

EXTREMAL AND STATIONARY DISCS FOR THE KOBAYASHI k -METRIC

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ABSTRACT. We generalize Lempert's and Poletsky's works [11, 14] on the description of extremal discs for the Kobayashi metric to a higher order setting with k -stationarity condition introduced in [1].

INTRODUCTION

The Kobayashi pseudodistance was introduced by S. Kobayashi [8, 9] for the purpose of finding a generalization of the Poincaré distance of the unit disc in more general complex spaces. It turned out later that the Kobayashi pseudodistance is an inner pseudodistance and H. L. Royden [15] defined an infinitesimal pseudometric, called the Kobayashi pseudometric, whose integrated pseudodistance coincides with the Kobayashi pseudodistance. Due to their invariance by biholomorphisms, the Kobayashi pseudodistance and metric are particularly adapted to study properties of holomorphic mappings and structures of complex spaces (cf. [10, 6]). In his celebrated paper, L. Lempert [11] described extremal discs for the Kobayashi metric in smooth bounded strongly convex domains of \mathbb{C}^n as stationary discs and used it to define a canonical representation of such domains onto the unit ball. Lempert's work has given rise to studies of the relation between extremality and stationarity conditions in more general situations (see for instance [14, 5, 13, 16, 3]). In particular, E. Poletsky [14] proved that the stationarity condition is the Euler-Lagrange equation corresponding to the Kobayashi extremal problem on bounded domains of \mathbb{C}^n defined by C^2 plurisubharmonic functions.

In [1], the first two authors introduced a higher order notion of stationarity condition in order to study finite jet determination of CR automorphisms of Levi degenerate hypersurfaces. Independently, the study on higher order Kobayashi metrics has been developed by several authors [17, 18]. In view of the works of Lempert and Poletsky, it seems natural to study the relation between generalized stationary discs and extremal discs for higher order Kobayashi metrics. In this paper, we prove that the stationarity condition introduced in [1] is the Euler-Lagrange equation corresponding to the higher order Kobayashi extremal problem on bounded domains of \mathbb{C}^n defined by C^2 plurisubharmonic functions (Theorem 1). Moreover, we show that on smooth strictly convex domains of \mathbb{C}^n , generalized stationary discs are locally extremal for the higher order Kobayashi metric (Theorem 3). Note that the present paper is focused on the higher order Kobayashi metric introduced by S. Venturini [17]. A study of the metric introduced by J. Yu [18], by means of extremal discs, was carried out by M. Jarnicki and P. Pflug in [6].

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1. PRELIMINARIES

For $r > 0$, we set $\Delta_r = \{\zeta \in \mathbb{C} \mid |\zeta| < r\}$ and we denote by $\Delta = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$ the unit disc in \mathbb{C} .

1.1. Jet bundles. Let M be a complex manifold of dimension n . Locally, we identify M with an open domain in \mathbb{C}^n by taking local coordinates (z^1, \dots, z^n) . Let U_1 and U_2 be two open neighborhoods of the origin in \mathbb{C} . For two holomorphic mappings $f : U_1 \rightarrow M$ and $g : U_2 \rightarrow M$ and for a positive integer k , we denote $f \sim_k g$ if

$$f^{(\ell)}(0) = g^{(\ell)}(0)$$

for all $\ell = 0, \dots, k$. Obviously, \sim_k is an equivalence relation, and the equivalence class of f will be denoted by $J_p^k(f)$, where $p = f(0) \in M$ and is called a k -jet at p . The space of all k -jets at p is denoted by $J_p^k(M)$ and we define by $J^k(M) = \bigcup_{p \in M} J_p^k(M)$ the k -jet bundle over M . Then the local coordinate system on M enables us to identify $J_p^k(M)$ with $(\mathbb{C}^n)^k$ by the mapping $J_p^k(f) \mapsto (f'(0), \dots, f^{(k)}(0))$.

Although the k -jet bundle is not a vector bundle over M unless $k = 1$, we can still define a complex multiplication as follows. For $\lambda \in \mathbb{C}$, we set

$$\lambda \cdot J_p^k(f) := J_p^k(f_\lambda)$$

where $f_\lambda(\zeta) = f(\lambda\zeta)$. Locally, if $J_p^k(f) = \xi = (\xi_1, \xi_2, \dots, \xi_k) \in (\mathbb{C}^n)^k$ then

$$\lambda \cdot J_p^k(f) = (\lambda\xi_1, \lambda^2\xi_2, \dots, \lambda^k\xi_k).$$

1.2. Higher order Kobayashi metrics. Let M be a complex manifold. Following S. Venturini [17], we define for a positive integer k , the *Kobayashi k -pseudometric* $K_M^k(p, \xi)$ by

$$K_M^k(p, \xi) := \inf \left\{ \frac{1}{\lambda} > 0 \mid f : \Delta \rightarrow M \text{ holomorphic, } f(0) = p, J_p^k(f) = \lambda \cdot \xi \right\}$$

for any $p \in M$ and any $\xi \in J_p^k(M)$. Note that K_M^1 is the classical Kobayashi pseudometric defined on the tangent bundle. From the definition, it is clear that $K_M^k(p, cv) = |c|K_M^k(p, v)$ for any $c \in \mathbb{C}$. The following properties are standard and their proofs are straightforward.

Proposition 1.1. (i) *Let M and N be two complex manifolds and $\Phi : M \rightarrow N$ a holomorphic mapping. Then for any $p \in M$ and $\xi \in J_p^k(M)$, we have*

$$K_N^k(\Phi(p), \Phi_*(\xi)) \leq K_M^k(p, \xi),$$

where $\Phi_*(\xi) = J_{\Phi(p)}^k(\Phi \circ g)$, if $\xi = J_p^k(g)$.

