ is a smooth defining function of D, strictly J-plurisubharmonic in a neighborhood of $\overline{D}$.

1.3. **Hypersurfaces of contact type.** Let (V, ω) be a symplectic manifold, namely ω is a closed, non-degenerate, skew symmetric 2-form on the smooth manifold V. Let Γ be a hypersurface contained in V. We say that Γ is of *contact type* if there is a vector field X, defined near Γ, transverse to Γ and pointing outwards, such that $d(i(X)\omega) = \omega$. The 1-form $\alpha = i(X)\omega$ is a *contact form* on Γ and it defines a *contact structure* $\zeta = \ker \alpha$ on Γ.

Let D be a relatively compact domain in an almost complex manifold. The *complex tangent space* of the boundary ∂D of D is by definition $T^J \partial D := T\partial D \cap J\partial D$. If D is strongly J-pseudoconvex, the complex tangent space $T^J \partial D$ is a contact structure. More precisely, let ρ be a defining function of ∂D. Since D is

strongly J-pseudoconvex there exists a positive constant c such that the function $\tilde{\rho} = \rho + c\rho^2$ is strictly J-plurisubharmonic in a neighborhood U of ∂D and such that $\partial D = \{\tilde{\rho} = 0\}$ and $D \cap U = \{\tilde{\rho} < 0\}$. Consider the one-form $d^c_J \tilde{\rho}$ and let α be its restriction to the tangent bundle $T\partial D$. It follows that $T^J \partial D = \mathrm{Ker}\,\alpha$. Due to the strict J-plurisubharmonicity of $\tilde{\rho}$, the 2-form $\omega := dd^c_J \tilde{\rho}$ is a symplectic form on U that tames J, $T^J \partial D$ is a contact structure and α is a contact form for $T^J \partial D$. If the boundary ∂D of D is connected, a theorem due to W.L.Chow [8] states that any two points in ∂D may be connected by a $\mathcal{C}^1$ *horizontal curve*, i.e. a curve $\gamma : [0,1] \to \partial D$ satisfying $\gamma'(s) \in T^J_{\gamma(s)} \partial D$ for every $s \in [0,1]$. This enables to define the *Carnot-Carathéodory metric* as follows (see [3, 17]):

$$d_H(p,q) := \inf\left\{ l(\gamma), \gamma : [0,1] \to \partial D \text{ horizontal }, \gamma(0) = p, \gamma(1) = q \right\},$$

where $l(\gamma)$ is the *Levi length* of the horizontal curve γ defined by $l(\gamma) := \int_0^1 \mathcal{L}_J \tilde{\rho}(\gamma(t), \gamma'(t))^{\frac{1}{2}} dt$.

1.4. The Kobayashi pseudometric.
The existence of local pseudoholomorphic discs proved in [27] enables to define the *Kobayashi pseudometric* $K_{(M,J)}$ for $p \in M$ and $v \in T_p M$:

$$K_{(M,J)}(p,v) := \inf\left\{ \frac{1}{r} > 0, u : \Delta \to (M,J) \ J\text{-holomorphic }, u(0) = p, d_0 u(\partial/\partial x) = rv \right\},$$

and its integrated pseudodistance $d_{(M,J)}$:

$$d_{(M,J)}(p,q) := \inf\left\{ l_K(\gamma), \gamma : [0,1] \to M, \gamma(0) = p, \gamma(1) = q \right\},$$

for $p, q \in M$, where where $l_K(\gamma)$ is the *Kobayashi length* of a $\mathcal{C}^1$-piecewise smooth curve γ defined by $l_K(\gamma) := \int_0^1 K_{(M,J)}(\gamma(t), \gamma'(t)) dt$. The manifold (M,J) is *Kobayashi hyperbolic* if $d_{(M,J)}$ is a distance.

Let h be any J-hermitian metric in M and denote by $\|\cdot\|_h$ the associated infinitesimal metric. A *Brody curve* is a nonconstant J-holomorphic curve from $\mathbb{C}$ to M satisfying $\|df\|_h \leq 1$. Notice that if D is a relatively compact strongly J-pseudoconvex domain in (M,J) or is compact, then D contains no Brody curve if and only if D is Kobayashi hyperbolic.

