JET DETERMINATION OF SMOOTH CR AUTOMORPHISMS AND GENERALIZED STATIONARY DISCS

FLORIAN BERTRAND, GIUSEPPE DELLA SALA AND BERNHARD LAMEL

Abstract. We prove finite jet determination for (finitely) smooth CR diffeomorphisms of (finitely) smooth Levi degenerate hypersurfaces in $\mathbb{C}^{n+1}$ by constructing generalized stationary discs glued to such hypersurfaces.

1. Introduction

Let $M, M' \subset \mathbb{C}^{n+1}$ be C^ℓ-smooth hypersurfaces. We recall that the complex tangent space T_p^cM, for $p \in M$, is defined by $T_p^cM = T_pM \cap iT_pM$, and is the largest complex subspace of $\mathbb{C}^{n+1}$ contained in T_pM. A map $H\colon M \to M'$ (of class C^1) is said to be a CR map if $H'(p)|_{T_p^cM}$ maps T_p^cM into T_p^cM' and is complex linear. We will only be concerned with germs of CR maps which are also diffeomorphisms (of some regularity).

CR maps possess strong rigidity properties. We are interested mostly in one particular aspect of this rigidity here, namely the finite determination property. In the setting where M and M' are real-analytic, and $H\colon M \to M'$ extends to a germ of a biholomorphic map or is given by a formal power series at a point $p \in M$, this is usually phrased in terms of the finite jet determination property, and we know that H is determined by finitely many of its derivatives at a point $p \in M$ in many circumstances (we refer the reader to the paper of Baouendi, Mir, and Rotschild [2] as well as to Juhlin's paper [18] and the discussion of the literature therein).

Actually, in the real-analytic setting one knows quite a bit more: Not only are (formal) biholomorphisms between sufficiently nondegenerate hypersurfaces determined by their jets, they can actually be reconstructed from their jets in an analytic manner (see e.g. the survey [22] and [23]). One of the appealing parts of such so-called jet parametrizations is that they provide insight into structural properties of the automorphism groups of manifolds. However, they depend on jets, and therefore pointwise information. If one tries to study CR diffeomorphisms which are a priori only smooth (of some regularity), these methods are not applicable.

Our goal in this paper is to study such (finitely) smooth CR automorphisms of (finitely) smooth hypersurfaces in $\mathbb{C}^{n+1}$. We shall assume that our hypersurfaces, in suitable coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$, pass through $0 \in \mathbb{C}^{n+1}$ and that their defining functions are perturbations of a *finite type model hypersurface* S_P of the form

$$\operatorname{Re} w = P_d(z, \bar{z}),$$

2010 *Mathematics Subject Classification.* 32H02, 32H12, 32V35.

Research of the first author was supported by an URB grant and a long-term faculty development grant both from the American University of Beirut.

Research of the second author was supported by an URB grant from the American University of Beirut and by the Center for Advanced Mathematical Sciences.

Research of the third author was supported by the Austrian Science Fund FWF, project AI1776 and the Qatar National Research Fund, NPRP 7-511-1-098.

where P is a weighted homogeneous polynomial. To be more exact, we consider (sufficiently smooth) perturbations of S_P given by

$$\operatorname{Re} w = P_d(z, \bar{z}) + O(d+1),$$

where we endow z with the weight 1 and w with the weight $(2d+1)/2d$. We can now state our main theorem as follows.

Theorem 1.1. *Let P be a weighted homogeneous polynomial such that S_P is generically Levi-nondegenerate and the set of Levi-degenerate points containing* 0 *has dimension at most* $2n-1$. *Then there exists an $\ell \in \mathbb{N}$ such that for any allowable perturbation M of S_P in any neighbourhood U of* 0, *every local CR diffeomorphism of class C^ℓ of M is determined by its ℓ-jet: If $H\colon M \to M$ and $\tilde{H}\colon M \to M$ are CR diffeomorphisms of class C^ℓ with $j_0^\ell H = j_0^\ell \tilde{H}$, then $H = \tilde{H}$.*

In fact, Theorem 1.1 holds under the less stringent (but more technical) condition that "there exists an allowable vector" $v \in T_0^c S_P$; this condition is explained in Definition 3.1. In order to not duplicate formulations of theorems, we shall use the definition of an allowable vector as well as the condition that M is an allowable deformation (which is discussed in Definition 4.7); both conditions are geometric conditions in a suitable sense to be defined below, in particular, they can be defined independently of coordinates. We shall simply say *allowable hypersurface* from now on to indicate an allowable perturbation based on a model hypersurface which possesses an allowable vector. Our main theorem then reads:

Theorem 1.2. *Let M be an allowable hypersurface. Then there exists an $\ell \in \mathbb{N}$ only depending on the associated model hypersurface such that every local CR diffeomorphism of class C^ℓ of M is determined by its ℓ-jet: If $H\colon M \to M$ and $\tilde{H}\colon M \to M$ are CR diffeomorphisms of class C^ℓ with $j_0^\ell H = j_0^\ell \tilde{H}$, then $H = \tilde{H}$.*

The number ℓ depends on the specific form of P and can, in essence, be computed given P; we shall give upper bounds on ℓ later. However, an especially interesting aspect of the current paper is its application to the problem of unique determination of smooth diffeomorphisms of smooth hypersurfaces, in which case one can use known jet determination results in the formal setting which already provide ways to compute $j_0^\ell H$ from $j^{p_0} H$ for $\ell \geq p_0$. To be precise, we need to introduce some notation. We define, for $\ell \in \mathbb{N} \cup \{\infty, \omega\}$ and for a CR manifold M of class C^k for $k \geq \ell$, the spaces

$$\operatorname{Aut}^\ell(M,p) = \{H\colon M \to M\colon H \text{ is a germ at } p \text{ of a CR diffeomorphism of class } C^\ell\},$$

and for a smooth CR manifold M,

$$\operatorname{Aut}^f(M,p) = \{H\colon M \to M\colon H \text{ is a formal CR diffeomorphism of } M\}.$$

Here the space of formal CR diffeomorphisms of M is defined to be the space of formal power series maps $H\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ which have the property that they are formal biholomorphisms of the associated formal manifold (given by the ideal generated by the Taylor series of the defining equations of M), i.e. for one (and hence every) defining function ϱ of M and for one (and hence every) local parametrisation $\mathbb{R}^{2n+1} \supset U \ni x \mapsto Z(x) \in M$ we have that for any $\ell \in \mathbb{N}$ it holds that $\varrho(H(Z(x)), \overline{H(Z(x))}) = O(|x|^{\ell+1})$.

In particular, by definition we have natural maps

$$j_p^k\colon \operatorname{Aut}^\ell(M,p) \to G_p^k(\mathbb{C}^{n+1}),\ k \leq \ell \text{ and } j_p^k\colon \operatorname{Aut}^f(M,p) \to G_p^k(\mathbb{C}^{n+1}),\ k \in \mathbb{N},$$

into the jet group of order k of germs of biholomorphisms at p. We know that if M is *formally holomorphically nondegenerate* and *formally minimal* then, by [23], the map j_p^k is injective for k large enough. Every allowable smooth hypersurface is, as the reader can easily convince herself or himself, formally holomorphically nondegenerate and formally nonminimal. Therefore there exists a smallest number $k_0(M)$ such that for $k \geq k_0(M)$, the map $j_p^k\colon \operatorname{Aut}^f(M,p) \to G_p^k(\mathbb{C}^{n+1})$ is injective; in particular, the k-jets of smooth CR diffeomorphisms of M are uniquely determined by their $p_0(M)$ jets.

Theorem 1.2 therefore has the following immediate corollary.

Corollary 1.3. *Let M be an allowable smooth hypersurface. Then there exists an $\ell_0 \in \mathbb{N}$ such that for $\ell \geq \ell_0$ the map $j_p^{k_0(M)}\colon \operatorname{Aut}^\ell(M,p) \to G_p^{k_0(M)}(\mathbb{C}^{n+1})$ is injective. Furthermore ℓ_0 depends only on the associated model S_P.*

We would like to point out that Corollary 1.3 seems to be the first case of a jet determination result for *(finitely) smooth* CR diffeomorphisms aside from the finitely nondegenerate case. The jet determination problem for real-analytic CR diffeomorphisms of real-analytic CR manifolds has been studied widely, see e.g. [8, 11, 24, 20]. In the smooth case, results have been restricted to the setting of *finitely nondegenerate* hypersurfaces (see e.g. [9, 10, 19]). Our approach is most akin to the use of extremal discs by Huang [17, 16]. Let us give an outline of our approach.

When studying the automorphisms of a geometric structure, it is often convenient to extend the action of these automorphisms to spaces of invariant objects, and study the transformation properties of these invariant objects. In the study of real-analytic CR manifolds, a suitable family of associated objects is the family of Segre varieties. However, these have the drawback that they really can only be defined for real-analytic or formal CR manifolds and thus becomes unavailable in the setting of smooth CR manifolds.

Our approach in this paper is to construct another family of associated invariant objects, namely generalized stationary discs, which we refer to as k-stationary discs. We show that for allowable hypersurfaces in $\mathbb{C}^{n+1}$, one can invariantly attach a finite-dimensional family of generalized stationary discs.

This approach has been pioneered by the first and the second author for hypersurfaces in $\mathbb{C}^2$ in [5], generalizing the notion L. Lempert [25] used in his study of the Kobayashi metric on strictly convex domains (see also [17, 26]). These classical stationary discs are special analytic discs, attached to hypersurfaces M of $\mathbb{C}^{n+1}$, which admit a lift (with a pole of order at most 1 at 0) to the conormal bundle of M. The conormal bundle can be seen as a real $2n+2$-dimensional submanifod of $\mathbb{C}^{2n+2}$ and, as it turns out, it is totally real if M is Levi-nondegenerate [28]. Consequently, if M is Levi-nondegenerate the study of stationary discs falls into the framework developed in [13, 14, 12], and indeed the first author and L. Blanc-Centi employed this method to construct stationary discs in [4], and used it to show finite determination of automorphisms.

If the Levi form degenerates at some points, the conormal bundle admits complex tangencies, and therefore the attachment of discs is more complicated. We shall overcome this difficulty by constructing an associated circle bundle $\mathcal{N}^k S_P$ (a bundle over $S^1 \times S_P$ whose fiber at (ζ, p) is $\zeta^k N_p S_P$) whose CR singularities allows for attaching discs which pass through the singularity with certain predescribed orders. Geometrically, one can think of this construction as allowing a higher winding of the conormal part of the disc (k instead of 1 in the case of a classical stationary disc). Our theorem on the existence of discs is now as follows.

Theorem 1.4. *If M is an admissible hypersurface, then there exists a $k_0 \in \mathbb{N}$ and a finite dimensional manifold of (small) k_0-stationary discs attached to M.*

Our approach is based on the Riemann-Hilbert problem which as we already pointed out is singular in our situation. The approach described above will allow that the problem can be studied with tools of [6].

We note that for $n = 2$, we recover the results in [5]. However let us stress that we cannot generalize the methods used in [5], where the results are achieved by using a rather "ad hoc" procedure. So in this paper we develop methods which allow to treat the more general setting, and which we in addition believe to be more geometric.

The paper is organized as follows. In Section 2, we describe the needed preliminaries. Section 3 is devoted to weighted homogeneous model hypersurfaces. In Section 4, we study the existence of generalized stationary discs attached to admissible hypersurfaces. Finally, Section 5 is devoted to the proof of the finite jet determination theorems for CR diffeomorphisms.

2. Preliminaries

In this section, we collect some standard notation and collect facts which we will need throughout the paper. We denote by Δ the unit disc in $\mathbb{C}$ and by $b\Delta$ its boundary. We use coordinates $(z, w) \in \mathbb{C}^{n+1}$, where $z = (z_1, \ldots, z_n)$ are the standard coordinates in $\mathbb{C}^n$.