(ii) *Let M be a complex manifold and $p \in M$. Let $\tilde{\xi} \in J_p^{k+1}(M)$ and let $\xi = \pi_k(\tilde{\xi}) \in J_p^k(M)$, where π_k is the canonical projection from $J_p^{k+1}(M)$ onto $J_p^k(M)$. Then we have*

$$K_M^k(p, \xi) \leq K_M^{k+1}(p, \tilde{\xi}).$$

As an example, we consider in the case of the unit disc $M = \Delta$ and $k = 2$. We first need the following lemma

Lemma 1.1 (The second order Schwarz lemma). *Let $f : \Delta \rightarrow \Delta$ be a holomorphic function that $f(0) = 0$. Then we have $|f'(0)| \leq 1$ and*

$$(1.1) \quad |f''(0)| \leq 2(1 - |f'(0)|^2).$$

Moreover, if the equality holds in (1.1), then

$$(1.2) \quad f(\zeta) = \zeta \varphi(e^{i\theta} \zeta)$$

for some $\theta \in \mathbb{R}$, where $\varphi(\zeta) = \frac{\zeta + f'(0)}{1 + \overline{f'(0)}\zeta}$.

Proof. The classical Schwarz lemma implies that $|f'(0)| \leq 1$ and if $|f'(0)| = 1$, then $f(\zeta) = e^{i\theta}\zeta$ for some $\theta \in \mathbb{R}$. There is nothing more to prove in this case. Now we assume $|f'(0)| < 1$. Let $g(\zeta) = f(\zeta)/\zeta$. Then the Schwarz lemma implies again that g is a holomorphic function from Δ into Δ . Let $a = f'(0) = g(0)$ and let

$$\psi(\zeta) = \frac{\zeta - a}{1 - \overline{a}\zeta}, \quad h(\zeta) = \psi \circ g(\zeta).$$

Since $h(0) = 0$, we have

$$|h'(0)| = \frac{|f''(0)|}{2(1 - |f'(0)|^2)} \leq 1,$$

which is equivalent to (1.1). If the equality holds in the above inequality, then $h(\zeta) = e^{i\theta}\zeta$ for some $\theta \in \mathbb{R}$. This yields that $f(\zeta) = \zeta \varphi(e^{i\theta}\zeta)$ with $\varphi = \psi^{-1}$. \square

Applying Lemma 1.1, we now compute $K_{\Delta}^2(0, \xi)$ for $\xi = (\xi_1, \xi_2) \in J_0^2(\Delta)$. Let $f : \Delta \rightarrow \Delta$ be a holomorphic function such that $f(0) = 0$ and

$$f'(0) = \lambda \xi_1, \quad f''(0) = \lambda^2 \xi_2$$

for some $\lambda \in \mathbb{C}$. By Lemma 1.1 we have

$$|\lambda \xi_1| \leq 1, \quad |\lambda^2 \xi_2| \leq 2 - 2|\lambda \xi_1|^2.$$

Therefore,

$$|\lambda| \leq \sqrt{2/(|\xi_2| + 2|\xi_1|^2)}$$

and there exists a disc f of the form (1.2) which attains the equality. This shows that for $\xi = (\xi_1, \xi_2) \in J_0^2(\Delta)$, we have

$$K_{\Delta}^2(0, \xi) = \sqrt{|\xi_1|^2 + \frac{|\xi_2|}{2}}.$$

Remark 1.1. *The following higher order Kobayashi metric was introduced by J. Yu in [18] (see also [19, 7, 12, 6])*

$$\chi_{\Omega}^k(p, v) := \inf \left\{ \frac{1}{\lambda} > 0 \mid f : \Delta \rightarrow \Omega \text{ holomorphic, } f = p + \zeta^k \Psi, \Psi(0) = \lambda v \right\}$$

for p in a domain $\Omega \subset \mathbb{C}^n$ and $v \in \mathbb{C}^n$. Note that for $\xi \in J_p^k(\Omega)$ of the form $\xi = (0, \dots, 0, v)$ we have

$$K_{\Omega}^k(p, \xi) = \chi_{\Omega}^k(p, v).$$

1.3. **k -stationary discs.** Let $\Omega = \{\rho < 0\} \subset \mathbb{C}^n$ be a smooth domain, where ρ is a defining function. We set $\partial\rho = \left(\frac{\partial\rho}{\partial z_1}, \dots, \frac{\partial\rho}{\partial z_n}\right)$. Let k be a positive integer. Following [1], we define

Definition 1.1. A map $f : \bar{\Delta} \rightarrow \mathbb{C}^n$, holomorphic on Δ and continuous up to $\bar{\Delta}$, is a k -stationary disc attached to $b\Omega$ if $f(b\Delta) \subset b\Omega$ and if there exists a continuous function $c : b\Delta \rightarrow \mathbb{R}^+$ such that the function $\zeta \mapsto \zeta^k c(\zeta) \partial\rho(f(\zeta)) \in \mathbb{C}^n$ defined on $b\Delta$ extends holomorphically to Δ .

These discs generalize the notion of stationary discs introduced by L. Lempert [11] and are particularly well adapted to study Levi degenerate hypersurfaces [1, 2]. In fact, in the present context, we will also need

Definition 1.2. A bounded holomorphic map $f : \Delta \rightarrow \mathbb{C}^n$ is a k -stationary disc attached to $b\Omega$ in the L^∞ sense if $f(b\Delta) \subset b\Omega$ a.e. and if there exists a L^∞ function $c : b\Delta \rightarrow \mathbb{R}^+$ such that the function $\zeta \mapsto \zeta^k c(\zeta) \partial\rho(f(\zeta)) \in \mathbb{C}^n$ defined on $b\Delta$ extends holomorphically to Δ .