1.5. Gromov hyperbolic spaces.
Let (X,d) be a metric space. The *Gromov product* of two points $x, y \in X$ with respect to a basepoint $\omega \in X$ is defined by $(x|y)_\omega := \frac{1}{2}(d(x,\omega) + d(y,\omega) - d(x,y))$. The Gromov product measures the failure of the triangle inequality to be an equality and is always nonnegative. The metric space X is *Gromov hyperbolic* if there is a nonnegative constant δ such that for any $x, y, z, \omega \in X$ one has:

$$(1.1) \qquad\qquad (x|y)_\omega \geq \min((x|z)_\omega, (z|y)_\omega) - \delta.$$

We point out that (1.1) can be also written as follows:

$$(1.2) \qquad d(x,y) + d(z,\omega) \leq \max(d(x,z) + d(y,\omega), d(x,\omega) + d(y,z)) + 2\delta,$$

for $x, y, z, \omega \in X$.

Definition 1.1. Let d and d' be two metrics on X.
  (1) (X,d) and (X,d') are roughly isometric if there exists a positive constant c such that for every $x, y \in X$:

$$d(x,y) - c \leq d'(x,y) \leq d(x,y) + c.$$

(2) (X, d) and (X, d') are quasi-isometric if there exist two positive constants λ and c such that for every $x, y \in X$:

$$\frac{1}{\lambda} d(x, y) - c \leq d'(x, y) \leq \lambda d(x, y) + c.$$

If (X, d) and (X, d') are roughly isometric then (X, d) is Gromov hyperbolic if and only if (X, d') is Gromov hyperbolic. In case we consider quasi-isometric spaces, the spaces need to be both geodesic. This is provided by the following theorem (see Theorem 12 p. 88 in [14]) which is crucial for us:

Theorem [Gh-dlH]. *Let (X, d) and (X, d') be two quasi-isometric geodesic metric spaces. Then (X, d) is Gromov hyperbolic if and only if (X, d') is.*

We recall that a metric space (X, d) is a *geodesic space* if any two distinct points $x, y \in X$ can be joined by a *geodesic segment*, that is the image of an isometry $\gamma : [0, d(x, y)] \to X$ with $\gamma(0) = x$ and $\gamma(d(x, y)) = y$.

2. PROOF OF THEOREM 1

The proof of Theorem 1 is based on the Morse theory and more precisely on the fact that the topology of a domain defined by a Morse function can change only at critical points of the function (see Section I.3 "Homotopy Type in Terms of Critical Values" in [25] for more details). We recall that a *Morse function* is a smooth function which Hessian is non-degenerate at its critical points. The *Morse index* of a non-degenerate critical point of such a function is the number of negative eigenvalues of its Hessian. A level set $\{\rho = c\}$ containing a critical point of the function ρ will be called a *critical set* and the corresponding value c will be called a *critical value* of ρ.

Let $D = \{\rho < 0\}$ be a relatively compact strongly J-pseudoconvex region. After a small perturbation of ρ that preserves both the strict J-plurisubharmonicity of the defining function and the number of connected components of the boundary $\partial D = \{\rho = 0\}$, we can suppose that ρ is a Morse function. Hence ρ has a finite number of critical points denoted by $p_1, \cdots, p_r$. We denote by k_j the Morse index of the critical point p_j, $j = 1, \cdots, r$. Moreover, we can always assume that every critical set contains only one critical point. We order the critical points in such a way such that the critical values $-c_j := \rho(p_j)$ satisfy $-c_j > -c_{j+1}$ for $j = 1, \ldots, r - 1$. The number of connected components of $\overline{D} \cap \{\rho = -t\}$ for $t \geq 0$ can change only at a critical level set. More precisely, it is sufficient to describe the topology of $\overline{D} \cap \{\rho = -t\}$ in a neighborhood of a critical level set near a critical point. We denote by $C_{j,t} \in \mathbb{N} \cup \{\infty\}$ the number of connected components of the set $D \cap \{\rho = -c_j + t\}$.