2.1. **Function spaces.** Let k be an integer and let $0 < \alpha < 1$. We write $\mathcal{C}^{k,\alpha} = \mathcal{C}^{k,\alpha}(b\Delta, \mathbb{R})$ for the space of real-valued functions defined on $b\Delta$ of class $C^{k,\alpha}$. We equip the space $\mathcal{C}^{k,\alpha}$ with its usual norm

$$\|v\|_{\mathcal{C}^{k,\alpha}} = \sum_{j=0}^{k} \|v^{(j)}\|_\infty + \sup_{\zeta \neq \eta \in b\Delta} \frac{\|v^{(k)}(\zeta) - v^{(k)}(\eta)\|}{|\zeta - \eta|^\alpha}$$

where $\|v^{(j)}\|_\infty = \max\limits_{b\Delta} \|v^{(j)}\|$.

We define $\mathcal{C}^{k,\alpha}_{\mathbb{C}} = \mathcal{C}^{k,\alpha} + i\mathcal{C}^{k,\alpha} = \mathcal{C}^{k,\alpha}(b\Delta, \mathbb{C})$. Therefore $v \in \mathcal{C}^{k,\alpha}_{\mathbb{C}}$ if and only if $\operatorname{Re} v, \operatorname{Im} v \in \mathcal{C}^{k,\alpha}$. We endow the space $\mathcal{C}^{k,\alpha}_{\mathbb{C}}$ with the norm

$$\|v\|_{\mathcal{C}^{k,\alpha}_{\mathbb{C}}} = \|\operatorname{Re} v\|_{\mathcal{C}^{k,\alpha}} + \|\operatorname{Im} v\|_{\mathcal{C}^{k,\alpha}}.$$

We denote by $\mathcal{A}^{k,\alpha}$ the subspace of $\mathcal{C}^{k,\alpha}_{\mathbb{C}}$ of functions f which possess a continuous extension $F : \overline{\Delta} \to \mathbb{C}$, with F holomorphic on Δ.

Let m be an integer. We denote by $\mathcal{A}^{k,\alpha}_{0m}$ the subspace of $\mathcal{C}^{k,\alpha}_{\mathbb{C}}$ of functions that can be written as $(1-\zeta)^m f$, with $f \in \mathcal{A}^{k,\alpha}$. Note that since $\mathcal{A}^{k,\alpha}_{0m}$ is not a closed subspace of $\mathcal{C}^{k,\alpha}_{\mathbb{C}}$, it is not a Banach space with the induced norm. Instead, we equip $\mathcal{A}^{k,\alpha}_{0m}$ with the following norm

$$\|(1-\zeta)^m f\|_{\mathcal{A}^{k,\alpha}_{0m}} = \|f\|_{\mathcal{C}^{k,\alpha}_{\mathbb{C}}} \tag{2.1}$$

which makes it a Banach space, isomorphic to $\mathcal{A}^{k,\alpha}$. Note that the inclusion of $\mathcal{A}^{k,\alpha}_{0m}$ in $\mathcal{A}^{k,\alpha}$ is a bounded operator.

Finally, we denote by $\mathcal{C}^{k,\alpha}_{0^m}$ the subspace of $\mathcal{C}^{k,\alpha}$ of functions that can be written as $(1-\zeta)^m v$ with $v \in \mathcal{C}^{k,\alpha}_{\mathbb{C}}$. The space $\mathcal{C}^{k,\alpha}_{0^m}$ is equipped with the norm

$$\|(1-\zeta)^m f\|_{\mathcal{C}^{k,\alpha}_{0^m}} = \|f\|_{\mathcal{C}^{k,\alpha}_{\mathbb{C}}}.$$

Notice that $\mathcal{C}^{k,\alpha}_{0^m}$ is a Banach space. Denote by τ_m the map $\mathcal{C}^{k,\alpha}_{0^m} \to \mathcal{C}^{k,\alpha}_{\mathbb{C}}$ given by $\tau_m((1-\zeta)^m v) = v$. We recall the following lemma from [6]:

Lemma 2.1. *Define the closed subspace $\mathcal{R}_m$ of $\mathcal{C}^{k,\alpha}_{\mathbb{C}}$ by*

$$\mathcal{R}_m = \{v \in \mathcal{C}^{k,\alpha}_{\mathbb{C}} \mid v(\zeta) = (-1)^m \zeta^{-m} \overline{v(\zeta)} \ \forall\, \zeta \in b\Delta\}.$$

Then

(i.) *τ_m maps $\mathcal{C}^{k,\alpha}_{0^m}$ isomorphically to $\mathcal{R}_m$;*

(ii.) *if $m = 2m'$ is even, the map $v \mapsto \zeta^{m'} v$ induces an isomorphism between $\mathcal{R}_m$ and $\mathcal{R}_0 = \mathcal{C}^{k,\alpha}$;*

(iii.) *if $m = 2m'+1$ is odd, the map $v \mapsto \zeta^{m'} v$ induces an isomorphism between $\mathcal{R}_m$ and $\mathcal{R}_1$.*

2.2. **Partial indices and Maslov index.** We denote by $\mathsf{GL}_N(\mathbb{C})$ the general linear group on $\mathbb{C}^N$. Let $G : b\Delta \to \mathsf{GL}_N(\mathbb{C})$ be a smooth map. We consider a Birkhoff factorization (see [27]) of $-\overline{G}^{-1} G$:

$$-\overline{G(\zeta)}^{-1} G(\zeta) = B^+(\zeta) \begin{pmatrix} \zeta^{\kappa_1} & & & (0) \\ & \zeta^{\kappa_2} & & \\ & & \ddots & \\ (0) & & & \zeta^{\kappa_N} \end{pmatrix} B^-(\zeta) \quad \text{for all } \zeta \in b\Delta$$

where $B^+ \colon \bar{\Delta} \to \mathsf{GL}_N(\mathbb{C})$ and $B^- \colon (\mathbb{C} \cup \infty) \setminus \Delta \to \mathsf{GL}_N(\mathbb{C})$ are smooth maps, holomorphic on Δ and $\mathbb{C} \setminus \overline{\Delta}$ respectively. The integers $\kappa_1, \ldots, \kappa_N$ are called the *partial indices* of $-\overline{G}^{-1} G$ and their sum $\kappa := \sum_{j=1}^N \kappa_j$ is called the *Maslov index* of $-\overline{G}^{-1} G$ and it is equal to the winding number of the map $\zeta \mapsto \det\left(-\overline{G(\zeta)}^{-1} G(\zeta)\right)$ around the origin.

2.3. k_0**-stationary discs.** Let $S = \{r = 0\}$ be a finitely smooth hypersurface defined in a neighborhood of the origin in $\mathbb{C}^{n+1}$. Let k, k_0 be integers and let $0 < \alpha < 1$. We recall that a holomorphic disc $f \in (\mathcal{A}^{k,\alpha})^{n+1}$ is *attached* to S if $f(\zeta) \in S$ for all $\zeta \in b\Delta$. The following definition was given in [5]:

Definition 2.2. A holomorphic disc $f \in (\mathcal{A}^{k,\alpha})^{n+1}$ attached to $S = \{r = 0\}$ is said to be k_0-stationary if there exists a continuous function $c \colon b\Delta \to \mathbb{R} \setminus \{0\}$ such that the map $\zeta \mapsto \zeta^{k_0} c(\zeta) \partial r(f(\zeta))$, defined on $b\Delta$, extends as a map in $(\mathcal{A}^{k,\alpha})^{n+1}$.

The set of such discs is invariant under CR diffeomorphisms.

Proposition 2.3. *Let $S \subset \mathbb{C}^{n+1}$ be a finitely smooth real hypersurface containing 0. There exists a neighborhood U of the origin in $\mathbb{C}^{n+1}$ such that if H is a CR diffeomorphism of class $\mathcal{C}^{k+1}$ sending $S \cap U$ to a real hypersurface $S' \subset \mathbb{C}^{n+1}$ and $f \colon \Delta \to U$ is a k_0-stationary disc in $(\mathcal{A}^{k,\alpha})^{n+1}$ attached to S then the disc $H \circ f$ extends as a k_0-stationary disc in $(\mathcal{A}^{k,\alpha})^{n+1}$ attached to S'.*

Proof. Using Theorem 6.2.2 in [1], we write $W = \bigcup \varphi\left(\overline{\Delta}\right)$ where the union is taken over all analytic discs f attached to S. The CR diffeomorphism H of class $\mathcal{C}^{k+1}$ admits a local holomorphic extension $\widetilde{H}$ to W continuous up to $S \cap W$. The image of any k_0-stationary disc $f \in (\mathcal{A}^{k,\alpha})^{n+1}$ attached to S is contained in W. The map $H \circ f \in \mathcal{C}^{k,\alpha}_{\mathbb{C}}$ defined on $\partial\Delta$ therefore extends as $\widetilde{H} \circ f \in (\mathcal{A}^{k,\alpha})^{n+1}$. The rest of the proof is the same computation as given in [5, Proposition 2.5]. $\square$

In our context, the following geometric version of Definition 2.2 is more convenient to work with.

Definition 2.4. A holomorphic disc $f \in (\mathcal{A}^{k,\alpha})^{n+1}$ attached to $S = \{r = 0\}$ is k_0-stationary if there exists a holomorphic lift $\boldsymbol{f} = (f, \tilde{f})$ of f to the cotangent bundle $T^*\mathbb{C}^{n+1}$, continuous up to the boundary and such that for all $\zeta \in b\Delta$, $\boldsymbol{f}(\zeta) \in \mathcal{N}^{k_0}S(\zeta)$ where

$$\mathcal{N}^{k_0}S(\zeta) := \{(z, w, \tilde{z}, \tilde{w}) \in T^*\mathbb{C}^{n+1} \mid (z, w) \in S, (\tilde{z}, \tilde{w}) \in \zeta^{k_0} N^*_z S \setminus \{0\}\}, \tag{2.2}$$

and where $N^*_z S = \operatorname{span}_{\mathbb{R}}\{\partial r(z)\}$ is the conormal fiber at z of the hypersurface S.

Indeed, one can consider k_0-stationary discs as sections of the circle bundle $\mathcal{N}^{k_0}S = \{(\zeta, \xi) \colon \xi \in \mathcal{N}^{k_0}S(\zeta)\} \subset S^1 \times \mathbb{C}^{2n+2}$. For a Levi-nondegenerate hypersurface S, this turns out to be totally real.

We are interested in constructing k_0-stationary discs for Levi-degenerate hypersurfaces. Notice that in such a situation, the submanifold $\mathcal{N}^{k_0}S(\zeta)$ is not totally real for all $\zeta \in b\Delta$. In fact we are precisely interested in discs passing through the degeneracy locus of $\mathcal{N}^{k_0}S$. For this purpose, we will restrict our attention to discs which satisfy certain pointwise constraints.

3. The model situation

3.1. **Weighted polynomial models.** A (real) polynomial $P : \mathbb{C}^n \to \mathbb{C}$ is *weighted homogeneous of weight* $M = (m_1, \cdots, m_n) \in \mathbb{N}^n$ *and (weighted) degree* $d \in \mathbb{N}$ if for any real number t and $z \in \mathbb{C}^n$ we have

$$P(t^{m_1}z_1, \cdots, t^{m_n}z_n, t^{m_1}\bar{z}_1, \cdots, t^{m_n}\bar{z}_n) = t^d P(z, \bar{z}).$$

With the abbreviated notation $t^M z = (t^{m_1}z_1, \cdots, t^{m_n}z_n)$, the condition can be written as $P(t^M z, t^M \bar{z}) = t^d P(z, \bar{z})$. Note that a weighted homogeneous polynomial P of weight $(1, \cdots, 1)$ is homogeneous. We shall encounter circumstances in which it is more convenient to assume that $m_1, \cdots, m_n$ are all even; since the actual size of the weights $(m_1, \ldots, m_n)$ is often not so important, we shall assume most of the time that we work with such an "even" weight system. We notice that in the case of an even weight system all linear combinations of weights and also of possible homogeneities are even; in particular, the numbers d and $d - m_i$ and $d - m_i - m_j$ for $1 \le i, j \le n$, are even.