Example 1. Consider the case $\Omega = \Delta$. Let

$$f(\zeta) = \prod_{j=1}^k \frac{\zeta - a_j}{1 - \bar{a}_j \zeta}$$

for some $a_1, \dots, a_k \in \Delta$. Let $\rho(\zeta) = |\zeta|^2 - 1$ be the standard defining function of the unit disc Δ . Then

$$\partial\rho(f(\zeta)) = \overline{f(\zeta)}$$

and so the function

$$b\Delta \ni \zeta \mapsto \zeta^k \partial\rho(f(\zeta)) = \zeta^k \overline{f(\zeta)}$$

has winding index 0 around the origin. Therefore, there exists a function g defined on $b\Delta$ such that $e^{g(\zeta)} = \zeta^k \overline{f(\zeta)}$. Let h be a continuous real-valued function on $b\Delta$ for that $g + h$ extends to a holomorphic function on Δ and let $c(\zeta) = e^{h(\zeta)}$. Then

$$\zeta^k c(\zeta) \partial\rho(f(\zeta)) = e^{(g+h)(\zeta)},$$

which is holomorphic on Δ . Therefore, the disc f is k -stationary.

2. EXTREMAL DISCS FOR THE KOBAYASHI k -PSEUDOMETRIC ARE k -STATIONARY

For $v = (v_1, \dots, v_n), w = (w_1, \dots, w_n) \in \mathbb{C}^n$, we set

$$\langle v, w \rangle = \sum_{j=1}^n v_j w_j.$$

For a domain $\Omega \subset \mathbb{C}^n$, we denote by $\text{Hol}(\Delta, \Omega)$ the space of holomorphic maps from Δ to Ω , by $H^\infty(\Delta, \mathbb{C}^n) = \text{Hol}(\Delta, \mathbb{C}^n) \cap L^\infty(\Delta, \mathbb{C}^n)$ and by $H^1(\Delta, \mathbb{C}^n) = \text{Hol}(\Delta, \mathbb{C}^n) \cap L^1(b\Delta, \mathbb{C}^n)$. Let k be a positive integer.

Definition 2.1. Let $\Omega \subset \mathbb{C}^n$ be a domain. A map $f : \Delta \rightarrow \Omega$ is a *extremal disc for the Kobayashi k -metric for the pair* $(p, \xi) \in \Omega \times J_p^k(\Omega)$ if $f(0) = p$, $J_p^k(f) = \lambda \cdot \xi$ with $\lambda > 0$ and if $g : \Delta \rightarrow \Omega$ is holomorphic and such that $g(0) = p$, $J_p^k(g) = \mu \cdot \xi$ with $\mu > 0$, then $\mu \leq \lambda$.

Note that in case $\Omega \subset \mathbb{C}^n$ is a bounded domain, Montel's theorem ensures the existence of extremal discs for any pair $(p, \xi) \in \Omega \times J_p^k(\Omega)$.

Theorem 1. *Let $\Omega = \{\rho < 0\} \subset \mathbb{C}^n$ be a bounded domain defined by a C^2 plurisubharmonic function ρ . Let $p \in \Omega$, $\xi \in J_p^k(\Omega) \setminus \{0\}$. Then any extremal disc f for the Kobayashi k -metric is almost proper, that is $f(b\Delta) \subset b\Omega$ a.e., and k -stationary in the L^∞ sense.*

The proof follows the variational approach developed by E. Poletsky in [14]. We point out that in order to describe complex geodesics in complex ellipsoids, A. Edigarian [4] generalized Poletsky's theory. It follows from M. Jarnicki and P. Pflug [6] (see in particular Remark 11.4.4), that using Edigarian-Poletsky theory, extremal discs for the Yu's k^{th} order Kobayashi metric are k -stationary in the L^∞ sense.

2.1. Poletsky extremal problem. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain defined by a C^2 plurisubharmonic function. Consider a real-valued functional Φ defined on $H^\infty(\Delta, \mathbb{C}^n)$ satisfying

(A) Φ is differentiable,

(B) for any $f \in H^\infty(\Delta, \mathbb{C}^n)$, the differential $d_f\Phi$ of Φ at f is of the form $d_f\Phi(h) = \text{Re} \int_{b\Delta} \langle h, \omega \rangle d\theta$ where ω is holomorphic on $\mathbb{C} \setminus \overline{\Delta_r}$ for some $r < 1$.

Consider $N + 1$ functionals Φ_0, \dots, Φ_N satisfying conditions (A) and (B) and real numbers a_1, \dots, a_N . The Poletsky extremal problem (P) consists in finding $f_0 \in \text{Hol}(\Delta, \Omega)$ which maximizes Φ_0 under the constraints $\Phi_j(f) = a_j$, $1 \leq j \leq N$, and $f \in \text{Hol}(\Delta, \Omega)$.

We also recall that, according to the definition given in [14] (see also [4]), real-valued linear functionals F_1, \dots, F_N defined on $H^1(\Delta, \mathbb{C}^n)$ and of the form $F_j(f) = \text{Re} \int_{b\Delta} \langle f, \omega_j \rangle d\theta$ are called *linearly independent over $b\Delta$* if $\sum_{j=1}^N \lambda_j \omega_j = g$ on $b\Delta$ for certain $\lambda_j \in \mathbb{R}$ and $g \in H^\infty(\Delta, \mathbb{C}^n)$ with $g(0) = 0$ only when $\lambda_j = 0$ for all $1 \leq j \leq N$ and $g \equiv 0$.