Lemma 2.1. *Under the above assumptions, we have for $\varepsilon > 0$, sufficiently small:*
  (1) *if $2 \leq k_j \leq n$ then $C_{j,\varepsilon} = C_{j,-\varepsilon}$,*
  (2) *if $k_j = 1$ then $C_{j,\varepsilon} \leq C_{j,-\varepsilon}$,*
  (3) *for every critical point p_j of Morse index $k_j = 0$, there is a neighborhood U_j of p_j such that $\{\rho = -c_j + \varepsilon\} \cap U_j$ is connected and $\{\rho = -c_j - \varepsilon\} \cap U_j$ is empty. Moreover $k_r = 0$ and $C_{r,\varepsilon} = 1$.*

Notice that according to this lemma, $C_{j,t}$ are finite for $j = 1, \ldots, r - 1$ and $t \in \mathbb{R}$. In order to prove Lemma 2.1, we need the following version of the Morse Lemma:

Lemma 2.2. *Let (M, J) be an almost complex manifold. Let ρ be a strictly J-plurisubharmonic Morse function on M. Let p be a critical point of ρ. Then there exist local (not necessarily holomorphic) coordinates $z = (z_1, \cdots, z_n)$, with $z_j = x_j + iy_j$, centered at p and defined in a neighborhood U_j of p_j, such that one has*

(2.1) $$\rho(z) = \rho(0) + \sum_{j=1}^{n} x_j^2 + \sum_{j=1}^{n} a_j y_j^2 + O(|z|^3)$$

where $a_j = \pm 1$. In particular the Morse index of a critical point is smaller than n.

Proof of Lemma 2.2. Let p be a critical point of ρ. We use a normalization due to K.Diederich and A.Sukhov (see Lemma 3.2 and Proposition 3.5 in [11]). In the associated coordinate system $w = (w_1, \cdots, w_n)$ centered at $p = 0$, the Levi forms of ρ at the origin, with respect to J_{st} and J, coincide. In particular ρ is strictly J_{st}-plurisubharmonic at the origin. Then

$$\rho(w) = \rho(0) + \sum_{i,j=1}^{n} a_{ij} w_i \overline{w}_j + \Re e \sum_{i,j=1}^{n} b_{ij} w_i w_j + O(|w|^3),$$

where $A = (a_{ij})$ and $B = (b_{ij})$ are respectively Hermitian and symmetric matrices. Applying a linear transformation $w \mapsto \tilde{w} := Lw$, we can reduce A to the identity $A = I_n$. Moreover we can make a unitary transformation $\tilde{w} \mapsto \tilde{z} := U\tilde{w}$ preserving $\sum_{j=1}^{n} |\tilde{w}_j|^2$ and changing the matrix B into a diagonal matrix $U^t BU$ with nonnegative elements (see Lemma 7.2 in [10]). Then the expression of ρ in the $\tilde{z} = (\tilde{z}_1, \cdots, \tilde{z}_n)$ coordinates reduces to

$$\rho(\tilde{z}) = \rho(0) + \sum_{j=1}^{n} |\tilde{z}_j|^2 + \sum_{j=1}^{n} \alpha_j \Re e \tilde{z}_j^2 + O(|\tilde{z}|^3),$$

where $\alpha_j \geq 0$ for $j = 1, \ldots, n$. According to the non-degeneracy of the Hessian of ρ at 0, we have $\alpha_j \neq \pm 1$ for $j = 1, \cdots, n$, and thus we can set $x_j = (1+\alpha_j)^{\frac{1}{2}} \Re e \tilde{z}_j$ and $y_j = |1-\alpha_j|^{\frac{1}{2}} \Im m \tilde{z}_j$ which gives (2.1). $\quad \square$

Proof of Lemma 2.1. The content of Lemma 2.1 is purely local. From now on we will denote by U_j a connected neighborhood of the critical point p_j. As pointed out it is sufficient to compute the number of connected components of the set $\{\rho = -c_j \pm \varepsilon\} \cap U_j$.

Let p_j be a critical point of ρ of Morse index $2 \leq k_j \leq n$. In a coordinate system centered at p_j given by Lemma 2.2, ρ can be written

$$\rho(z) = -c_j + \sum_{l=1}^{n} x_l^2 + \sum_{l=1}^{n} a_l y_l^2 + O(|z|^3)$$

with k_j negative coefficients among $a_1 \ldots, a_n$, say for $I = \{l_1, \cdots, l_{k_j}\}$. For $t_\varepsilon > 0$ sufficiently small, the set $\{\rho = -c_j + \varepsilon\} \cap \{\sum_{l \in I} y_l^2 = t_\varepsilon\} \cap U_j$ is a perturbation of a $(2n - k_j)$-sphere which varies smoothly with the real parameter t_ε. It follows that set $\{\rho = -c_j + \varepsilon\} \cap U_j$ is connected for small positive ε. Moreover, for $t'_\varepsilon > 0$ small enough, the set $\{\rho = -c_j + \varepsilon\} \cap \{\sum_{l=1}^{n} x_l^2 + \sum_{l \notin I} y_l^2 = t'_\varepsilon\} \cap U_j$ is a perturbation of a k_j-sphere, and thus the set $\{\rho = -c_j - \varepsilon\} \cap U_j$ is connected for small positive ε. This proves (1).