For two multi-indices $M = (m_1, \cdots, m_n)$ and $J = (j_1, \cdots, j_n)$ we write

$$M \cdot J = \sum_{i=1}^{n} m_i j_i.$$

We now fix a weight (vector) $M = (m_1, \cdots, m_n)$ and a real-valued, weighted homogeneous polynomial P of (weighted) degree d, written as

$$P(z, \bar{z}) = \sum_{\substack{M\cdot J + M\cdot K = d \\ d-k_0 \le M\cdot J \le k_0}} \alpha_{JK} z^J \bar{z}^K = \sum_{\ell = d-k_0}^{k_0} \underbrace{\left(\sum_{\substack{M\cdot J + M\cdot K = d \\ M\cdot K = \ell}} \alpha_{JK} z^J \bar{z}^K \right)}_{:= P^{d-\ell,\ell}(z,\bar{z})} \tag{3.1}$$

where k_0 is the largest k with $\dfrac{d}{2} \le k \le d-1$ for which there exists two multi-indices $\tilde J, \tilde K$ with $M \cdot \tilde K = k$ satisfying $\alpha_{\tilde J \tilde K} \ne 0$. The $P^{d-\ell,\ell}$ are the "bihomogeneous" components of P, satisfying $P^{d-\ell,\ell}(t^M z, s^M \bar z) = t^{d-\ell} s^\ell P^{d-\ell,\ell}(z,\bar z)$. Since P is assumed to be real-valued, we have that $\alpha_{JK} = \overline{\alpha}_{KJ}$ for all multi-indices J, K, and also, that $P^{d-\ell,\ell}(z,\bar z) = \bar P^{\ell,d-\ell}(\bar z, z)$. We define the *model hypersurface* $S_P = \{\rho = 0\} \subset \mathbb{C}^{n+1}$ where

$$\rho(z,w) = -\operatorname{Re} w + P(z,\overline{z}) = -\operatorname{Re} w + \sum_{\substack{M\cdot J + M\cdot K = d \\ d-k_0 \le M\cdot J \le k_0}} \alpha_{JK} z^J \overline{z}^K. \tag{3.2}$$

Define for $v = (v_1, \cdots, v_n) \in \mathbb{C}^n$ the analytic disc $h^v : \Delta \to \mathbb{C}^n$

$$h^v(\zeta) = (1-\zeta)^M v = ((1-\zeta)^{m_1} v_1, (1-\zeta)^{m_2} v_2, \dots, (1-\zeta)^{m_n} v_n).$$

In analogy with the case of hypersurfaces in $\mathbb{C}^2$ [5], we will need to control the Levi form of S_P along the boundary of h^v,

$$P_{z\overline{z}}(h^v(\zeta), \overline{h^v(\zeta)}) = \begin{pmatrix} P_{z_1\overline{z}_1}(h^v(\zeta), \overline{h^v(\zeta)}) & \cdots & P_{z_1\overline{z}_n}(h^v(\zeta), \overline{h^v(\zeta)}) \\ \vdots & \ddots & \vdots \\ P_{z_n\overline{z}_1}(h^v(\zeta), \overline{h^v(\zeta)}) & \cdots & P_{z_n\overline{z}_n}(h^v(\zeta), \overline{h^v(\zeta)}) \end{pmatrix}.$$

For $\zeta \in b\Delta$ we have

$$\begin{aligned}
\zeta^{k_0} P_{z_i\overline{z}_j}(h^v(\zeta), \overline{h^v(\zeta)}) &= \sum_{\ell=d-k_0}^{k_0} P^{d-\ell,\ell}_{z_i\overline{z}_j}((1-\zeta)^M v, (1-\bar\zeta)^M \bar v) \\
&= \sum_{\ell=d-k_0}^{k_0} (1-\zeta)^{d-\ell-m_i} (1-\bar\zeta)^{\ell-m_j} \zeta^{k_0} P^{d-\ell,\ell}_{z_i\overline{z}_j}(v,\bar v) \\
&= (1-\zeta)^{d-m_i-m_j} \sum_{\ell=d-k_0}^{k_0} (-1)^{\ell-m_j} \zeta^{k_0-\ell+m_j} P^{d-\ell,\ell}_{z_i\overline{z}_j}(v,\bar v) \\
&= (1-\zeta)^{d-m_i-m_j} \zeta^{k_0} P_{z_i\overline{z}_j}(v, (-\bar\zeta)^M \bar v).
\end{aligned}$$

and with a similar computation for the $P_{z_iz_j}$ derivatives, we can thus write

$$\begin{cases} \zeta^{k_0} P_{z_i\overline{z}_j}(h^v(\zeta), \overline{h^v(\zeta)}) = (1-\zeta)^{d-m_i-m_j} Q^v_{i\overline{j}}(\zeta) \\ \\ \zeta^{k_0} P_{z_iz_j}(h^v(\zeta), \overline{h^v(\zeta)}) = (1-\zeta)^{d-m_i-m_j} S^v_{ij}(\zeta) \end{cases}$$

where $Q^v_{i\overline{j}}$ and S^v_{ij} are holomorphic polynomials, and where each $Q^v_{i\overline{j}}$ has degree at most $2k_0 - d + m_j$ and each S^v_{ij} has degree at most $2k_0 - d$. Furthermore, each $Q^v_{i\overline{j}}$ is divisible by ζ^{m_j}; this observation will turn out to be crucial in the proof of our main result. Our crucial assumption is now that not only does h^v only pass through Levi-nondegenerate points for $\zeta \ne 1$, but also, that the Levi form of S_P along h^v has the generic order of vanishing at 1 (so that the order of vanishing of the Levi form is going to stay constant under small perturbations of both P and v). To be exact:

Definition 3.1. We say that v is *admissible for P* if there exists g^v such that for $f^v = (h^v, g^v)$ we have that $f^v(\partial\Delta) \subset S_P$, but $f^v(\Delta) \not\subset S_P$ and if for $\zeta \in b\Delta$

$$Q^v(\zeta) = \det \begin{pmatrix} Q_{1\bar{1}}(\zeta) & \dots & Q_{1\bar{n}}(\zeta) \\ \vdots & \ddots & \vdots \\ Q_{n\bar{1}}(\zeta) & \dots & Q_{n\bar{n}}(\zeta) \end{pmatrix} \neq 0. \tag{3.3}$$

We also note that for a generic P, $Q(\zeta)$ has exactly degree $n(2k_0 - d) + \sum_{i=1}^n m_i$. Under generic conditions, we do find admissible vectors:

Lemma 3.2. *Assume that S_P is generically Levi-nondegenerate, and that the set of Levi-degenerate points $\Sigma_P = \{(z, w) \in S_P \colon \det P_{z_i\bar{z}_j}(z, \bar{z}) = 0\}$ does not have any branches of dimension $2n-1$ near 0. Then there exists an admissible vector v for P.*

Proof. We first claim that for an open, dense subset of v's, we have that their associated Q^v vanishes only at 1. Since

$$\begin{aligned}(1-\zeta)^{nd-2|M|} Q^v(\zeta) &= \zeta^{nk_0} \det \begin{pmatrix} P_{z_1\bar{z}_1}(h^v(\zeta), \overline{h^v(\zeta)}) & \dots & P_{z_1\bar{z}_n}(h^v(\zeta), \overline{h^v(\zeta)}) \\ \vdots & & \vdots \\ P_{z_n\bar{z}_1}(h^v(\zeta), \overline{h^v(\zeta)}) & \dots & P_{z_n\bar{z}_n}(h^v(\zeta), \overline{h^v(\zeta)}) \end{pmatrix} \\ &=: \zeta^{nk_0} D^v(\zeta, \bar{\zeta}),\end{aligned}$$

the zeroes of Q^v for $\zeta \neq 1$ are exactly those points $\zeta \in \partial\Delta$ for which $(h^v(\zeta), \operatorname{Re} P(h^v(\zeta), \overline{h^v(\zeta)}) \in \Sigma_P$ is a Levi-degenerate point. Indeed, assume on the contrary that there exists an open set of v's each of which has a $\zeta = \zeta_v$ with $D(\zeta_v) = 0$. Passing to a smooth point of the real algebraic variety Σ_P we see that therefore its dimension would need to be at least $2n - 1$, which is excluded by assumption.

We next study the behaviour of Q^v at 1 and claim that for v which satisfy that $D^v(1, 1) \neq 0$ we have that $Q^v(1) \neq 0$. In order to see this, we replace the variable $\zeta \in \Delta$ with a variable t in the upper half plane by the coordinate change $\zeta = \frac{i-t}{i+t}$. We then have that $(1 - \zeta) = 2t + O(t^2)$, and the boundary $\partial\Delta$ corresponds to $\mathbb{R}$, so that

$$\begin{aligned} D^v(\zeta, \bar{\zeta}) &= D^v(2t + O(t^2), 2t + O(t^2)) \\ &= (2t)^{nd-2|M|} D^v(1, 1) + O(t^{nd-2|M|+1}). \end{aligned}$$

It follows that $Q^v(1) = D^v(1, 1) \neq 0$. The set of all v's for which $D^v(1, 1) \neq 0$ is by assumption open and dense.

Lastly, we claim that the set of vectors v for which $h^v(\Delta) \not\subset S_P$ is contained in the set of v's for which $P(v, \bar{v}) \neq 0$. The Lemma follows with that claim: Admissible vectors lie in the intersection of the three dense, open sets we have discussed. So assume that $f^v(\Delta) \subset S_P$. Then it is easy to see that $g^v(\zeta) = 0$ for $\zeta \in \Delta$. Hence $P((1 - \zeta)^M v, (1 - \bar{\zeta})^M \bar{v}) = 0$ throughout Δ and therefore $P(v, \bar{v}) = 0$. ☐

In particular, the disc f^0

$$f^0 = (h^v, g^0) = ((1 - \zeta)^{m_1} v_1, \dots, (1 - \zeta)^{m_n} v_n), g^0) \tag{3.4}$$

is a k_0-stationary disc attached to S_P and satisfies $f^0(1) = 0$. We shall henceforth use f^0 to denote a (fixed) k_0 stationary disc associated with an admissible v.

4. Construction of k_0-stationary discs

In this section, we aim to construct k_0-stationary discs for suitable deformations of the model hypersurface studied in Section 3. To this end, we first define a space X parametrizing allowed deformations.

4.1. Space of allowed deformations.

Let $S_P = \{\rho = 0\}$ be a weighted polynomial model of the form (3.2). Let $k > 0$ be an integer. Choose $\delta > 0$ large enough so that $f^0\left(\overline{\Delta}\right)$, for f^0 defined in (3.4), is contained in the polydisc $\delta\Delta^{n+1} \subset \mathbb{C}^{n+1}$. Following [5], we consider the affine Banach space X of functions $r \in \mathcal{C}^{k+3}\left(\overline{\delta\Delta^{n+1}}\right)$ which can be written as

$$r(z,w) = \rho(z,w) + \theta(z, \operatorname{Im} w)$$

with
(4.1)

$$\theta(z,\operatorname{Im} w) = \sum_{M\cdot J+M\cdot K=d+1} (z^J\overline{z}^K)\cdot r_{JK0}(z) + \sum_{l=1}^{d}\sum_{M\cdot J+M\cdot K=d-l} z^J\overline{z}^K(\operatorname{Im} w)^l\cdot r_{JKl}(z,\operatorname{Im} w)$$

where $r_{JK0} \in \mathcal{C}^{k+3}_{\mathbb{C}}\left(\overline{\delta\Delta^n}\right)$ and $r_{JKl} \in \mathcal{C}^{k+3}_{\mathbb{C}}\left(\overline{\delta\Delta^n}\times[-\delta,\delta]\right)$. Furthermore, we equip X with the following norm

$$\|r\|_X = \sup \|r_{JKl}\|_{\mathcal{C}^{k+3}}$$

so that X is isomorphic to a real closed subspace of a suitable power of $\mathcal{C}^{k+3}_{\mathbb{C}}\left(\overline{\delta\Delta^n}\times[-\delta,\delta]\right)$ and, hence is a Banach space.