Remark 2.1. *In [14], linear independence is actually defined as a stronger property; the condition $g(0) = 0$ is omitted. However, an inspection of the proof of Theorem 3 in [14] shows that the definition above is enough for its application. Note that in [4, 6], linear independence is defined as well with the condition $g(0) = 0$.*

E. Poletsky proved the following theorem

Theorem 2 (Theorem 3 p. 330 [14]). *Suppose f_0 is a solution of the extremal problem (P) and suppose that the differentials $d_{f_0}\Phi_1, \dots, d_{f_0}\Phi_N$ are linearly independent over $b\Delta$. Then:*

- i. f_0 is almost proper.
- ii. *There exist real numbers $\lambda_1, \dots, \lambda_N$, a L^∞ function $c : b\Delta \rightarrow \mathbb{R}^+$ and $g \in H^\infty(\Delta, \mathbb{C}^n)$ with $g(0) = 0$ such that*

$$\sum_{j=1}^N \lambda_j \omega_j + g = c \partial \rho(f_0)$$

a.e. on $b\Delta$, where $d_{f_0}\Phi_j = \text{Re} \int \langle \cdot, \omega_j \rangle d\theta$ for some ω_j which is holomorphic on $\mathbb{C} \setminus \overline{\Delta_r}$ for some $r < 1$.

2.2. Proof of Theorem 1. Let $p \in \mathbb{C}^n$ and $\xi = (\xi_1, \dots, \xi_k) \in J_p^k(\Omega) \setminus \{0\}$. We write $\xi_j = (\xi_{j1}, \dots, \xi_{jn}) \in \mathbb{C}^n$ and for any holomorphic map $f : \Delta \rightarrow \mathbb{C}^n$, we write $f = (f_1, \dots, f_n)$. Let $1 \leq j_0 \leq k$ be the smallest integer such that $\xi_{j_0} \neq 0$.

We formulate the extremality condition as Poletsky extremal problem (P), as follows. For any j such that $\xi_j \neq 0$ we choose linearly independent vectors $\eta_j^1, \dots, \eta_j^{n-1}$ (with $\eta_j^\ell = (\eta_{j1}^\ell, \dots, \eta_{jn}^\ell)$) such that $\langle \xi_j, \eta_j^\ell \rangle = 0$ for all $1 \leq \ell \leq n-1$. We define, for $1 \leq h \leq n$, $1 \leq \ell \leq n-1$, $j_0 \leq m \leq k$ and $j_0 + 1 \leq m' \leq k$ such that $\xi_m \neq 0$, $\xi_{m'} \neq 0$, continuous real functionals on $H^\infty(\Delta, \mathbb{C}^n)$:

$$\left\{ \begin{array}{l} \Phi_{0,h}^1(f) = \frac{1}{2\pi} \operatorname{Re} \int_{b\Delta} f_h(e^{i\theta}) d\theta, \quad \Phi_{0,h}^2(f) = \frac{1}{2\pi} \operatorname{Re} \int_{b\Delta} -i f_h(e^{i\theta}) d\theta, \\ \Phi_{m,\ell}^1(f) = \frac{1}{2\pi} \operatorname{Re} \int_{b\Delta} \frac{\langle f(e^{i\theta}), \eta_m^\ell \rangle}{\zeta^m} d\theta, \quad \Phi_{m,\ell}^2(f) = \frac{1}{2\pi} \operatorname{Re} \int_{b\Delta} -i \frac{\langle f(e^{i\theta}), \eta_m^\ell \rangle}{\zeta^m} d\theta, \\ \Phi_{j_0}^1(f) = \frac{(j_0)!}{2\pi \|\xi_{j_0}\|^2} \operatorname{Re} \int_{b\Delta} \frac{\langle f(e^{i\theta}), \bar{\xi}_{j_0} \rangle}{\zeta^{j_0}} d\theta, \\ \Phi_{m'}^1(f) = \frac{m'!}{2\pi \|\xi_{m'}\|^2} \operatorname{Re} \int_{b\Delta} \frac{\langle f(e^{i\theta}), \bar{\xi}_{m'} \rangle}{\zeta^{m'}} d\theta - (\Phi_{j_0}^1(f))^{\frac{m'}{j_0}}, \\ \Phi_m^2(f) = \frac{1}{2\pi} \operatorname{Re} \int_{b\Delta} -i \frac{\langle f(e^{i\theta}), \bar{\xi}_m \rangle}{\zeta^m} d\theta. \end{array} \right.$$

For all $j_0 + 1 \leq m \leq k$ such that $\xi_m = 0$ and all $j_0 + 1 \leq m' \leq k$ such that $\xi_{m'} = 0$ we define instead

$$\left\{ \begin{array}{l} \Phi_{m,\ell}^1(f) = \frac{1}{2\pi} \operatorname{Re} \int_{b\Delta} \frac{f_\ell(e^{i\theta})}{\zeta^m} d\theta, \quad \Phi_{m,\ell}^2(f) = \frac{1}{2\pi} \operatorname{Re} \int_{b\Delta} -i \frac{f_\ell(e^{i\theta})}{\zeta^m} d\theta, \\ \Phi_m^2(f) = \frac{1}{2\pi} \operatorname{Re} \int_{b\Delta} -i \frac{f_n(e^{i\theta})}{\zeta^m} d\theta, \quad \Phi_{m'}^1(f) = \frac{1}{2\pi} \operatorname{Re} \int_{b\Delta} \frac{f_n(e^{i\theta})}{\zeta^{m'}} d\theta. \end{array} \right.$$