Let p_j be a critical point of ρ of Morse index 1. In a coordinate system centered at p_j provided by Lemma 2.2, ρ has the form

$$\rho(z) = -c_j + \sum_{l=1}^{n} x_l^2 + \sum_{l \neq l_0} y_l^2 - y_{l_0}^2 + O(|z|^3)$$

for some l_0. The set $\{\rho = -c_j + \varepsilon\} \cap U_j$ is connected whereas the set $\{\rho = -c_j - \varepsilon\} \cap U_j$ is not. This proves (2).

Let p_j be a critical point of ρ of Morse index 0. In a coordinate system centered at p_j provided by Lemma 2.2, ρ has the form

$$\rho(z) = -c_j + \sum_{l=1}^{n} x_l^2 + \sum_{l=1}^{n} y_l^2 + O(|z|^3),$$

and so the set $\{\rho = -c_j + \varepsilon\} \cap U_j$ is connected and the set $\{\rho = -c_j - \varepsilon\} \cap U_j$ is empty. Moreover, since the domain D is relatively compact then $k_r = 0$. Finally since p_r is the last critical point of ρ, the level set $D \cap \{\rho = -c_r + \varepsilon\}$ is connected and thus $C_{r,\varepsilon} = 1$. $\quad \square$

Proof of Theorem 1. Assume by contradiction that the boundary $\partial D = \{\rho = 0\}$ of D has m connected components with $m \geq 2$. For $\varepsilon > 0$ small enough, $D \cap \{\rho = -\varepsilon\}$ has m connected components. We claim that the number of connected components of $D \cap \{\rho = -t\}$ stays larger than or equal to m as $t \to c_r$, which contradicts the connectedness of the level sets nearby $\{\rho = -c_r\}$. According to Lemma 2.1, critical points of ρ of Morse index k_j with $2 \leq k_j \leq n$ preserve the number of connected components of level sets nearby the critical level set $\{\rho = -c_j\}$. Critical points of Morse index 1 may actually increase the number of connected components of level sets. Now assume that $p_j \neq p_r$ is a critical point of Morse index 0 and denote by D_j the connected component of $D \cap \{\rho \geq c_j - 2\varepsilon\}$ containing p_j. The number of connected components of $\{\rho = -c_j - \varepsilon\} \cap D_j$ is positive otherwise D_j would be a connected component of D different from D contradicting the connectedness of D. By part (3) of Lemma 2.1, the number of connected components of $\{\rho = -c_j + \varepsilon\} \cap D_j$ is larger than 1 since the index of p_j is 0. Hence by connectedness of D_j, there exists a critical point $p_{j'} \in D_j$ with $j' < j$ such that $C_{j',\varepsilon} < C_{j',-\varepsilon}$. This implies that $C_{j',\varepsilon} \leq C_{j,-\varepsilon}$. $\qquad\square$

3. PROOF OF THEOREM 2

We state a stronger version of Theorem 2, namely:

Theorem 3. *Let D be a smooth relatively compact domain in an almost complex manifold (M, J). We assume that the boundary ∂D of D is connected and that ∂D is strongly J-pseudoconvex. If D does not contain any Brody J-holomorphic curve then $(D, d_{(D,J)})$ is Gromov hyperbolic.*

Notice that a relatively compact domain D, defined by a global strictly J-plurisubharmonic function, does not contain any Brody curves and thus is Kobayashi hyperbolic. Therefore Theorem 2 follows from Theorem 1 and Theorem 3.

3.1. **Proof of Theorem 3.** If D does not contain any Brody J-holomorphic curve then we first point out that (D, J) is Kobayashi hyperbolic. Indeed, assuming by contradiction that (D, J) is not Kobayashi hyperbolic we may construct on D a sequence of J-holomorphic discs with derivatives exploding at the centers. Since ∂D is strongly J-pseudoconvex, it follows from estimates of the Kobayashi infinitesimal pseudometric (see [13]) that such discs do not approach the boundary ∂D of D. Hence by a Brody renormalization process we construct a Brody curve contained in D, which is a contradiction.