Remark 4.1. Equivalently, a defining function r (of class $\mathcal{C}^{k+3}$) is an allowed deformation of S_P if and only if

$$r_{z^J\overline{z}^K s^\ell}(0) = \begin{cases} J!K!\alpha_{J,K} & M(J+K)=d,\ \ell=0\\ 0 & M(J+K)+\ell<d.\end{cases}$$

One can show that these conditions are independent of the choice of suitably adapted holomorphic coordinates (actually, they are independent with respect to CR diffeomorphisms of class C^ℓ whose linear parts preserve weights, for ℓ large enough), and hence, that the definition of "allowed deformation" actually gives rise to a well-defined class of real hypersurfaces, independent of the coordinates used.

4.2. Defining equations of $\mathcal{N}^{k_0}S$ and singular Riemann-Hilbert problems.

Let

$$S_P = \{\rho = 0\} = \{-\operatorname{Re} w + P(z,\overline{z}) = 0\} \subset \mathbb{C}^{n+1}$$

be a weighted model hypersurface of the form (3.2). For $\zeta \in b\Delta$, the submanifold $\mathcal{N}^{k_0}S_P(\zeta) \subset \mathbb{C}^{2n+2}$ (see (2.2)) may be defined by $2n+2$ explicit real equations.

Indeed, we have

$$(z,w,\tilde z,\tilde w)\in \mathcal{N}^{k_0}S_P(\zeta) \;\Leftrightarrow\; \begin{cases} \rho(z,w)=0 \\ \text{there exists } \ c: b\Delta \to \mathbb{R}\setminus\{0\} \text{ such that} \\ (\tilde z,\tilde w)=\zeta^{k_0}c(\zeta)\left(P_z(z,\overline{z}),-\frac{1}{2}\right) \end{cases}$$

$$\Leftrightarrow\; \begin{cases} \rho(z,w)=0 \\ \dfrac{\tilde w}{\zeta^{k_0}}\in\mathbb{R} \\ \tilde z_i+2\tilde w P_{z_i}(z,\overline{z})=0 \ \text{ for } 1\le i\le n. \end{cases}$$

It follows that a set of $2n+2$ real defining equations for the submanifold $\mathcal{N}^{k_0}S_P(\zeta)\subset\mathbb{C}^{2n+2}$ is given by

$$\begin{cases} \tilde\rho_1(\zeta)(z,w,\tilde z,\tilde w) = -\mathrm{Re}\, w+P(z,\overline{z})=0 \\ \tilde\rho_2(\zeta)(z,w,\tilde z,\tilde w) = (\tilde z_1+2\tilde w P_{z_1}(z,\overline{z}))+\left(\overline{\tilde z_1+2\tilde w P_{z_1}(z,\overline{z})}\right)=0 \\ \tilde\rho_3(\zeta)(z,w,\tilde z,\tilde w) = i\,(\tilde z_1+2\tilde w P_{z_1}(z,\overline{z}))-i\left(\overline{\tilde z_1+2\tilde w P_{z_1}(z,\overline{z})}\right)=0 \\ \qquad\vdots \\ \tilde\rho_{2n}(\zeta)(z,w,\tilde z,\tilde w) = (\tilde z_n+2\tilde w P_{z_n}(z,\overline{z}))+\left(\overline{\tilde z_n+2\tilde w P_{z_n}(z,\overline{z})}\right)=0 \\ \tilde\rho_{2n+1}(\zeta)(z,w,\tilde z,\tilde w) = i\,(\tilde z_n+2\tilde w P_{z_n}(z,\overline{z}))-i\left(\overline{\tilde z_n+2\tilde w P_{z_n}(z,\overline{z})}\right)=0 \\ \tilde\rho_{2n+2}(\zeta)(z,w,\tilde z,\tilde w) = i\dfrac{\tilde w}{\zeta^{k_0}}-i\zeta^{k_0}\overline{\tilde w}=0. \end{cases}$$

We set

$$\tilde\rho := (\tilde\rho_1,\cdots,\tilde\rho_{2n+2}).$$

For a general hypersurface $S=\{r=0\}$ with $r\in X$ in the space of allowed deformations, we denote by $\tilde r(\zeta)$ the corresponding defining functions of $\mathcal{N}^{k_0}S(\zeta)$. This allows to consider lifts of stationary discs as solutions of a nonlinear Riemann-Hilbert type problem with singularities. More precisely, a holomorphic disc $\boldsymbol{f}\in\left(\mathcal{A}^{k,\alpha}\right)^{2n+2}$ is the lift of a k_0-stationary disc attached to S if and only if

$$\tilde r(\boldsymbol{f})=0 \ \text{ on } b\Delta. \tag{4.2}$$

The next section is devoted to the study of the nonlinear problem (4.2). Its linearization leads to a singular linear Riemann-Hilbert problem which can be treated with the techniques developed in [6].

4.3. **Construction of k_0-stationary discs.** Let

$$S_P=\{\rho=0\}=\{-\mathrm{Re}\, w+P(z,\overline{z})=0\}\subset\mathbb{C}^{n+1}$$

be a weighted model hypersurface of the form (3.2) with weight $M=(m_1,\cdots,m_n)$ and degree d. Let $v=(v_1,\cdots,v_n)$ be an admissible vector for P. Consider a real hypersurface $S=\{r=0\}$ with $r\in X$. We introduce the following space of maps

$$Y^{M,d}:=\prod_{i=1}^{n}\left(\mathcal{A}^{k,\alpha}_{0^{m_i}}\right)\times\mathcal{A}^{k,\alpha}_{0}\times\prod_{i=1}^{n}\left(\mathcal{A}^{k,\alpha}_{0^{d-m_i}}\right)\times\mathcal{A}^{k,\alpha} \tag{4.3}$$

endowed with the product norm defined in Equation (2.1). We denote by $\mathcal{S}^{k_0,r}$ the set of lifts $\boldsymbol{f} \in Y^{M,d}$ of k_0-stationary discs for the hypersurface $S = \{r = 0\}$. Following Section 3, we consider the initial k_0-stationary disc attached to S_P given by

$$\boldsymbol{f^0} = (h^0, g^0, \tilde{h}^0, \tilde{g}^0) = ((1-\zeta)^{m_1}v_1, \cdots, (1-\zeta)^{m_n}v_n, g^0, \tilde{h}^0, -\zeta^{k_0}/2) \in Y^{M,d}$$

where $\tilde{h}^0(\zeta) = \zeta^{k_0}P_z(h^0, \overline{h^0})$. We have:

Theorem 4.2. *Under the above assumptions, there exist an integer N, open neighborhoods V of ρ in X and U of 0 in $\mathbb{R}^N$, a real number $\eta > 0$ and a map*

$$\mathcal{F} : V \times U \to Y^{M,d}$$

of class $\mathcal{C}^1$ such that:

i. $\mathcal{F}(\rho, 0) = \boldsymbol{f^0}$,

ii. for all $r \in V$ the map

$$\mathcal{F}(r, \cdot) : U \to \{\boldsymbol{f} \in \mathcal{S}^{k_0,r} \;\; | \; \|\boldsymbol{f} - \boldsymbol{f^0}\|_{Y^{M,d}} < \eta\}$$

is one-to-one and onto.

Remark 4.3. In the proof of Theorem 4.2, we show that the dimension N is estimated above by $2(n+1)(k_0+1) + 2nk_0 - 2dn$. Since this dimension depends on the choice of the weights $(m_1, \cdots, m_n)$, a precise computation of N is not relevant for our approach.

Proof. In a neighborhood of $(\rho, \boldsymbol{f^0})$ in $X \times Y^{M,d}$, we define the following map between Banach spaces

$$\mathcal{H} : X \times Y^{M,d} \to \mathcal{C}_0^{k,\alpha} \times \prod_{i=1}^{n} \left(\left(\mathcal{C}^{k,\alpha}_{0^{d-m_i}} \right)^2 \right) \times \mathcal{C}^{k,\alpha}$$

by

$$\mathcal{H}(r, \boldsymbol{f}) := \tilde{r}(\boldsymbol{f}).$$

Here we use the notation

$$\tilde{r}(\boldsymbol{f})(\zeta) = \tilde{r}(\zeta)(\boldsymbol{f}(\zeta)).$$

It follows from the definition of the Banach spaces X and $Y^{M,d}$ that the map $\mathcal{H}$ is of class $\mathcal{C}^1$; the proof of this claim is analogous to the proof of Lemma 3.3 in [5] (see also Lemma 5.1 in [15] and Lemma 11.1 in [13]). Recall that a holomorphic disc $\boldsymbol{f} \in Y^{M,d}$ is the lift of a k_0-stationary disc attached to $S = \{r = 0\}$ if and only if it solves the nonlinear Riemann-Hilbert problem (4.2). In other words, for any fixed $r \in X$, the zero set of $\mathcal{H}(\tilde{r}, \cdot)$ coincides with $\mathcal{S}^{k_0,r}$. In order to show Theorem 4.2 we apply the implicit function theorem to the map $\mathcal{H}$. To this end, we need to consider the partial derivative of $\mathcal{H}$ with respect to $Y^{M,d}$ at $(\rho, \boldsymbol{f^0})$

$$\boldsymbol{f'} \mapsto 2\mathrm{Re}\left[\overline{G(\zeta)}\boldsymbol{f'}\right] \tag{4.4}$$

where the matrix

$$G(\zeta) := \left(\tilde{\rho}_{\overline{z}}(\boldsymbol{f^0}), \tilde{\rho}_{\overline{w}}(\boldsymbol{f^0}), \tilde{\rho}_{\overline{\tilde{z}}}(\boldsymbol{f^0}), \tilde{\rho}_{\overline{\tilde{w}}}(\boldsymbol{f^0})\right) \in M_{2n+2}(\mathbb{C})$$

has the following expression

$$G(\zeta)=\begin{pmatrix} P_{\bar z_1}(h^0,\overline{h^0}) & \dots & P_{\bar z_n}(h^0,\overline{h^0}) & -1/2 & 0 & \cdots & 0 & 0\\ & & & 0 & 1 & \ddots & 0 & 2\overline{P_{z_1}(h^0,\overline{h^0})}\\ & & & 0 & -i & \ddots & 0 & -2i\overline{P_{z_1}(h^0,\overline{h^0})}\\ & B(\zeta) & & 0 & 0 & \ddots & 0 & 2\overline{P_{z_2}(h^0,\overline{h^0})}\\ & & & \vdots & \vdots & \ddots & \vdots & \vdots\\ & & & 0 & 0 & \ddots & -i & -2i\overline{P_{z_n}(h^0,\overline{h^0})}\\ 0 & \cdots & 0 & 0 & 0 & \ddots & 0 & -i\zeta^{k_0}\end{pmatrix}.$$

Using the notation $d_{\ell j} := d - m_\ell - m_j$, the entries of the $2n\times n$ matrix $B(\zeta)$ are given by

$$B_{2\ell-1,j}(\zeta) = -(1-\zeta)^{d_{\ell j}}\left(Q_{\ell\bar j}(\zeta)+\frac{\overline{S}_{\ell j}(\zeta)}{\zeta^{d_{\ell j}}}\right)$$

for odd $1\le 2l-1\le 2n-1$ and

$$B_{2\ell,j}(\zeta) = -i(1-\zeta)^{d_{\ell j}}\left(Q_{\ell\bar j}(\zeta)-\frac{\overline{S}_{\ell j}(\zeta)}{\zeta^{d_{\ell j}}}\right)$$

for even $2\le 2\ell\le 2n$.