Lemma 2.1. *A holomorphic disc $f : \Delta \rightarrow \Omega$ is extremal for the Kobayashi k -metric for the pair $(p, \xi) = (p, 0, \dots, 0, \xi_{j_0}, \dots, \xi_k) \in \Omega \times J_p^k(\Omega)$ if and only if it maximizes $\Phi_{j_0}^1$ under the constraints*

$$\left\{ \begin{array}{l} \Phi_{0,h}^1(f) = \operatorname{Re} p_h \\ \Phi_{0,h}^2(f) = \operatorname{Im} p_h \\ \Phi_{m,\ell}^1(f) = \Phi_{m,\ell}^2(f) = \Phi_m^2(f) = \Phi_{m'}^1(f) = 0 \\ f \in \operatorname{Hol}(\Delta, \Omega). \end{array} \right.$$

Proof. Expanding each component of f in its Fourier series, we obtain that the conditions $\Phi_{0,h}^1(f) = \operatorname{Re} p_h$, $\Phi_{0,h}^2(f) = \operatorname{Im} p_h$ are equivalent to $f(0) = p$. Moreover, the choice of the

vectors η_j^ℓ implies that, for certain $\mu_m \in \mathbb{C}$, $f^{(m)}(0) = \mu_m \xi_m$ for all $j_0 \leq m \leq k$ whenever $\Phi_{m,\ell}^1(f) = \Phi_{m,\ell}^2(f) = 0$ for all $j_0 \leq m \leq k$, $1 \leq \ell \leq n-1$. The conditions $\Phi_m^2(f) = 0$, ensure further that $\mu_m \in \mathbb{R}$ for all $j_0 \leq m \leq k$. Setting $\mu_{j_0} = \lambda^{j_0}$, the equations $\Phi_{m'}^1(f) = 0$ amount to $\mu_{m'} = \lambda^{m'}$ whenever $\xi_{m'} \neq 0$, and to $f^{(m')}(0) = 0$ if $\xi_{m'} = 0$. Since then $\Phi_{j_0}^1(f) = \lambda^{j_0}$, a disc f is extremal if it maximizes $\Phi_{j_0}^1$ under the above constraints and $f \in \text{Hol}(\Delta, \Omega)$. \square

Lemma 2.2. *The problem in Lemma 2.1 satisfies the conditions of the Poletsky extremal problem (P).*

Proof of Lemma 2.2. Note that the functionals under considerations are all linear with the exception of $\Phi_{m'}^1$ for any $j_0 + 1 \leq m' \leq k$ such that $\xi_{m'} \neq 0$. A straightforward computation yields

$$d_{f^0} \Phi_{m'}^1(f) = \frac{1}{2\pi} \text{Re} \int_{b\Delta} \left\langle f(e^{i\theta}), \left(\frac{m'!}{\|\xi_{m'}\|^2 \zeta^{m'}} \bar{\xi}_{m'} - C_{m'}(f^0) \frac{j_0!}{\|\xi_{j_0}\|^2 \zeta^{j_0}} \bar{\xi}_{j_0} \right) \right\rangle d\theta$$

where

$$C_{m'}(f^0) = \frac{m'}{j_0} (\Phi_{j_0}^1(f^0))^{\frac{m'}{j_0}-1}.$$

for any $j_0 + 1 \leq m' \leq k$ such that $\xi_{m'} \neq 0$. Now, let e_1, \dots, e_n be the standard basis of \mathbb{C}^n and let $f^0 \in H^\infty(\Delta, \mathbb{C}^n)$. Define for $1 \leq h \leq n$, $1 \leq \ell \leq n-1$, $j_0 \leq m \leq k$ and $j_0 + 1 \leq m' \leq k$ such that $\xi_m \neq 0$, $\xi_{m'} \neq 0$, holomorphic maps $\mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}^n$ as follows:

$$\left\{ \begin{array}{l} \omega_{0,h}^1 = \frac{1}{2\pi} e_h, \quad \omega_{0,h}^2 = -\frac{i}{2\pi} e_h, \\ \omega_{m,\ell}^1 = \frac{1}{2\pi \zeta^m} \eta_m^\ell, \quad \omega_{m,\ell}^2 = -\frac{i}{2\pi \zeta^m} \eta_m^\ell \\ \omega_{j_0}^1 = \frac{j_0!}{2\pi \|\xi_{j_0}\|^2 \zeta^{j_0}} \bar{\xi}_{j_0} \\ \omega_{m'}^1 = \frac{m'!}{2\pi \|\xi_{m'}\|^2 \zeta^{m'}} \bar{\xi}_{m'} - C_{m'}(f^0) \omega_{j_0}^1 \\ \omega_m^2 = -\frac{i}{2\pi \zeta^m} \bar{\xi}_m \end{array} \right.$$

For all $j_0 + 1 \leq m \leq k$ such that $\xi_m = 0$ and all $j_0 + 1 \leq m' \leq k$ such that $\xi_{m'} = 0$ we define instead

$$\omega_{m,\ell}^1 = \frac{1}{2\pi \zeta^m} e_\ell, \quad \omega_{m,\ell}^2 = -\frac{i}{2\pi \zeta^m} e_\ell, \quad \omega_m^2 = -\frac{i}{2\pi \zeta^m} e_n, \quad \omega_{m'}^1 = \frac{1}{2\pi \zeta^{m'}} e_n.$$