We equip M with an arbitrary smooth Riemannian metric and we denote by dist the associated distance. For $p \in D$ we define a boundary projection map $\pi : D \to \partial D$ by $\mathrm{dist}(p, \pi(p)) = \mathrm{dist}(p, \partial D) =: \delta(p)$. Notice that the map π is uniquely defined near the boundary. Set $N_\varepsilon(\partial D) := \{q \in D, \delta(q) \leq \varepsilon\}$ where ε is a sufficiently small positive real number such that π is uniquely defined on $N_\varepsilon(\partial D)$ (see [28], Chapter 9, for the existence of such a tubular neighborhood). We emphasize that the construction of tubular neighborhoods using the exponential map gives a Riemannian analogue of some Euclidean lemmas in [1] (Lemma 2.1 and Lemma 2.2 in particular). For points $p \in D \backslash N_\varepsilon(\partial D)$, $dist(p, \partial D)$ may be reached at different points. We pick up arbitrarily a point $\tilde{p} \in \partial D$ such that $dist(p, \partial D) = dist(p, \tilde{p})$ and we set $\pi(p) = \tilde{p}$. The projection π so defined is not intrinsic but this is not important for our goal.

We define the *height* of p by $h(p) := \sqrt{\delta(p)}$. Then we define a metric $g : D \times D \to [0, +\infty)$ by:

$$g(p, q) := 2 \log \left(\frac{d_H(\pi(p), \pi(q)) + \max\{h(p), h(q)\}}{\sqrt{h(p)h(q)}} \right),$$

for $p, q \in D$ (see [1]). It is important to notice that different choices of a Riemannian metric and of a projection π give a different metric that coincides with g up to an additive bounded term. This is not a matter of fact since we deal with roughly and quasi-isometric spaces.

Z.M.Balogh and M.Bonk proved in [1] that the metric g satisfies (1.2) and thus that the metric space (D, g) is Gromov hyperbolic. Their proof in the Euclidean space extends directly to the almost complex case. Hence we have the following:

Lemma 3.1. *The metric space (D, g) is Gromov hyperbolic.*

However the space (D, g) is not geodesic. In order to construct a geodesic Gromov hyperbolic metric space, we need to perturb the metric g. The following construction is due to L.Blanc-Centi [5] in $\mathbb{R}^{2n}$. For p (resp. q) in D denoted by p_ε (resp. q_ε) the point on the fiber $\pi^{-1}(\pi(p))$ (resp. $\pi^{-1}(\pi(q))$) with height $\sqrt{\varepsilon}$ and let $l_g(\gamma) := \sup_{0=t_0 < t_1 < \cdots < t_n = 1} \sum_{i=1}^{n} g(\gamma(t_{i-1}), \gamma(t_i))$. Then we define a new metric d as follows:

$$
d(p, q) := \begin{cases}
\inf\{l_g(\gamma), \gamma : [0, 1] \to N_\varepsilon(\partial D) \text{smooth curve joining } p \text{ and } q\} \text{ for } p, q \in N_\varepsilon(\partial D), \\[2mm]
d(p, q_\varepsilon) + \operatorname{dist}(q, q_\varepsilon) \text{ for } p \in N_\varepsilon(\partial D), q \in D \setminus N_\varepsilon(\partial D), \\[2mm]
\operatorname{dist}(p, q) \text{ for } p, q \in D \setminus N_\varepsilon(\partial D) \text{ such that } \pi(p) = \pi(q), \\[2mm]
\operatorname{dist}(p, p_\varepsilon) + d(p_\varepsilon, q_\varepsilon) + \operatorname{dist}(q, q_\varepsilon) \text{ for } p, q \in D \setminus N_\varepsilon(\partial D) \text{ such that } \pi(p) \neq \pi(q).
\end{cases}
$$

Then we have the following:

Proposition 3.1. (i) *The metric space (D, d) is Gromov hyperbolic.*
(ii) *(D, d) is geodesic.*

To prove Proposition 3.1 (i) it is sufficient to prove that (D, d) and (D, g) are roughly isometric since (D, g) is Gromov hyperbolic. The proof of (ii) consists essentially in constructing geodesic curves joining any two points. The proof of Proposition 3.1 follows [5]. It was obtained there in case M is the Euclidean space $\mathbb{R}^{2n}$ endowed with some almost complex structure. However the arguments remain valid for domains in an almost complex manifold. For convenience we include the key points of the proof.