In order to apply the implicit function theorem, we need to study the kernel and surjectivity of the map $\boldsymbol{f'}\mapsto 2\mathrm{Re}\left[\overline{G(\zeta)}\boldsymbol{f'}\right]$. After permuting columns of $G(\zeta)$, we consider the following operator

$$L_1:\mathcal{A}_0^{k,\alpha}\times\prod_{i=1}^n\left(\mathcal{A}_{0^{d-m_i}}^{k,\alpha}\times\mathcal{A}_{0^{m_i}}^{k,\alpha}\right)\times\mathcal{A}^{k,\alpha}\to\mathcal{C}_0^{k,\alpha}\times\prod_{i=1}^n\left(\left(\mathcal{C}_{0^{d-m_i}}^{k,\alpha}\right)^2\right)\times\mathcal{C}^{k,\alpha}$$

given by

$$L_1(g',\tilde h_1',h_1',\cdots,\tilde h_n',h_n',\tilde g') := 2\mathrm{Re}\left[\overline{G_1(\zeta)}(g',\tilde h_1',h_1',\cdots,\tilde h_n',h_n',\tilde g')\right],$$

where

$$G_1(\zeta)=\begin{pmatrix} -1/2 & & (*)\\ & A(\zeta) & \\ (0) & & -i\zeta^{k_0}\end{pmatrix}$$

and where $A(\zeta)$ is

$$\begin{pmatrix} 1 & B_{1,1}(\zeta) & \dots & 0 & B_{1,n}(\zeta)\\ -i & B_{2,1}(\zeta) & \dots & 0 & B_{2,1}(\zeta)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & B_{2n-1,1}(\zeta) & \dots & 1 & B_{2n-1,n}(\zeta)\\ 0 & B_{2n,1}(\zeta) & \dots & -i & B_{2n,n}(\zeta)\end{pmatrix}.$$

The kernels of the differential map (4.4) and L_1 are of the same dimension, and the map (4.4) is onto if and only if L_1 is onto.

Note that since $G_1(1)$ is not invertible, the classical techniques developed in [12, 13, 14] to study the corresponding linear Riemann-Hilbert problem cannot be directly applied. Therefore the following step is crucial in our approach since it allows one to reduce a linear singular Riemann-Hilbert problem to a regular one with homogeneous pointwise constraints, and allows then the use of Theorem 2.1 in [6]. For $\varphi \in \mathcal{C}_0^{k,\alpha} \times \prod_{i=1}^n \left(\left(\mathcal{C}_{0^{d-m_i}}^{k,\alpha} \right)^2 \right) \times \mathcal{C}^{k,\alpha}$, we manipulate the linear system

$$2\mathrm{Re}\left[\overline{G_1(\zeta)}(g', \tilde{h}_1', h_1', \cdots, \tilde{h}_n', h_n', \tilde{g}')\right] = \varphi$$

in the following way. We divide the first line by $(1-\zeta)$ and the $(2\ell-1)^{th}$ and $(2\ell)^{th}$ lines by $(1-\zeta)^{d-m_\ell}$, for $l = 1, \cdots, n$. Following Lemma 2.1, we then multiply the $(2\ell-1)^{th}$ and $(2\ell)^{th}$ lines by ζ^{s_ℓ}, where $s_\ell := \dfrac{d-m_\ell}{2}$, $\ell = 1, \cdots, n$. The resulting linear operator

$$L_2 : (\mathcal{A}^{k,\alpha})^{2n+2} \to \mathcal{R}_1 \times (\mathcal{R}_0)^{2n} \times \mathcal{C}^{k,\alpha} \tag{4.5}$$

is equivalent to L_1 with respect to the properties we are interested in, namely its surjectivity and the description of its kernel. The new linear operator L_2, and its corresponding matrix G_2, are of the form considered in Theorem 2.1 and Theorem 2.2 [6]. We have thus reduced the problem to studying the linear operator

$$L_3 : (\mathcal{A}^{k,\alpha})^{2n} \to (\mathcal{R}_0)^{2n}$$

defined by

$$L_3(\tilde{h}_1', h_1', \cdots, \tilde{h}_n', h_n') := 2\mathrm{Re}\left[\overline{A(\zeta)}(\tilde{h}_1', -h_1', \cdots, \tilde{h}_n', -h_n')\right]$$

where the corresponding matrix, still denoted by $A(\zeta)$, is

$$\begin{pmatrix} \overline{\zeta}^{s_1} & Q_{1\bar{1}}\zeta^{s_1-m_1} + \overline{S}_{11}\overline{\zeta}^{s_1} & \dots & 0 & Q_{1\bar{n}}\zeta^{s_1-m_n} + \overline{S}_{1n}\overline{\zeta}^{s_1} \\ -i\overline{\zeta}^{s_1} & iQ_{1\bar{1}}\zeta^{s_1-m_1} - i\overline{S}_{11}\overline{\zeta}^{s_1} & \dots & 0 & iQ_{1\bar{n}}\zeta^{s_1-m_n} - i\overline{S}_{1n}\overline{\zeta}^{s_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & Q_{n\bar{1}}\zeta^{s_n-m_1} + \overline{S}_{n1}\overline{\zeta}^{s_n} & \dots & \overline{\zeta}^{s_n} & Q_{n\bar{n}}\zeta^{s_n-m_n} + \overline{S}_{nn}\overline{\zeta}^{s_n} \\ 0 & iQ_{n\bar{1}}\zeta^{s_n-m_1} - i\overline{S}_{n1}\overline{\zeta}^{s_n} & \dots & -i\overline{\zeta}^{s_n} & iQ_{n\bar{n}}\zeta^{s_n-m_n} - i\overline{S}_{nn}\overline{\zeta}^{s_n} \end{pmatrix}.$$

Out of convenience, we set $Q'_{\ell\bar{j}} = Q_{\ell\bar{j}}\zeta^{-m_j}$ and therefore

$$A(\zeta) = \begin{pmatrix} \overline{\zeta}^{s_1} & Q'_{1\bar{1}}\zeta^{s_1} + \overline{S}_{11}\overline{\zeta}^{s_1} & \dots & 0 & Q'_{1\bar{n}}\zeta^{s_1} + \overline{S}_{1n}\overline{\zeta}^{s_1} \\ -i\overline{\zeta}^{s_1} & iQ'_{1\bar{1}}\zeta^{s_1} - i\overline{S}_{11}\overline{\zeta}^{s_1} & \dots & 0 & iQ'_{1\bar{n}}\zeta^{s_1} - i\overline{S}_{1n}\overline{\zeta}^{s_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & Q'_{n\bar{1}}\zeta^{s_n} + \overline{S}_{n1}\overline{\zeta}^{s_n} & \dots & \overline{\zeta}^{s_n} & Q'_{n\bar{n}}\zeta^{s_n} + \overline{S}_{nn}\overline{\zeta}^{s_n} \\ 0 & iQ'_{n\bar{1}}\zeta^{s_n} - i\overline{S}_{n1}\overline{\zeta}^{s_n} & \dots & -i\overline{\zeta}^{s_n} & iQ'_{n\bar{n}}\zeta^{s_n} - i\overline{S}_{nn}\overline{\zeta}^{s_n} \end{pmatrix}.$$

Note that by manipulating rows of A one shows that

$$\det A(\zeta) = (2i)^n Q'(\zeta) \tag{4.6}$$

where

$$Q'(\zeta) = \zeta^{-(m_1+\cdots+m_n)}Q(\zeta).$$

Lemma 4.4. *The linear operator $L_3 : (\mathcal{A}^{k,\alpha})^{2n} \to (\mathcal{R}_0)^{2n}$ is onto.*

Proof of Lemma 4.4. According to Theorem 2.1 in [6] (with $m = 0$), we need to show that the partial indices of the matrix

$$\overline{A^{-1}(\zeta)}A(\zeta) = \frac{1}{\det A(\zeta)}A'(\zeta) = \frac{1}{(2i)^n Q'(\zeta)}A'(\zeta)$$

are greater than or equal to -1. For $1 \leq j, \ell \leq 2n$ we denote by $A'_{j\ell}$ the (j,ℓ)-entry of A'. A direct computation gives for $\ell, p = 1, \cdots, n$

$$A'_{2\ell-1,2p} = (-2i)^n \zeta^{s_1+\cdots+s_n-s_\ell} \det \begin{pmatrix} Q'_{l\overline{p}}\zeta^{s_\ell} & S_{l1}\zeta^{s_\ell} & S_{l2}\zeta^{s_\ell} & \cdots & S_{ln}\zeta^{s_\ell} \\ \overline{S}_{1p}\overline{\zeta}^{s_1} & \overline{Q}'_{1\overline{1}}\overline{\zeta}^{s_1} & \overline{Q}'_{1\overline{2}}\overline{\zeta}^{s_1} & \cdots & \overline{Q}'_{1\overline{n}}\overline{\zeta}^{s_1} \\ \overline{S}_{2p}\overline{\zeta}^{s_2} & \overline{Q}'_{2\overline{1}}\overline{\zeta}^{s_2} & \overline{Q}'_{2\overline{2}}\overline{\zeta}^{s_2} & \cdots & \overline{Q}'_{2\overline{n}}\overline{\zeta}^{s_2} \\ \vdots & \vdots & \vdots & \vdots\cdots & \vdots \\ \overline{S}_{np}\overline{\zeta}^{s_n} & \overline{Q}'_{n\overline{1}}\overline{\zeta}^{s_n} & \overline{Q}'_{n\overline{2}}\overline{\zeta}^{s_n} & \cdots & \overline{Q}'_{n\overline{n}}\overline{\zeta}^{s_n} \end{pmatrix}$$

$$= (-2i)^n \det \underbrace{\begin{pmatrix} Q'_{\ell\overline{p}} & S_{\ell 1} & S_{\ell 2} & \cdots & S_{\ell n} \\ \overline{S}_{1p} & \overline{Q}'_{1\overline{1}} & \overline{Q}'_{1\overline{2}} & \cdots & \overline{Q}'_{1\overline{n}} \\ \overline{S}_{2p} & \overline{Q}'_{2\overline{1}} & \overline{Q}'_{2\overline{2}} & \cdots & \overline{Q}'_{2\overline{n}} \\ \vdots & \vdots & \vdots & \vdots\cdots & \vdots \\ \overline{S}_{np} & \overline{Q}'_{n\overline{1}} & \overline{Q}'_{n\overline{2}} & \cdots & \overline{Q}'_{n\overline{n}} \end{pmatrix}}_{:=B_{2\ell-1,2p}}$$

$$= (-2i)^n a'_{2\ell-1,2p}$$

where $a'_{2\ell-1,2p} = \det B_{2\ell-1,2p}$. For a square matrix B, we write $C_{j\ell}(B)$ for its (j,ℓ)-cofactor. Notice that for all $j = 1, \cdots, n$ and any $p, p' = 1, \cdots, n$, we have

$$C_{j,1}(B_{2\ell-1,2p}) = C_{j1}(B_{2\ell-1,2p'}).$$

We denote this cofactor by $C_{j,1;\ell}$. We also have

$$C_{1,j}(B_{2\ell-1,2p}) = C_{1,j}(B_{2\ell'-1,2p})$$

for any $p = 1, \ldots, n$ and every $\ell, \ell' = 1, \ldots, n$ which will be denoted by $C^p_{1,j}$. A straightforward computation leads to

$$A'_{2\ell-1,2p-1} = (-2i)^n C_{p+1,1;\ell}$$

and

$$A'_{2\ell,2p} = (-2i)^n C^p_{1,\ell+1}$$

for $\ell, p = 1, \cdots, n$. Denote by $D_{\ell p}$ the $n \times n$ matrix obtained by removing the first row and $(\ell+1)^{\text{th}}$ column of $B_{2\ell-1,2p}$, namely

$$D_{\ell p} = \begin{pmatrix} \overline{S}_{1p} & \overline{Q}'_{1\overline{1}} & \overline{Q}'_{1\overline{2}} & \cdots & \overline{Q}'_{1\overline{\ell-1}} & \overline{Q}'_{1\overline{\ell+1}} & \cdots & \overline{Q}'_{1\overline{n}} \\ \overline{S}_{2p} & \overline{Q}'_{2\overline{1}} & \overline{Q}'_{2\overline{2}} & \cdots & \overline{Q}'_{2\overline{\ell-1}} & \overline{Q}'_{2\overline{\ell+1}} & \cdots & \overline{Q}'_{2\overline{n}} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \overline{S}_{np} & \overline{Q}'_{n\overline{1}} & \overline{Q}'_{n\overline{2}} & \cdots & \overline{Q}'_{n\overline{\ell-1}} & \overline{Q}'_{n\overline{\ell+1}} & \cdots & \overline{Q}'_{n\overline{n}} \end{pmatrix}$$