With these definitions, we have

$$\begin{aligned} \Phi_{0,h}^1(f) &= \text{Re} \int_{b\Delta} \langle f, \omega_{0,h}^1 \rangle d\theta, & \Phi_{0,h}^2(f) &= \text{Re} \int_{b\Delta} \langle f, \omega_{0,h}^2 \rangle d\theta, \\ \Phi_{m,\ell}^1(f) &= \text{Re} \int_{b\Delta} \langle f, \omega_{m,\ell}^1 \rangle d\theta, & \Phi_{m,\ell}^2(f) &= \text{Re} \int_{b\Delta} \langle f, \omega_{m,\ell}^2 \rangle d\theta, \end{aligned}$$

$$\Phi_{j_0}^1(f) = \operatorname{Re} \int_{b\Delta} \langle f, \omega_{j_0}^1 \rangle d\theta, \quad d_{f_0} \Phi_{m'}^1(f) = \operatorname{Re} \int_{b\Delta} \langle f, \omega_{m'}^1 \rangle d\theta, \quad \Phi_m^2(f) = \operatorname{Re} \int_{b\Delta} \langle f, \omega_m^2 \rangle d\theta,$$

from which the conclusion follows immediately. \square

In order to apply Theorem 2, it remains to prove the following lemma

Lemma 2.3. *For any disc $f^0 \in H^\infty(\Delta, \mathbb{C}^n)$, the linear functionals $\Phi_{0,h}^1$, $\Phi_{0,h}^2$, $\Phi_{m,\ell}^1$, $\Phi_{m,\ell}^2$, $\Phi_{j_0}^1$, $d_{f_0} \Phi_{m'}^1$ and Φ_m^2 are linearly independent on $b\Delta$.*

Proof. Assume that $\xi_m \neq 0$ for all $j_0 + 1 \leq m \leq k$ (the case when $\xi_m = 0$ for some m is simpler). Using the exact expression of the ω 's obtained in the proof of Lemma 2.2, we need to consider the vector valued equation on $b\Delta$

$$\begin{aligned} & \sum_{h=1}^n (\lambda_{0,h}^1 - i\lambda_{0,h}^2) e_h + \sum_{m=1}^{j_0-1} \frac{1}{\zeta^m} \sum_{\ell=1}^{n-1} (\lambda_{m,\ell}^1 - i\lambda_{m,\ell}^2) \eta_m^\ell + \\ & \frac{1}{\zeta^{j_0}} \left(\left(\frac{j_0! \lambda_{j_0}^1}{\|\xi_{j_0}\|} - i\lambda_{j_0}^2 \right) \bar{\xi}_{j_0} + \sum_{\ell=1}^{n-1} (\lambda_{j_0,\ell}^1 - i\lambda_{j_0,\ell}^2) \eta_{j_0}^\ell - \sum_{m'=j_0+1}^k \frac{j_0! C_{m'}(f^0) \lambda_{m'}^1 \bar{\xi}_{j_0}}{\|\xi_{j_0}\|^2} \right) + \\ & + \sum_{m'=j_0+1}^k \frac{1}{\zeta^{m'}} \left(\left(\frac{m'! \lambda_{m'}^1}{\|\xi_{m'}\|^2} - i\lambda_{m'}^2 \right) \bar{\xi}_{m'} + \sum_{\ell=1}^{n-1} (\lambda_{m',\ell}^1 - i\lambda_{m',\ell}^2) \eta_{m'}^\ell \right) = g \end{aligned}$$

for suitable $\lambda_{0,h}^1, \lambda_{0,h}^2, \lambda_{m,\ell}^1, \lambda_{m,\ell}^2, \lambda_m^1, \lambda_{j_0}^1, \lambda_{m'}^1 \in \mathbb{R}$ and $g \in H^\infty(\Delta, \mathbb{C}^n)$ such that $g(0) = 0$. Since the Fourier expansion of g is of the form $\sum_{j \geq 1} g_j \zeta^j$, we have immediately that $g \equiv 0$, $\lambda_{0,h}^1 = \lambda_{0,h}^2 = 0$ for $1 \leq h \leq n$ and $\lambda_{m,\ell}^1 = \lambda_{m,\ell}^2 = 0$ for $1 \leq m \leq j_0 - 1$. Now observe that for all $j_0 \leq m \leq k$ the vectors $\bar{\xi}_m, \eta_m^1, \dots, \eta_m^{n-1}$ are a basis of \mathbb{C}^n over \mathbb{C} . Indeed, if $\bar{\xi}_m = \sum_{\ell=1}^{n-1} \mu_\ell \eta_m^\ell$ we would have

$$\|\xi_m\|^2 = \langle \xi_m, \bar{\xi}_m \rangle = \sum_{\ell=1}^{n-1} \mu_\ell \langle \xi_m, \eta_m^\ell \rangle = 0$$

which is a contradiction. It follows that $\lambda_{m'}^1 = \lambda_{m'}^2 = \lambda_{m',\ell}^1 = \lambda_{m',\ell}^2 = 0$ for all $j_0 + 1 \leq m' \leq k$, $1 \leq \ell \leq n - 1$, and finally that $\lambda_{j_0}^1 = \lambda_{j_0}^2 = \lambda_{j_0,\ell}^1 = \lambda_{j_0,\ell}^2 = 0$. \square