Proof of Proposition 3.1. (i) Since (D, g) is Gromov hyperbolic according to Lemma 3.1, we just need to prove that (D, g) and (D, d) are roughly isometric. It is sufficient to prove that these spaces are roughly isometric near the boundary, namely that there is a positive constant c such that for all $p, q \in N_\varepsilon(\partial D)$:

$$g(p, q) - c \leq d(p, q) \leq g(p, q) + c.$$

Restricting to $N_\varepsilon(\partial D)$ we define two types of curves in (D, J):
 - Normal curves are curves defined on the fibers of the projection. They are purely local objects since points considered belong to $N_\varepsilon(\partial D)$ and have the same boundary projection (see Lemma 1 in [5]).
 - Horizontal curves are curves joining two points $p, q \in N_\varepsilon(\partial D)$ with same height $h(p) = h(q)$. They are defined as follows: since $(\partial D, d_H)$ is geodesic, we consider a geodesic curve α in ∂D joining $\pi(p)$ and $\pi(q)$. For each t, we consider the point $\gamma(t) \in N_\varepsilon(\partial D)$ in the fiber $\pi^{-1}(\alpha(t))$ with height $h(p) = h(q)$. Then γ defines a smooth horizontal curve in $N_\varepsilon(\partial D)$.

These two kinds of curves being defined for manifolds the proof that (D, g) and (D, d) are roughly isometric is then straightforward, see [5]. Hence the metric space (D, d) is Gromov hyperbolic.

(ii) The proof that (D, d) is geodesic is achieved by studying the relative positions of any two points $p, q \in D$. That was first achieved in [1]. By the definition of the metric d, it reduces to the two following cases.

<u>Case 1</u>. $p, q \in D \setminus N_\varepsilon(\partial D)$ satisfy $\pi(p) = \pi(q)$. Then d coincides with the metric dist induced by a Riemannian metric. Since $(\overline{D \setminus N_\varepsilon(\partial D)}, \operatorname{dist})$ is compact, it is complete and thus according to the Hopf-Rinow Theorem, it is geodesic.

<u>Case 2</u>. $p, q \in N_\varepsilon(\partial D)$. One can prove that there exists a geodesic curve contained in $N_\varepsilon(\partial D)$, namely a curve $\gamma : [0, 1] \to N_\varepsilon(\partial D)$ such that $l_g(\gamma) = d(x, y)$. This is the content of Lemma 4 in [5]. $\square$

Remark 3.1. *It follows from the construction of the metric g that the boundary $\partial_G D$ of (D, g), as a Gromov hyperbolic space, may be identified with the boundary ∂D of the domain D. The same holds for the boundary of (D, d) as a Gromov hyperbolic space, since the spaces (D, g) and (D, d) are roughly isometric.*

The space (D, d) being geodesic and Gromov hyperbolic by Proposition 3.1, it remains to show that the metric space $(D, d_{(D,J)})$ is geodesic and quasi-isometric to (D, d), in order to complete the proof of Theorem 3. We will use the following precise estimates of the Kobayashi metric obtained in [13]:

Theorem [G-S]. *Let D be a relatively compact domain in an almost complex manifold (M, J). Assume that $\partial D = \{\rho = 0\}$ where ρ is strictly J-plurisubharmonic in a neighborhood U of ∂D. Then there exists a positive constant C such that :*
(3.1)
$$\frac{1}{C} \left[\frac{|\partial_J \rho(p)(v - iJ(p)v)|^2}{|\rho(p)|^2} + \frac{|v|^2}{|\rho(p)|} \right]^{1/2} \leq K_{(D,J)}(p, v) \leq C \left[\frac{|\partial_J \rho(p)(v - iJ(p)v)|^2}{|\rho(p)|^2} + \frac{|v|^2}{|\rho(p)|} \right]^{1/2}$$
for every $p \in D \cap U$ and every $v \in T_p M$.

We emphasize that as pointed out in Subsection 1.3, the existence of such a function ρ is ensured as soon as D is strongly J-pseudoconvex. As a classical consequence of the lower estimate of the Kobayashi metric (3.1), we obtain the completeness of the metric space $(D, d_{(D,J)})$. According to the Hopf-Rinow Theorem (see [18] p. 9) the metric space $(D, d_{(D,J)})$ is geodesic. Moreover we have:

Proposition 3.2. *The metric spaces $(D, d_{(D,J)})$ and (D, d) are quasi-isometric.*

The proof of Proposition 3.2 is essentially given by [1] and remains valid in our setting. For seek of completeness we include the key steps of the proof.

