Discrete Time Markov Chains (3)

- **Long-run Properties of MC (Stationary Solution)**
  - Consider the two-state MC of the weather condition in Example 4.

$$\mathbf{P}^4 = \begin{pmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{pmatrix} \Rightarrow \mathbf{P}^8 = \begin{pmatrix} 0.5715 & 0.4285 \\ 0.5714 & 0.4286 \end{pmatrix}$$

$$\Rightarrow \mathbf{P}^{16} = \begin{pmatrix} 0.5714 & 0.4286 \\ 0.5714 & 0.4286 \end{pmatrix} \Rightarrow \mathbf{P}^{32} = \begin{pmatrix} 0.5714 & 0.4286 \\ 0.5714 & 0.4286 \end{pmatrix}$$

  - We see that the transition probability, $p_{ij}^{(n)}$, is converging (as $n \to \infty$) to some *limiting probabilities* π_j.
  - Other terms used to describe π_j are the *steady-state* or the *stationary* probabilities.
  - That is, π_j represents the probability that the process will be in state j after that the system has been "operational" for a long time (i.e., after two may transitions).
  - Alternatively, we can interpret π_j as the long-run mean fraction of time the MC is in state j.
  - E.g., in the two-state weather MC, we conclude that it rains 57.14% of the time.
  - Do the limiting probabilities exist for every MC?
  - No!

- E.g., For the two-state chain with

$$\mathbf{P}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\Rightarrow \mathbf{P}^2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\mathbf{I}\Rightarrow \mathbf{P}^3=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=\mathbf{P}.$$

- Therefore, for n even $\mathbf{P}^n = \mathbf{I}$, and for n odd $\mathbf{P}^n = \mathbf{P}$, regardless of how large n is.
- That is, the limiting probabilities π_j do not exist.
- When do the limiting probabilities exist?

Theorem 1. *For a finite state MC (with state space S) which is **irreducible and aperiodic**, $\lim_{n\to\infty} p_{ij}^{(n)}$ exists and is independent of i. Letting $\pi_j = \lim_{n\to\infty} p_{ij}^{(n)}$, then π_j is the **unique** solution to*

$$\pi_j = \sum_{i\in S}\pi_i p_{ij}, \quad i\in S,$$
$$\sum_{j\in S}\pi_j = 1.$$

- In matrix form, letting $\pi = (\pi_1,\dots,\pi_j,\dots,\pi_{|S|})$, then the equations in the theorem reduce to

$$\pi = \pi\mathbf{P},$$
$$\sum_{j\in S}\pi_j = 1.$$

- Because $\pi_j = \lim_{n\to\infty} p_{ij}^{(n)}$ is independent of i, then we can formally write that

$$\pi_j = \lim_{n\to\infty} P\{X_n = j\}.$$

- **Mean Recurrence Time and Cost Models**
  - The interpretation of π_j as the long run fraction of time spent in state j allows for the following useful facts.
  - Suppose that each visit to state j incurs a cost C_j. Then, the long-run mean cost per unit time is $\sum_{j\in S} \pi_j C_j$.
  - The mean recurrence time of state j, μ_{jj}, is the mean number of transitions until the MC, starting in state j, returns to j.
  - On average, the MC will have one transition out of state j, for every μ_{jj} transition from states other than j.
  - If one measures time in number of transitions (e.g., when a transition happens once a day), then $\pi_j = 1/\mu_{jj} \Rightarrow \mu_{jj} = 1/\pi_j$.