We are now in a position to apply Theorem 2. We obtain that a disc f which is extremal for the problem in Lemma 2.1, and thus for the Kobayashi k -metric, must satisfy the Euler-Lagrange equations

$$\begin{aligned} & \rho(f) = 0 \text{ a.e. on } b\Delta, \\ & \sum_{h=1}^n (\lambda_{0,h}^1 \omega_{0,h}^1 + \lambda_{0,h}^2 \omega_{0,h}^2) + (\lambda_{j_0}^1 \omega_{j_0}^1 + \lambda_{j_0}^2 \omega_{j_0}^2) + \sum_{\ell=1}^{n-1} (\lambda_{1,\ell}^1 \omega_{1,\ell}^1 + \lambda_{1,\ell}^2 \omega_{1,\ell}^2) + \\ & + \sum_{m'=2}^k \left(\lambda_{m'}^1 \omega_{m'}^1 + \lambda_{m'}^2 \omega_{m'}^2 + \sum_{\ell=1}^{n-1} \lambda_{m',\ell}^1 \omega_{m',\ell}^1 + \lambda_{m',\ell}^2 \omega_{m',\ell}^2 \right) + g = c \partial \rho(f) \text{ a.e. on } b\Delta \end{aligned}$$

for a suitable L^∞ function $c : b\Delta \rightarrow \mathbb{R}^+$ and a certain $g \in H^\infty(\Delta, \mathbb{C}^n)$ such that $g(0) = 0$. Since the maps ω are all meromorphic with a pole (of order at most k) only at 0, it follows that the map $\zeta^k c \partial \rho(f)$ extends holomorphically to Δ , hence f is k -stationary in the L^∞ -sense. This concludes the proof of Theorem 1. \square

3. k -STATIONARY DISCS ARE LOCALLY EXTREMAL FOR THE KOBAYASHI k -PSEUDOMETRIC

Theorem 3. *Let $\Omega \subset \mathbb{C}^n$ be a smooth strictly convex domain. Then k -stationary discs for Ω are locally extremal for the Kobayashi k -metric.*

Proof. We follow here the approach developed by L. Lempert (see Proposition 1 in [11]). Let $f : \Delta \rightarrow \mathbb{C}^n$ be a k -stationary disc. Let $g : \Delta \rightarrow \mathbb{C}^n$ be a holomorphic map, close enough to f , such that $g \neq f$, $g(0) = f(0) = p$ and $J_p^k(g) = \lambda \cdot J_p^k(f)$ for some positive λ . We wish to prove that $\lambda \leq 1$. Due to the convexity of Ω we have a.e. on $b\Delta$

$$\operatorname{Re}\langle f(\zeta) - g(\zeta), \partial\rho(f(\zeta)) \rangle > 0$$

and thus

$$\operatorname{Re}\langle f(\zeta) - g(\zeta), c(\zeta)\partial\rho(f(\zeta)) \rangle = \operatorname{Re}\langle \zeta^{-k}(f(\zeta) - g(\zeta)), \tilde{f}(\zeta) \rangle > 0.$$

This implies that

$$\operatorname{Re} \sum_{j=1}^k \left\langle \frac{1-\lambda^j}{j!} f^{(j)}(0) \zeta^{j-k}, \tilde{f}(\zeta) \right\rangle + \operatorname{Re} \sum_{j=k+1}^{\infty} \left\langle \frac{1}{j!} (f^{(j)}(0) - g^{(j)}(0)) \zeta^{j-k}, \tilde{f}(\zeta) \right\rangle > 0$$

and so

$$\operatorname{Re} \sum_{j=1}^k \frac{1-\lambda^j}{j!} \left\langle f^{(j)}(0) e^{i(j-k)\theta}, \tilde{f}(e^{i\theta}) \right\rangle + \operatorname{Re} \sum_{j=k+1}^{\infty} \frac{1}{j!} \left\langle (f^{(j)}(0) - g^{(j)}(0)) e^{i(j-k)\theta}, \tilde{f}(e^{i\theta}) \right\rangle > 0.$$

Since $\operatorname{Re} \sum_{j=k+1}^{\infty} \left\langle \frac{1}{j!} (f^{(j)}(0) - g^{(j)}(0)) \zeta^{j-k}, \tilde{f}(\zeta) \right\rangle$ extends as a harmonic function u on Δ and since $u(0) = 0$, we have

$$\sum_{j=1}^k \frac{1-\lambda^j}{j!} \operatorname{Re} \int_0^{2\pi} \left\langle f^{(j)}(0), \tilde{f}(e^{i\theta}) \right\rangle e^{i(j-k)\theta} d\theta > 0$$

which can be written as

$$(1-\lambda) \left[\sum_{j=1}^k \frac{1+\lambda+\dots+\lambda^{j-1}}{j!} \operatorname{Re} \int_0^{2\pi} \left\langle f^{(j)}(0), \tilde{f}(e^{i\theta}) \right\rangle e^{i(j-k)\theta} d\theta \right] > 0.$$

We have

$$(1-\lambda) \left[\sum_{j=1}^k \frac{1+\lambda+\dots+\lambda^{j-1}}{j!} \operatorname{Re} \left\langle f^{(j)}(0), \int_0^{2\pi} \tilde{f}(e^{i\theta}) e^{i(j-k)\theta} d\theta \right\rangle \right] > 0,$$

and therefore

$$(3.1) \quad (1-\lambda) \operatorname{Re} \left[\sum_{j=1}^k \frac{1+\lambda+\dots+\lambda^{j-1}}{j!(j-k)!} \left\langle f^{(j)}(0), \tilde{f}^{(k-j)}(0) \right\rangle \right] > 0.$$

Lemma 3.1. *For $\lambda < 1$, let $g_\lambda(\zeta) := f(\lambda\zeta)$. Then we have*

$$\operatorname{Re}\langle f(\zeta) - g_\lambda(\zeta), \partial\rho(f(\zeta)) \rangle = (1-\lambda)k_\lambda(\zeta)$$

with $k_\lambda(\zeta) \rightarrow k_1(\zeta) \neq 0$ for all $\zeta \in b\Delta$.