for $\ell, p = 1, \cdots, n$. Note that

$$\det(D_{\ell p}) = (-1)^\ell C^p_{1,\ell+1}$$

and

$$C_{j,1}(D_{\ell p}) = C_{j,1}(D_{\ell p'})$$

which we denote by $c_{j,1;l}$. A direct computation gives

$$A'_{2\ell,2p-1} = (-1)^{\ell+1}(-2i)^n c_{p,1;\ell}.$$

Therefore

$$
(4.7) \qquad \frac{A'(\zeta)}{(-2i)^n} = \begin{pmatrix}
C_{2,1;1} & a'_{1,2} & C_{3,1;1} & a'_{1,4} & \cdots & C_{n+1,1;1} & a'_{1,2n} \\
c_{1,1;1} & C^1_{1,2} & c_{2,1;1} & C^2_{1,2} & \cdots & c_{n,1;1} & C^n_{1,2} \\
C_{2,1;2} & a'_{3,2} & C_{3,1;2} & a'_{3,4} & \cdots & C_{n+1,1,2} & a'_{3,2n} \\
-c_{1,1;2} & C^1_{1,3} & -c_{2,1;2} & C^2_{1,3} & \cdots & -c_{n,1;2} & C^n_{1,3} \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
C_{2,1;n} & a'_{2n-1,2} & C_{3,1;n} & a'_{2n-1,4} & \cdots & C_{n+1,1;n} & a'_{2n-1,2n} \\
\frac{c_{11,n}}{(-1)^{n+1}} & C^1_{1,n+1} & \frac{c_{2,1;n}}{(-1)^{n+1}} & C^2_{1,n+1} & \cdots & \frac{c_{n,1;n}}{(-1)^{n+1}} & C^n_{1,n+1}
\end{pmatrix}.
$$

Denote by C_p the p^{th} column of $A'(\zeta)$. Notice that performing the following column operation

$$
(4.8) \qquad C_{2p} \to C_{2p} - \sum_{j=1}^{n} \overline{S}_{jp} C_{2j-1}
$$

for each $p = 1, \cdots, n$ transforms $A'(\zeta)$ into

$$
(4.9) \quad A'(\zeta) = (-2i)^n \begin{pmatrix}
C_{2,1;1} & Q'_{1\overline{1}}\overline{Q'} & C_{3,1;1} & Q'_{1\overline{2}}\overline{Q'} & \cdots & C_{n+1,1;1} & Q'_{l\overline{n}}\overline{Q'} \\
c_{1,1;2} & 0 & c_{2,1;2} & 0 & \cdots & c_{n,1;2} & 0 \\
C_{2,1;2} & Q'_{2\overline{1}}\overline{Q'} & C_{3,1;2} & Q'_{2\overline{2}}\overline{Q'} & \cdots & C_{n+1,1;2} & Q'_{2\overline{n}}\overline{Q'} \\
-c_{1,1;3} & 0 & -c_{2,1;3} & 0 & \cdots & -c_{n,1;3} & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
C_{2,1;n} & Q'_{n\overline{1}}\overline{Q'} & C_{3,1;n} & Q'_{n\overline{2}}\overline{Q'} & \cdots & C_{n+1,1;n} & Q'_{n\overline{n}}\overline{Q'} \\
c\frac{{}_{1,1;n}}{(-1)^{n+1}} & 0 & \frac{c_{2,1;n}}{(-1)^{n+1}} & 0 & \cdots & \frac{c_{n,1;n}}{(-1)^{n+1}} & 0
\end{pmatrix}.
$$

Now let $\kappa_1 \geq \ldots \geq \kappa_{2n}$ be the partial indices of $\overline{A^{-1}}A$, and let Λ be the diagonal matrix with entries $\zeta^{\kappa_1}, \ldots, \zeta^{\kappa_{2n}}$. According to Lemma 5.1 in [13] there exists a smooth map $\Theta : \overline{\Delta} \to \mathsf{GL}_{2n}(\mathbb{C})$, holomorphic on Δ, such that

$$
(4.10) \qquad \Theta\overline{A^{-1}}A = \Lambda\overline{\Theta}.
$$

Denote by $\lambda = (\lambda_1, \mu_1, \ldots, \lambda_n, \mu_n)$ the last row of the matrix Θ. Using (4.7) and (4.10), we get the following system:

$$
\left.\begin{aligned}
\sum_{k=1}^{n} C_{j+1,1;k}\lambda_k + \sum_{k=1}^{n}(-1)^{k+1}c_{j,1;k+1}\mu_k &= \overline{Q'}\zeta^{\kappa_{2n}}\overline{\lambda}_j \\
\sum_{k=1}^{n} a'_{2k-1,2j}\lambda_k + \sum_{k=1}^{n} C^j_{1,k+1}\mu_k &= \overline{Q}'\zeta^{\kappa_{2n}}\overline{\mu}_j
\end{aligned}\right\} j = 1, \ldots, n
$$

Performing operations (4.8), and considering only the lines of the system coming from the second line above, we obtain the following (see (4.9)):

$$
\overline{Q'}\sum_{k=1}^{n} Q'_{k\overline{j}}\lambda_k = \overline{Q'}\zeta^{\kappa_{2n}}\overline{\mu}_j - \sum_{k=1}^{n}\overline{S}_{kj}\overline{Q'}\zeta^{\kappa_{2n}}\overline{\lambda}_k, \quad j = 1, \ldots, n
$$

Dividing by $\overline{Q'}$, which by assumption (3.3) is non-vanishing for all $\zeta \in b\Delta$, we have

$$
\sum_{k=1}^{n} Q'_{k\overline{j}}\lambda_k = \zeta^{\kappa_{2n}}\overline{\mu}_j + \sum_{k=1}^{n}\overline{S}_{kj}\zeta^{\kappa_{2n}}\overline{\lambda}_k, \quad j = 1, \ldots, n.
$$

Recall that $Q_{i\bar{j}}$ is divisible by ζ^{m_j} (see Section 3.1) and thus $Q'_{i\bar{j}} = Q_{i\bar{j}}\zeta^{-m_j}$ is holomorphic. Now if $\kappa_{2n} \leq -1$, the right hand side of each one of the equations above is antiholomorphic (and divisible by $\overline{\zeta}$), while the left hand side is holomorphic. Thus they must both vanish, leading to the system

$$\sum_{k=1}^{n} Q'_{k\bar{j}}\lambda_k = 0, \quad j = 1, \ldots, n.$$

which implies that each λ_j vanishes identically since the determinant of the system is $Q' \neq 0$. From this we obtain immediately that each μ_j also vanishes identically. In summary, the arguments above show that either $\kappa_{2n} \geq 0$ or $\lambda_j = \mu_j = 0$ for all $j = 1, \cdots, n$. Since Θ is invertible, the latter would be a contradiction, hence we conclude that $\kappa_{2n} \geq 0$. This proves Lemma 4.4. □

Lemma 4.5. *The kernel of the linear operator $L_3 : (\mathcal{A}^{k,\alpha})^{2n} \to (\mathcal{R}_0)^{2n}$ has finite real dimension less than or equal to $2n(2k_0 - d) + 2n$.*

Proof of Lemma 4.5. According to Theorem 2.1 [6] (with $m = 0$), the dimension of $\ker L_3$ is equal to $\kappa + 2n$, where κ is the Maslov index of $\overline{A^{-1}}A$, namely

$$\operatorname{ind}\det\left(-\overline{A^{-1}}A\right) = \frac{1}{2i\pi}\int_{b\Delta}\frac{\left[\det\left(-\overline{A(\zeta)}^{-1}A(\zeta)\right)\right]'}{\det\left(-\overline{A(\zeta)}^{-1}A(\zeta)\right)}\mathrm{d}\zeta.$$

Using (4.6) we have

$$\det\overline{A^{-1}}A = (-1)^n\frac{Q'(\zeta)}{\overline{Q'(\zeta)}} = (-1)^n\zeta^{-2(m_1+\cdots+m_n)}\frac{Q(\zeta)}{\overline{Q(\zeta)}}.$$

Therefore

$$\begin{aligned}\operatorname{ind}\det\left(-\overline{A^{-1}}A\right) &= -2\sum_{i=1}^{n} m_i + 2\operatorname{ind}Q \\ &\leq -2\sum_{i=1}^{n} m_i + 2\left(n(2k_0 - d) + \sum_{i=1}^{n} m_i\right) = 2n(2k_0 - d).\end{aligned}$$

□

Finally, according to Lemma 4.4, Lemma 4.5 in the present paper and Theorem 2.2 in [6], the linear operator L_2 defined in (4.5) is onto and its kernel has finite real dimension N less than or equal to $2k_0 + 2n(2k_0 - d) + 2n + 2 = 2(n+1)(k_0+1) + 2nk_0 - 2dn$. This concludes the proof of Theorem 4.2. □

4.4. **The case of homogeneous hypersurfaces.** Consider now the case of a model hypersurface defined as $S_P = \{\rho = 0\} = \{-\mathrm{Re}\, w + P(z, \overline{z}) = 0\}$ with P a polynomial written as in (3.1) of even degree d and $m_1 = m_2 = \ldots = m_n = 1$, that is

$$P(z, \bar{z}) = \sum_{\substack{|J|+|K|=d \\ d-k_0\leq|J|\leq k_0}} \alpha_{JK} z^J \overline{z}^K.$$

In this situation we just say that S_P is a homogeneous (rather than weighted homogeneous) hypersurface. We will assume the existence of an admissible vector in the sense of Definition 3.1.

The method followed in the previous section for the proof of Theorem 4.2 does not apply directly to S_P. In particular, in order to define the operator L_2 in Equation (4.5) one needs the weights m_j to be even. However a slight modification of the procedure is possible: we apply the same rescaling as before to the system (i.e. we divide every line except the first and the last one by $(1-\zeta)^{d-m_j} = (1-\zeta)^{d-1}$) and then we multiply every line except the first and the last by ζ^s, where $s = \frac{d-2}{2}$. By Lemma 2.1 the resulting linear operator is of the kind

$$L_2 : (\mathcal{A}^{k,\alpha})^{2n+2} \to \mathcal{R}_1 \times (\mathcal{R}_1)^{2n} \times \mathcal{C}^{k,\alpha}$$

and the corresponding matrix G_2 is still of the form considered in Theorem 2.1 and Theorem 2.2 of [6]. The proofs of Lemmas 4.4 and 4.5 are essentially the same, and the estimate on the dimension of the kernel in Lemma 4.5 can be given as $2n(2k_0-d)$.

In fact, stronger assumptions on the geometry of S_P allow to be more precise on the dimension of the kernel, since it is possible in some cases to determine the Maslov index of Q exactly. For instance, the following assumption is analogous to the one considered in [5] for hypersurfaces of $\mathbb{C}^2$:

Lemma 4.6. *Suppose that the Levi form $P_{z\overline{z}}$ is positive definite outside of* 0. *Then the index of Q is $n(k_0 - \frac{d}{2} + 1)$.*

Proof. For any homogeneous polynomial $P(z,\overline{z})$ of degree d, denote by $Q_P(\zeta)$ the holomorphic polynomial obtained by applying the procedure of section 3.1 to P. For a small $\epsilon \geq 0$ we define P_ϵ as

$$P_\epsilon(z,\overline{z}) = |z_1|^d + \ldots + |z_n|^d + \epsilon \|z\|^d.$$

Note that the Levi form of P_ϵ is positive definite outside 0 if $\epsilon > 0$. One can compute directly that $Q_{P_0}(\zeta) = C\zeta^{n(k_0-\frac{d}{2}+1)}$ for a certain constant C, hence the index of $Q_{P_\epsilon}(\zeta)$ is equal to $n(k_0 - \frac{d}{2} + 1)$ for $\epsilon > 0$ small enough. On the other hand, the set of the homogeneous polynomials P of degree d such that $P_{z\overline{z}}$ is positive definite outside 0 is a connected (and indeed convex) subset of the space of the polynomials of degree d, and since $Q_P(\zeta)$ depends continuously on P it follows that its index is constant on this set. ☐

Following the proof of Lemma 4.5 we have that the dimension of $\ker L_3$ is given by the Maslov index of $\overline{A^{-1}}A$ (since Theorem 2.1 from [6] must be applied with $m = 1$), which in this case is just $2\mathrm{ind}Q = n(2k_0 - d + 2)$. Accordingly, the dimension N in Theorem 4.2 can be computed exactly as $2k_0+n(2k_0-d+2)+2 = 2(n+1)(k_0+1)-dn$, which is lower than the estimate given in the general case by roughly a factor of 2.