- **Example 8**
  - Every year at the beginning of the gardening season (March), a gardener assesses the soil condition as (1) good, (2) fair, and (3) poor, and applies a fertilizer accordingly. The gardener estimates that the transition probabilities of the soil condition from one year to another by the following matrix:

$$\mathbf{P} = \begin{pmatrix} 0.3 & 0.6 & 0.1 \\ 0.1 & 0.60 & 0.30 \\ 0.05 & 0.40 & 0.55 \end{pmatrix}$$

- If fertilizing costs \$100, \$125, and \$160 in good, fair, and poor years respectively. What is the expected annual cost?
- The gardener situation can be represented by a MC with states {1,2,3} and a transition probability matrix **P**. It can be easily verified that this MC is irreducible and aperiodic.
- The limiting probabilities for the MC are given by

$$\pi\mathbf{P} = \pi \Rightarrow \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} \begin{pmatrix} 0.3 & 0.6 & 0.1 \\ 0.1 & 0.60 & 0.30 \\ 0.05 & 0.40 & 0.55 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix},$$

$\pi_1 + \pi_2 + \pi_3 = 1$,

which implies that the π_i's are the solution to the following equations:

$$\begin{array}{c} 0.3\pi_1 + 0.1\pi_2 + 0.05\pi_3 = \pi_1 \\ 0.6\pi_1 + 0.6\pi_2 + 0.4\pi_3 = \pi_2 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{array} \Rightarrow \begin{cases} -0.7\pi_1 + 0.1\pi_2 + 0.05\pi_3 = 0 \\ 0.6\pi_1 - 0.4\pi_2 + 0.4\pi_3 = 0 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

Solving this system of linear equations gives $\pi_1 = 0.102$, $\pi_2 = 0.525$, and $\pi_3 = 0.373$.

- The expected fertilizing cost is then

$$0.102\times100 + 0.525\times125 + 0.373\times160 = \$135.51.$$

- If the soil is good this year, after how many years it is expected to be good again?
- $\mu_{11} = 1/\pi_1 = 1/0.102 = 9.8$ years.

- **Example 9**
  - An auto insurance company is using the Bonus Malus (Latin for Good-Bad) system. Each policyholder is given a positive integer valued state and the annual premium is a function of this state. A policyholder's state changes from year in response to number of claims made by that policy holder. Suppose there are four states in the Bonus Malus system. The policyholder will move from one state this year to another state next year based on the number of claims made this year as indicated in the following table.

		Next State if			
State	Annual Premium	0 claims	1 claim	2 claims	≥ 3 claims
1	$200	1	2	3	4
2	$250	1	3	4	4
3	$400	2	4	4	4
4	$600	3	4	4	4

Suppose the number of claims a policyholder makes in a year has a Poisson distribution with mean 1/2.

  - What is the average premium paid by a policyholder?
  - The policy holder situation can be modeled with a MC states {1,2,3,4}. The transition probability matrix for this MC is

$$\mathbf{P} = \begin{pmatrix} e^{-0.5} & 0.5e^{-0.5} & (0.5^2/2)e^{-0.5} & 1-e^{-0.5}-0.5e^{-0.5}-(0.5^2/2)e^{-0.5} \\ e^{-0.5} & 0 & 0.5e^{-0.5} & 1-e^{-0.5}-0.5e^{-0.5} \\ 0 & e^{-0.5} & 0 & 1-e^{-0.5} \\ 0 & 0 & e^{-0.5} & 1-e^{-0.5} \end{pmatrix}$$

Then,

$$\mathbf{P} = \begin{pmatrix} 0.6065 & 0.3033 & 0.0758 & 0.0144 \\ 0.6065 & 0 & 0.3033 & 0.092 \\ 0 & 0.6065 & 0 & 0.3935 \\ 0 & 0 & 0.6065 & 0.3935 \end{pmatrix}$$

- The limiting probabilities of this MC are the solution to the following system of equations (obtained by setting $\pi\mathbf{P} = \mathbf{P}$, and $\Sigma\pi_j = 1$).

$$\begin{aligned} -0.3935\pi_1 + 0.6065\pi_2 \quad\quad &= 0 \\ 0.3033\pi_1 - \pi_2 + 0.6065\pi_3 \quad &= 0 \\ 0.0758\pi_1 + 0.3033\pi_2 - \pi_3 + 6065\pi_4 &= 0 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1 \end{aligned}$$

- Then,

  $\pi_1 = 0.3692$, $\pi_2 = 0.2396$, $\pi_3 = 0.2103$, and $\pi_4 = 0.1809$.