Proof of Proposition 3.2. Since (D, d) and (D, g) are roughly isometric, it is sufficient to prove that $(D, d_{(D,J)})$ and (D, g) are quasi-isometric, namely that there exists two positive constants λ and c such that for all $p, q \in D$

(3.2)
$$\frac{1}{\lambda} g(p, q) - c \leq d_{(D,J)}(p, q) \leq \lambda g(p, q) + c.$$

This is obtained by considering various cases depending on the relative positions of p and q.

First notice that by integration, we obtain from (3.1) some estimates of the Kobayashi length of curves near the boundary. There are positive constants C_1 and C_2 such that for any smooth curve $\gamma : [0, 1] \to N_\varepsilon(\partial D)$ we have:

(3.3)
$$\frac{1}{C_1} \left| \log \frac{h(\gamma(1))}{h(\gamma(0))} \right| - C_2 \leq l_K(\gamma).$$

Moreover in case γ is a normal curve, the estimates are local since points on γ have the same boundary projection and $\gamma([0, 1]) \subseteq N_\varepsilon(\partial D)$. Thus the situation is similar to the Euclidean case considered by [1] and we have

(3.4)
$$l_K(\gamma) \leq C_1 \left| \log \frac{h(\gamma(1))}{h(\gamma(0))} \right| + C_2,$$

increasing C_1 and C_2 if necessary.

<u>Case 1</u>. $p, q \in D \setminus N_\varepsilon(\partial D)$. Since $d_{(D,J)}$ and g are non negative and uniformly bounded on $D \setminus N_\varepsilon(\partial D)$ we obtain (3.2).

<u>Case 2</u>. $p \in N_\varepsilon(\partial D)$ and $q \in D \setminus N_\varepsilon(\partial D)$. By definition we have

$$g(p,q) := 2 \log \left(\frac{d_H(\pi(p), \pi(q)) + h(q)}{\sqrt{h(p)h(q)}} \right)$$

and thus

$$\log \frac{1}{h(p)} - c \le g(p,q) \le \log \frac{1}{h(p)} + c.$$

Since $d_{(D,J)}$ is uniformly bounded on $D \setminus N_\varepsilon(\partial D)$, we have

$$d_{(D,J)}(p,q) \le d_{(D,J)}(p, p_\varepsilon) + d_{(D,J)}(p_\varepsilon, q) \le d_{(D,J)}(p, p_\varepsilon) + c,$$

for some uniform positive constant still denoted by c. Here p_ε is the unique point on $\pi^{-1}(\pi(p))$ with $h(p_\varepsilon) = \sqrt{\varepsilon}$ (or equivalently $\delta(p_\varepsilon) = \varepsilon$). Since $\pi(p) = \pi(p_\varepsilon)$, we obtain with (3.4)

$$d_{(D,J)}(p, p_\varepsilon) \le C_1 \log \frac{1}{h(p)} + C_2.$$

Therefore we obtain the right inequality of (3.2).

Now consider a smooth curve γ joining p and q and let q' be the first point on the curve with $h(q') = \sqrt{\varepsilon}$. Let β be the sub-curve of γ joining p and q'. According to (3.3) we get

$$\frac{1}{C_1} \log \frac{1}{h(p)} - C_2 \le l_K(\beta) \le l_K(\gamma)$$

which implies, after taking the infimum over all possible curves γ joining p and q, that $\frac{1}{C_1} \log \frac{1}{h(p)} - C_2 \le d_{(D,J)}(p,q)$. Hence we obtain the left inequality of (3.2).

<u>Case 3</u>. $p, q \in N_\varepsilon(\partial D)$ and $\max\{h(p), h(q)\} \ge d_H(\pi(p), \pi(q))$. Assume for instance that $h(p) \le h(q)$. We have

$$\log \frac{h(q)}{h(p)} - c \le g(p,q) \le \log \frac{h(q)}{h(p)} + c.$$

Denote by p' the point in the fiber $\pi^{-1}(\pi(p))$ such that $h(p') = h(q)$. Then (3.4) provides

$$d_{(D,J)}(p, p') \le C_1 \log \frac{h(q)}{h(p)} + C_2.$$

Consider a horizontal curve γ joining p' and q. Since $d_H(\pi(p'), \pi(q)) = d_H(\pi(p), \pi(q)) \le \max\{h(p), h(q)\} \le \varepsilon$, the situation is local and thus can be reduced to the Euclidean case. Hence following [1], we obtain

$$K_{(D,J)}(\gamma(t), \gamma'(t))^2 \le C \frac{\mathcal{L}_J \rho((\pi \circ \gamma)(t), (\pi \circ \gamma)'(t))}{\delta(q)}$$

and by integration

$$d_{(D,J)}(p', q) \le C \frac{d_H(\pi(p), \pi(q))}{h(q)}.$$

Since $h(q) \ge d_H(\pi(p), \pi(q))$ we obtain $d_{(D,J)}(p', q) \le C$. This gives the right inequality of (3.2).