4.5. **The case of decoupled hypersurfaces.** For a subset $I = \{i_1, \cdots, i_l\} \subset \{1, \cdots, n\}$, we set $z_I = (z_{i_1}, \cdots, z_{i_l})$. Let $\{I_1, ..., I_k\}$ be a partition of $\{1, \cdots, n\}$. We consider a model hypersurface S_P of the form $S_P = \{\rho = 0\} \subset \mathbb{C}^{n+1}$, where

$$\rho(z,w) = -\mathrm{Re}\, w + P(z,\overline{z}) = -\mathrm{Re}\, w + \sum_{j=1}^{k} P_j(z_{I_j}, \overline{z_{I_j}})$$

where $P_j : \mathbb{C}^{|I_j|} \to \mathbb{C}$, $j = 1, \cdots k$, is a (real) weighted homogeneous polynomial of (vector) weight $M_j \in \mathbb{N}^{|I_j|}$ and (weighted) degree $d_j \in \mathbb{N}$ written as in (3.1). We denote by $k_0^1, \cdots k_0^k$ the corresponding integers. We assume that there exists an admissible vector v for P.

Consider a real smooth hypersurface $S = \{r = 0\} \subset \mathbb{C}^{n+1}$ allowed in the sense of Section 3.1, namely

$$r(z,w) = \rho(z,w) + \sum_{j=1}^{k} \theta_j(z_{I_j}, \operatorname{Im} w)$$

where θ_j, $j = 1, \cdots, k$ is of the form (4.1). In such case, following the proof of Theorem 4.2, the differential of the corresponding map $\mathcal{H}$ at $(\rho, \boldsymbol{f}^0)$ is block upper triangular after permutation of coordinates. In this case, the corresponding operator L_2 (see (4.5)) is of the form considered in Theorem 2.2 of [6]. Therefore if S is close enough to S_P in the sense Section 3.1 then Theorem 4.2 applies and provides a Banach manifold of stationary discs of real dimension at most $\sum_{j=1}^{k} 2(|I_j|+1)(k_0^j+1) - 2d_j|I_j|$.

Note that in principle such a model can be directly treated as a weighted homogeneous hypersurface by choosing different weights. However, in such case, the Banach manifold of stationary discs provided by Theorem 4.2 is of much greater dimension than the one obtained by considering the model as decoupled.

4.6. Construction of k_0-stationary discs for admissible hypersurfaces.

Definition 4.7. Let $S \subset \mathbb{C}^{n+1}$ be a finitely smooth real hypersurface through $0 \in \mathbb{C}^{n+1}$, and assume that $T_0^c S = \{w = 0\}$; write $w = u + iv$. We say that S is admissible if for a (sufficiently smooth) defining function (and hence for all sufficiently smooth defining functions) $r(z, \bar z, \operatorname{Re} w, \operatorname{Im} w)$ for S near 0 we have

$$r_u(0) = -1, \quad r_{z^J \bar z^K s^\ell}(0) = \begin{cases} J!K!\alpha_{J,K} & \ell = 0, M(J+K) = d \\ 0 & M(J+K) + \ell < d. \end{cases}$$

Equivalently, S is admissible if any defining function may be locally written as

$$r(z,w) = \rho(z,w) + O\left(|z|^{d+1}\right) + \operatorname{Im} w \; O\left(|z, \operatorname{Im} w|^{d-1}\right)$$

where $\rho(z,w) = -\operatorname{Re} w + P(z,\overline{z})$, and $P(z,\overline{z})$ is of the form (3.1) and admits an admissible vector.

We remark that the preceding definition is *independent of the choice of defining function*. It is also independent of the choice of holomorphic coordinates as long as the linear tangential part (the "z-part") preserves the weights. Being an admissible hypersurface is therefore a geometric concept.

Here $O\left(|z|^{d+1}\right)$ and $O\left(|z, \operatorname{Im} w|^{d-1}\right)$ are understood to be weighted orders where $z_j, \overline{z_j}$ and w have respective weights m_j, m_j and 1. The following lemma is obtained exactly as Lemma 5.2 in [5]:

Lemma 4.8. *Let $S \subset \mathbb{C}^{n+1}$ be an admissible real hypersurface of class $\mathcal{C}^{d+k+4}$. Consider the scaling $\Lambda_t(z,w) = (t^{m_1} z_1, \cdots, t^{m_n} z_n, t^d w)$. For $t > 0$ small enough, the defining function $r_t = \dfrac{1}{t^d} r \circ \Lambda_t$ belongs to the neighborhood V in Theorem 4.2.*

Our main result about existence of discs follows now directly from the previous lemma and Theorem 4.2.

Theorem 4.9. *Let $S \subset \mathbb{C}^{n+1}$ be an admissible real hypersurface of class $\mathcal{C}^{d+k+4}$. There exists a finitely dimensional biholomorphically invariant manifold of small k_0-stationary discs of class $\mathcal{C}^{k,\alpha}$ attached to S.*

Remark 4.10. In case S is admissible with P homogeneous or decoupled, the corresponding versions of Theorem 4.9 is sharper and provides a family of discs of smaller dimension. Note that for the decoupled case, the scaling Λ_t should be modified; more precisely, following notations of Section 4.5, for $i \in I_j$, the variable z_i must be scaled by $t^{m_i \Pi_{l\neq j} d_l}$.

5. Finite jet determination of CR maps

5.1. **Statement of the result.** The existence of k_0-stationary discs obtained in Theorem 4.9 allows us to obtain finite jet determination results for CR diffeomorphisms, generalizing the result from [5] to higher dimension.

Theorem 5.1. *Let $P(z,\bar z)$ be a weighted homogeneous polynomial, of degree d. Then there exists an integer $\ell_0 \leq 6nd$ such that the following holds. Let $S \subset \mathbb{C}^{n+1}$ be an admissible real hypersurface of class $\mathcal{C}^{d+\ell_0+4}$ through $0 \in \mathbb{C}^{n+1}$, with model S_P. If H is a germ of a CR diffeomorphism of class $\mathcal{C}^{\ell_0+1}$ of S satisfying $j_0^{\ell_0+1}H = I$, then $H = \mathrm{id}$.*

Theorem 5.1 implies immediately Theorem 1.2, which in conjunction with Lemma 3.2 implies Theorem 1.1. We will see how ℓ_0 can be chosen in Lemma 5.3. However, the intention of this paper is not to give optimal bounds on the jet order needed for determination. This can be done better by considering purely formal constructions.

Remark 5.2. Assume that a jet determination result of order k' holds in the formal setting, in the sense that every ℓ-jet of a formal biholomorphisms which preserves a formal hypersurface (up to the order ℓ) and is trivial up to order k' necessarily coincides with the ℓ-jet of the identity map. Then the conclusion of Theorem 5.1 holds for k'-jet determination as long as the smoothness of S is at least $\mathcal{C}^{\max\{k',d+\ell_0+4\}}$. Indeed, the (ℓ_0+1)-order Taylor expansion of H represents a (ℓ_0+1)-order biholomorphism jet which preserves the polynomial hypersurface induced by the Taylor expansion of S up to order (ℓ_0+1), thus if it is trivial up to order k' it must be trivial up to order ℓ_0+1: from the theorem it follows in turn that H is the identity.

It follows for instance that, for the version of Theorem 5.1 in $\mathbb{C}^2$ (see Theorem 1.2 in [5]), we can always achieve 2-jet determination of CR diffeomorphisms as in the real-analytic case (see [11, 20]). In higher dimension we can achieve the order of jet determination established in the formal setting, see for instance [23, 24] and for the model case [21].

The proof of Theorem 5.1 is achieved by putting together several facts, following the approach taken in [5]:

1. According to Proposition 2.3, the family of k_0-stationary discs is invariant under CR diffeomorphisms.
2. By Lemma 4.8, the pullback r_t of the local defining function r of S under a suitable scaling method Λ_t belongs to the neighborhood V in Theorem 4.2.
3. Similarly, the pullback $H_t = \Lambda_t^{-1} \circ H \circ \Lambda_t$ of the CR diffeomorphism H can be made arbitrarily close to the identity (in the $\mathcal{C}^1$-norm) for t small enough.
4. There exist an integer ℓ_0, such that the lifts of k_0-stationary discs attached to r_t and passing through 0 are determined by their ℓ_0-jet at 1.
5. The union of the images of k_0-stationary discs obtained in Theorem 4.2 is an open set of $\mathbb{C}^{n+1}$.

Similarly to Lemma 4.8 which is obtained exactly as Lemma 5.2 in [5], the point 3. is proved in the same way as Lemma 5.3 in [5]. To prove point 4., note that it is sufficient to show that the restriction of $\mathfrak{j}_{\ell_0}$ to the tangent space $T_{\boldsymbol{f^0}}\mathcal{S}^{k_0,\rho}$ of $\mathcal{S}^{k_0,\rho}$ at the point $\boldsymbol{f^0} = (f^0, \widetilde{f}^0)$ is injective: the statement then follows from Theorem 4.2. Recall that here $\mathcal{S}^{k_0,\rho}$ denotes the set of lifts $\boldsymbol{f} \in Y^{M,d}$ (see (4.3)) of k_0-stationary discs for the model hypersurface $\{\rho = 0\}$ (see Definition 4.7). Since by the implicit function theorem $T_{\boldsymbol{f^0}}\mathcal{S}^{k_0,\rho}$ is kernel of the operator $\boldsymbol{f}' \mapsto 2\mathrm{Re}\left[\overline{G(\zeta)}\boldsymbol{f}'\right]$ (see 4.4), the claim is a consequence of Lemma 5.3 proved in the next section. We will prove point 5. in Lemma 5.4.

Finally, the proof of Theorem 5.1 follows from the points above with the same argument as in Section 5.2 of [5]: the only difference is that one needs to apply the argument to the lift of H_t to the conormal bundle rather than to H_t itself, and this is achieved as in Section 4.2 [4].

5.2. **Injectivity of the jet map.** Let $\ell_0, m, N \in \mathbb{N}$. We want to consider the linear map $\mathfrak{j}_{\ell_0} : Y^{M,d} \to \mathbb{C}^{(2n+2)(\ell_0+1)}$ sending $\boldsymbol{f}$ to its ℓ_0-jet at $\zeta = 1$

$$\mathfrak{j}_{\ell_0}(\boldsymbol{f}) = (\boldsymbol{f}(1), \partial\boldsymbol{f}(1), \ldots, \partial_{\ell_0}\boldsymbol{f}(1)) \in \mathbb{C}^{N(\ell_0+1)}$$

where $\partial_\ell\boldsymbol{f}(1) \in \mathbb{C}^N$ denotes the vector $\dfrac{\partial^\ell \boldsymbol{f}}{\partial\zeta^\ell}(1)$ for all $\ell = 1, \cdots, \ell_0$.