- Finally, the average annual premium is

$$200\pi_1 + 250\pi_2 + 400\pi_3 + 600\pi_4 = \$326.38 .$$

- If a policyholder is in state 2 now, after how many years this policyholder is expected to be in state 2 again?
- $\mu_{22} = 1/\pi_2 = 1/0.2396 = 4.17$ years.

- **Example 10**
  - The failure probability of a computer component is p_i when the age of the component is i years. Suppose the system starts fresh with a new component. In an effort to minimize failures during service, an age replacement policy is instituted. This policy calls for replacing the part upon its failure or upon reaching the age of four years, whichever happens first. Suppose replacement costs C dollars. Furthermore, replacement during service incurs an additional interruption cost of K dollars.
  - How would you find the expected annual cost of this system?
  - This system can be modeled with a MC with states {1, 2, 3, 4} representing the age of the components at the end of a year. The transition probability matrix of this MC is

$$\mathbf{P} = \begin{pmatrix} p_1 & 1-p_1 & 0 & 0 \\ p_2 & 0 & 1-p_2 & 0 \\ p_3 & 0 & 0 & 1-p_3 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

  - Then, we can solve for the limiting probabilities π_1, π_2, π_3, and π_4 .
  - Note that π_1 is the fraction of years where the component is replaced (due to failure or reaching age 4).

- Note also that π_4 is the fraction of years where the part is replaced due to reaching age 4.
- Therefore, the expected annual cost is

$$C\pi_1 + K(\pi_1 - \pi_4).$$

- **Example 11**
  - The transition matrix, **P,** of a MC with state space S, having n states, is said to be double-stochastic if $\sum_{i=1}^{n} p_{ij} = 1,\ j = 1, 2, \dots, n$, assuming that states are numbered from 1 to n.
  - Show that the limiting probabilities for a MC with a double-stochastic transition matrix are given by

$$\pi = (1/n, 1/n, \dots, 1/n)\ .$$

  - Note that π satisfy the equations for the limiting probabilities

$$\pi\mathbf{P} = (1/n, 1/n, \dots, 1/n)\begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}$$

$$= (1/n\sum_{i=1}^{n} p_{i1}, 1/n\sum_{i=1}^{n} p_{i2}, \dots, 1/n\sum_{i=1}^{n} p_{in}) = \pi\ .$$

In addition, $\sum_{j=1}^{n} \pi_j = 1$.

  - Therefore, the limiting probabilities are given by $\pi = (1/n, 1/n, \dots, 1/n)$ since the equations for the limiting probabilities have a *unique* solution.

- **Example 12**
  - Consider a production line where each item has a probability p of being defective, independent of the condition of other items. Initially, every item is sampled as it is produced (100% inspection). This procedure continues until 4 consecutive non-defective items are found. Then, the sampling plan calls for sampling only 1 out of 3 items at random (1-out-of-3 inspection), until a defective item is found. When this happens, the plan calls for reverting to 100% inspection until 4 consecutive non-defective items are found. The process continues in the same way.
  - What is the average fraction of items inspected (AFI)?
  - The inspection process can be modeled as a MC with states, 0,1,2,3 representing the number of consecutive non-defective items found when 100% inspection is adopted, and state 4 representing the 1-out-of-3 inspection. The transition probability matrix for this MC is

$$\mathbf{P} = \begin{pmatrix} p & 1-p & 0 & 0 & 0 \\ p & 0 & 1-p & 0 & 0 \\ p & 0 & 0 & 1-p & 0 \\ p & 0 & 0 & 0 & 1-p \\ \frac{p}{3} & 0 & 0 & 0 & 1-\frac{p}{3} \end{pmatrix}$$

- Once can then determine, the limiting probabilities, π_0, π_1, π_2, π_3, and π_4 . The average fraction inspected is then

$$AFI = \pi_0 + \pi_1 + \pi_2 + \pi_3 + (\pi_4 / 3) .$$

- Assume that each item found defective is replaced with a good item. What is the fraction of defective items in the output of the process (aka the average outgoing quality)?

$$AOQ = p(1-AFI) .$$