The lower bound, and therefore the left inequality of (3.2), is easily obtained by applying (3.3).

<u>Case 4</u>. $p, q \in N_\varepsilon(\partial D)$ and $\max\{h(p), h(q)\} < d_H(\pi(p), \pi(q))$. Assume for instance that $h(p) \le h(q)$. We have

$$2 \log \frac{d_H(\pi(p), \pi(q))}{\sqrt{h(p)h(q)}} - c \le g(p,q) \le 2 \log \frac{d_H(\pi(p), \pi(q))}{\sqrt{h(p)h(q)}} + c.$$

Set $h_0 := \min\{\sqrt{\varepsilon}, d_H(\pi(p), \pi(q))\}$ and denote by p' (resp. q') the point on the fiber $\pi^{-1}(\pi(p))$ (resp. $\pi^{-1}(\pi(q))$) with height h_0. As previously, one can show that

$$d_{(D,J)}(p,q) \leq C_1 \log \frac{h_0}{\sqrt{h(p)h(q)}} + C_2.$$

by considering a normal curve joining p and p' (resp. q and q') and a horizontal curve joining p' and q'. This gives the right inequality of (3.2).

Let γ be a curve in $N_\varepsilon(\partial D)$ joining p and q. Set $H := \max_{t \in [0,1]} h(\gamma(t)) = h(\gamma(t_0))$ and denote by γ_1 (resp. γ_2) the sub-curve of γ joining p and $\gamma(t_0)$ (resp. $\gamma(t_0)$ and q).

If $H \geq h_0$, the left inequality of (3.2) follows immediately by applying (3.3) to the sub-curves γ_1 and γ_2. In case $H < h_0$ we follow precisely [1], p. 521-523. Lower estimates of $d_{(D,J)}(p,q)$ are obtained by considering a dyadic decomposition of the curve γ by height levels and some estimates relating the Kobayashi lengths of those pieces and the Carnot-Carathéodory lengths of the projected curve $\pi \circ \gamma$, known as the box ball estimates (see [26]). The crucial fact used in [1] and that can be applied directly here is that the size of balls for the Carnot-Carathéodory distance can be approximated by Euclidean polydiscs. This ends the proof of Proposition 3.2.

□

Finally, Theorem [Gh-dlH] implies that the space $(D, d_{(D,J)})$ is Gromov hyperbolic which proves Theorem 3.

□

3.2. **A corollary of Theorem 3.**

Corollary 1. *Let (V, ω) be a symplectic manifold and let D be a smooth relatively compact domain contained in V. Assume that ∂D is connected and of contact type. Then there exists an almost complex structure J on V, tamed by ω, for which the following equivalence is satisfied:*

$(D, d_{(D,J)})$ is Gromov hyperbolic if and only if D does not contain Brody J-holomorphic curves.

Proof. Let $\zeta = Ker(\alpha)$ be the contact structure on ∂D (see Subsection 1.3). Since $\omega_{|\zeta}$ is nondegenerate, there exists an almost complex structure J defined on a neighborhood of ∂D and tamed by ω, such that ζ is J-invariant and $d\alpha(v, Jv) > 0$ for all non-zero $v \in \zeta$. According to [24] let $\rho : V \to \mathbb{R}$ be a smooth function such that $\partial D = \rho^{-1}(0)$ and $d\rho(X) > 0$ where X is given by Subsection 1.3. Then for $v \in \zeta$ we have $Jv \in \zeta$ and $Ker(d_J^c \rho_{|\partial D}) = Ker(\alpha_{|\partial D})$. This implies that there is a positive function μ defined in a neighborhood of ∂D such that $dd_J^c \rho = \mu \, d\alpha$ on ζ (see Lemma 2.4 in [24]). In particular ∂D is strongly J-pseudoconvex. Since ∂D is connected, we may apply Theorem 3 to conclude the proof of Corollary 1.

□

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