Lemma 5.3. *There exists an integer $\ell_0 \leq 6nd$ such that the restriction of $\mathfrak{j}_{\ell_0}$ to the kernel of the operator $\boldsymbol{f}' \mapsto 2\mathrm{Re}\left[\overline{G(\zeta)}\boldsymbol{f}'\right]$ (see 4.4) is injective.*

Proof. Following the notation of the proof of Theorem 4.2, we prove that there exists an integer ℓ_0 such that the restriction of $\mathfrak{j}_{\ell_0}$ to the kernel of L_2 (see (4.5)) is injective. According to Lemma 5.1 in [13] we write

$$-\overline{G_2^{-1}}G_2 = \Theta_2^{-1}\Lambda\overline{\Theta}_2$$

where $\Theta_2 : \overline{\Delta} \to GL_{2n+2}(\mathbb{C})$ is a smooth map holomorphic on Δ, and Λ is the diagonal matrix with entries $\zeta^{k_1}, \ldots, \zeta^{k_{2n+2}}$ where $k_1, \cdots k_{2n+2}$ are the partial indices of $\overline{G_2^{-1}}G_2$. Let $\boldsymbol{f} \in \ker L_1$. We can write

$$\boldsymbol{f} = -\overline{G_2^{-1}}G_2\overline{\boldsymbol{f}} = \Theta_2^{-1}\Lambda\overline{\Theta}_2\overline{\boldsymbol{f}}$$

and therefore

$$\Theta_2\boldsymbol{f} = \Lambda\overline{\Theta_2\boldsymbol{f}}.$$

It follows that the j^{th}-component of $\Theta_2\boldsymbol{f}$ is a polynomial of degree at most k_j. Hence $\Theta_2\boldsymbol{f}$ is determined by its $\ell_0 := \max\{k_1, \cdots, k_{2n+2}\}$-jet at 1. It remains to prove that the restriction of $\mathfrak{j}_{\ell_0}$ to $\ker L_1$ is injective. Indeed, for any $\ell \geq 0$ we have

$$\partial_\ell(\Theta_2\boldsymbol{f})(1) = \Theta_2(1)\partial_\ell\boldsymbol{f}(1) + R_{\ell-1}$$

where R is a linear function of the $(\ell-1)$-jet of f at 1. It follows that the (well-defined) linear map $\Theta_{\ell_1} : \mathbb{C}^{(2n+2)(\ell_0+1)} \to \mathbb{C}^{(2n+2)(\ell_0+1)}$ which sends the ℓ_0-jet of $\boldsymbol{f}$ at 1 to the ℓ_0-jet of $\Theta_2\boldsymbol{f}$ at 1 has a block-triangular matrix representation whose $(2n+2)\times(2n+2)$ blocks in the diagonal are all equal to the non-singular matrix $\Theta_2(1)$. Therefore Θ_{ℓ_2} is invertible, and the claim follows from the fact that $\mathfrak{j}_{\ell_0} \circ \Theta_2 = \Theta_{\ell_2} \circ \mathfrak{j}_{\ell_0}$ and that $\mathfrak{j}_{\ell_0}$

is injective on $\Theta_2(\ker L_1)$. To conclude the proof we estimate ℓ_0 by the Maslov index of $-\overline{G_2^{-1}}G_2$, namely

$$\begin{aligned}\operatorname{ind}\det\left(-\overline{G_2^{-1}}G_2\right) &= -2\sum_{i=1}^{n} m_i + 2\operatorname{ind}Q + 2k_0 \\ &\leq 4n(2k_0-d)) + \sum_{i=1}^{n} m_i + 2k_0 \leq 6nd.\end{aligned}$$

□

5.3. **An extended family of discs; covering of an open subset.** We choose an allowable vector v as described subsection 3.1, and consider the disk $f^v = (h^v, g^v)$ associated with it. This disk is k_0-stationary, since

$$\partial\rho \circ f^v = \left(P_{z_1}(h^v, \bar{h}^v), \dots, P_{z_n}(h^v, \bar{h}^v), -\frac{1}{2}\right),$$

and the degree in $\bar{\zeta}$ of each of the components is at most k^0; hence $\zeta^{k_0}\partial\rho \circ f^v$ does extend holomorphically to Δ. Consider, for every $a \in \Delta$, also the disk $f_a^v = f^v \circ \varphi_a$, where

$$\varphi_a(\zeta) = \frac{1-\bar{a}}{1-a}\frac{\zeta - a}{1-\bar{a}\zeta}.$$

This extended family of disks is useful, because we can compute the rank of its center evaluation map $(v, a) \mapsto C(v, a) = f_a^v(0) = (v, g_a^v(0))$. By construction, the (real) Jacobian of this map at $(v, a) = (v, 0)$ is given by

$$\begin{aligned}\det\begin{pmatrix} \frac{\partial}{\partial a}\Big|_0 g_a^v(0) & \frac{\partial}{\partial \bar{a}}\Big|_0 g_a^v(0) \\ \frac{\partial}{\partial a}\Big|_0 \overline{g_a^v(0)} & \frac{\partial}{\partial \bar{a}}\Big|_0 \overline{g_a^v(0)} \end{pmatrix} &= \det\begin{pmatrix} (g^v)'(0)\frac{\partial}{\partial a}\Big|_0 \varphi_a(0) & (g^v)'(0)\frac{\partial}{\partial \bar{a}}\Big|_0 \varphi_a(0) \\ \overline{(g^v)'(0)}\frac{\partial}{\partial a}\Big|_0 \overline{\varphi_a(0)} & \overline{(g^v)'(0)}\frac{\partial}{\partial \bar{a}}\Big|_0 \overline{\varphi_a(0)} \end{pmatrix} \\ &= \det\begin{pmatrix} -(g^v)'(0) & 0 \\ 0 & -\overline{(g^v)'(0)} \end{pmatrix} \\ &= |(g^v)'(0)|^2.\end{aligned}$$

We therefore have that the center evaluation map $(v, a) \mapsto C(v, a)$ is of full rank at $(v, a) = (v_0, 0)$ if and only if $(g^{v_0})'(0) \neq 0$.

However, we can also *compute* g^v: g^v is the holomorphic function which satisfies $g^v(1) = 0$ and $\operatorname{Re} g^v(\zeta) = P(h^v(\zeta), \overline{h^v(\zeta)})$ if $\zeta\bar{\zeta} = 1$. Therefore,

$$\begin{aligned}\operatorname{Re} g^v(\zeta) &= \operatorname{Re} \sum_{j=d-k_0}^{k_0} (1-\zeta)^j (1-\bar{\zeta})^{d-j} P^{j,d-j}(v, \bar{v}) \\ &= \operatorname{Re} \sum_{j=d-k_0}^{k_0} \left(\sum_{\ell} \binom{j}{\ell}\binom{d-j}{\ell}\right) P^{j,d-j}(v, \bar{v}) \\ &\quad + 2\operatorname{Re} \sum_{j=d-k_0}^{k_0} \sum_{e=1}^{|d-2j|} (-1)^e \zeta^e \left(\sum_{\ell} \binom{j}{e+\ell}\binom{d-j}{\ell}\right) P^{j,d-j}(v, \bar{v}).\end{aligned}$$

From this equality it is easy to see that

$$(g^v)'(0) = -2 \sum_{j=d-k_0}^{k_0} \left(\sum_{\ell} \binom{j}{1+\ell} \binom{d-j}{\ell} \right) P^{j,d-j}(v, \bar{v}).$$

Hence, $(g^v)'(0) \neq 0$ for a dense, open subset of the v's.

In particular, since the image of the model stationary disc f^v is the same as the image of the f_a^v for $a \in \Delta$, we have the following

Lemma 5.4. *The set $\cup_v f^v(\Delta)$ contains an open subset of $\mathbb{C}^{n+1}$.*

References

[1] M.S. Baouendi, P. Ebenfelt, L.P. Rothschild, *Real submanifolds in complex space and their mappings*, Princeton Mathematical Series, **47**. Princeton University Press, Princeton, NJ, 1999. xii+404 pp.

[2] M.S. Baouendi, N. Mir, and L. Rothschild, *Reflection ideals and mappings between generic submanifolds in complex space*. J. Geom. Anal. 12 (2002), no. 4, 543580.

[3] L. Blanc-Centi, *Stationary discs glued to a Levi non-degenerate hypersurface*, Trans. Amer. Math. Soc. **361** (2009), 3223-3239.

[4] F. Bertrand, L. Blanc-Centi, *Stationary holomorphic discs and finite jet determination problems*, Math. Ann. **358** (2014), 477-509.

[5] F. Bertrand, G. Della Sala, *Stationary discs for smooth hypersurfaces of finite type and finite jet determination*, J. Geom. Anal. **25** (2015), 2516-2545.

[6] F. Bertrand, G. Della Sala, *Riemann-Hilbert problems with constraints*, preprint.

[7] M. Černe, *Analytic discs attached to a generating CR-manifold*, Ark. Mat. **33** (1995), 217-248.

[8] S.S. Chern, J.K. Moser, *Real hypersurfaces in complex manifolds*, Acta math. **133** (1975), 219-271.

[9] P. Ebenfelt, *Finite jet determination of holomorphic mappings at the boundary*, Asian J. Math. **5** (2001), 637-662.

[10] P. Ebenfelt, B. Lamel, *Finite jet determination of CR embeddings*, J. Geom. Anal. **14** (2004), 241-265.

[11] P. Ebenfelt, B. Lamel, D. Zaitsev, *Finite jet determination of local analytic CR automorphisms and their parametrization by 2-jets in the finite type case*, Geom. Funct. Anal. **13** (2003), 546-573.

[12] F. Forstnerič, *Analytic disks with boundaries in a maximal real submanifold of $\mathbb{C}^2$*, Ann. Inst. Fourier **37** (1987), 1-44.

[13] J. Globevnik, *Perturbation by analytic discs along maximal real submanifolds of $\mathbb{C}^N$*, Math. Z. **217** (1994), 287-316.

[14] J. Globevnik, *Perturbing analytic discs attached to maximal real submanifolds of $\mathbb{C}^N$*, Indag. Math. **7** (1996), 37-46.

[15] C.D. Hill, G. Taiani, *Families of analytic discs in $\mathbb{C}^n$ with boundaries on a prescribed CR submanifold*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **5** (1978), 327-380.

[16] X. Huang, *A preservation principle of extremal mappings near a strongly pseudoconvex point and its applications*, Illinois J. Math. **38** (1994), 283-302.

[17] X. Huang, *A non-degeneracy property of extremal mappings and iterates of holomorphic self-mappings*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **21** (1994), 399-419.

[18] Juhlin, Robert, *Determination of formal CR mappings by a finite jet*, Adv. Math. 222 (2009), no. 5, 16111648.

[19] S.-Y. Kim, D. Zaitsev, *Equivalence and embedding problems for CR-structures of any codimension*, Topology **44** (2005), 557-584.

[20] M. Kolář, F. Meylan, *Infinitesimal CR automorphisms of hypersurfaces of finite type in $\mathbb{C}^2$*, Arch. Math. (Brno) **47** (2011), 367-375.

[21] M. Kolář, F. Meylan, D. Zaitsev. *Chern-Moser operators and polynomial models in CR geometry*. Adv. Math. 263 (2014), 321–356.

[22] B. Lamel. *Jet embeddability of local automorphism groups of real-analytic CR manifolds.* In: Geometric analysis of several complex variables and related topics, 89108, Contemp. Math., 550, Amer. Math. Soc., Providence, RI, 2011.

[23] R. Juhlin, B. Lamel, *Automorphism groups of minimal real-analytic CR manifolds.* Journal of the European Mathematical Society (JEMS), 15(2), 509-537.

[24] B. Lamel, N. Mir, *Finite jet determination of local CR automorphisms through resolution of degeneracies*, Asian J. Math. **11** (2007), 201-216.

[25] L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France **109** (1981), 427-474.

[26] A. Tumanov, *Extremal discs and the regularity of CR mappings in higher codimension*, Amer. J. Math. **123** (2001), 445-473.

[27] N.P. Vekua, *Systems of singular integral equations*, Noordhoff, Groningen (1967) 216 pp.

[28] S. Webster, *On the reflection principle in several complex variables*, Proc. Amer. Math. Soc. **71** (1978), 26-28.

Florian Bertrand
Department of Mathematics
American University of Beirut, Beirut, Lebanon
E-mail address: fb31@aub.edu.lb

Giuseppe Della Sala
Department of Mathematics
American University of Beirut, Beirut, Lebanon
E-mail address: gd16@aub.edu.lb

Bernhard Lamel
Department of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, Vienna, 1090, Austria
E-mail address: bernhard.lamel@univie.ac.at