Proof. The function $v = \rho \circ f$ is (strictly) subharmonic and negative on Δ , while it vanishes on $b\Delta$. By the Hopf lemma the radial derivative $\frac{\partial v}{\partial r}(\zeta)$ is non-vanishing for $\zeta \in b\Delta$. Using the chain rule we can write

$$\frac{\partial v}{\partial r}(\zeta) = \partial\rho(f(\zeta)) \cdot \frac{\partial f}{\partial r}(\zeta) + \bar{\partial}\rho(f(\zeta)) \cdot \frac{\partial \bar{f}}{\partial r}(\zeta) = 2\operatorname{Re} \left\langle \frac{\partial f}{\partial r}(\zeta), \partial\rho(f(\zeta)) \right\rangle.$$

On the other hand

$$\lim_{\lambda \rightarrow 1^-} \frac{f(\zeta) - g_\lambda(\zeta)}{1 - \lambda} = \lim_{\lambda \rightarrow 1^-} \frac{f(\zeta) - f(\lambda\zeta)}{1 - \lambda} = |\zeta| \frac{\partial f}{\partial r}(\zeta),$$

hence the statement of the lemma follows by setting $k_1(\zeta) = \frac{1}{2} \frac{\partial v}{\partial r}(\zeta)$ for $\zeta \in b\Delta$. \square

Now, we turn back to the proof of Theorem 3. Putting $g_\lambda(\zeta) = f(\lambda\zeta)$, with $\lambda < 1$, in place of $g(\zeta)$ in the previous computation, and using Lemma 3.1, we obtain

$$0 \neq \lim_{\lambda \rightarrow 1^-} \operatorname{Re} \underbrace{\left[\sum_{j=1}^k \frac{1 + \lambda + \dots + \lambda^{j-1}}{j!(j-k)!} \left\langle f^{(j)}(0), \tilde{f}^{(k-j)}(0) \right\rangle \right]}_{I_\lambda} = \operatorname{Re} \left[\sum_{j=1}^k \frac{j}{j!(j-k)!} \left\langle f^{(j)}(0), \tilde{f}^{(k-j)}(0) \right\rangle \right].$$

Due to (3.1) and $\lambda < 1$, the terms I_λ are positive and therefore the expression above is positive. Let then $g : \Delta \rightarrow \mathbb{C}^n$ be any holomorphic map as above which is close enough to f ; in particular $J_p^k(g) = \lambda \cdot J_p^k(f)$ with λ close enough to 1, so that we have

$$\operatorname{Re} \left[\sum_{j=1}^k \frac{1 + \lambda + \dots + \lambda^{j-1}}{j!(j-k)!} \left\langle f^{(j)}(0), \tilde{f}^{(k-j)}(0) \right\rangle \right] > 0,$$

which implies together with (3.1) that $\lambda < 1$. \square

REFERENCES

- [1] F. Bertrand, G. Della Sala, *Stationary discs for smooth hypersurfaces of finite type and finite jet determination*, J. Geom. Anal. **25** (2015), 2516-2545.
- [2] F. Bertrand, B. Lamel, G. Della Sala, *Jet determination of smooth CR automorphisms and generalized stationary discs*, submitted for publication.
- [3] B. Coupet, H. Gaussier, A. Sukhov, *Riemann maps in almost complex manifolds*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **2** (2003) 761-785.
- [4] A. Edigarian, *On extremal mappings in complex ellipsoids*, Ann. Polon. Math. **62** (1995), 83-96.
- [5] X. Huang, *A non-degeneracy property of extremal mappings and iterates of holomorphic self-mappings*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **21** (1994), 399-419.
- [6] M. Jarnicki, P. Pflug, *Invariant distances and metrics in complex analysis*. Second extended edition. De Gruyter Expositions in Mathematics, 9. Walter de Gruyter GmbH & Co. KG, Berlin, 2013. xviii+861 pp.
- [7] J. J. Kim, I. G. Hwang, J. G. Kim, and J. S. Lee, *On the higher order Kobayashi metrics*, Honam Math. J. **26** (2004), 549-557.
- [8] S. Kobayashi, *Distance, holomorphic mappings and the Schwarz lemma*, J. Math. Soc. Japan **19** (1967), 481-485.
- [9] S. Kobayashi, *Invariant distances on complex manifolds and holomorphic mappings*, J. Math. Soc. Japan **19** (1967), 460-480.
- [10] S. Kobayashi, *Hyperbolic complex spaces*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **318**. Springer-Verlag, Berlin, 1998. xiv+471 pp.

- [11] L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule* (French. English summary) [The Kobayashi metric and the representation of domains on the ball], Bull. Soc. Math. France **109** (1981), 427-474.
- [12] N. Nikolov, *Stability and boundary behavior of the Kobayashi metrics*, Acta Math. Hungar. **90** (2001), 283-291.
- [13] M.-Y. Pang, *Smoothness of the Kobayashi metric of nonconvex domains*, Internat. J. Math. **4** (1993), 953-987.
- [14] E. Poletsky, *The Euler-Lagrange equations for extremal holomorphic mappings of the unit disk*, Michigan Math. J. **30** (1983), 317-333.
- [15] H. L. Royden, *Remarks on the Kobayashi metric*, Several complex variables, II (Proc. Internat. Conf., Univ. Maryland, College Park, Md., 1970), pp. 125-137. Lecture Notes in Math., Vol. 185, Springer, Berlin, 1971.
- [16] A. Tumanov, *Extremal discs and the regularity of CR mappings in higher codimension*, Amer. J. Math. **123** (2001), 445-473.
- [17] S. Venturini, *The Kobayashi metric on complex spaces*, Math. Ann. **305** (1996), 25-44.
- [18] J. Yu, *Singular Kobayashi metrics and finite type conditions*, Proc. Amer. Math. Soc. **123** (1995), 121-130.
- [19] J. Yu, *Weighted boundary limits of the generalized Kobayashi-Royden metrics on weakly pseudoconvex domains*, Trans. Amer. Math. Soc. **347** (1995), 587-614.

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