Discrete Time Markov Chains (1)

- **Stochastic processes**
  - A stochastic (random) process, X_t, $t \in T$, is a collection (family) of random variables, where T is an index set.
  - A stochastic process is a representation of a system that evolves over time in a probabilistic manner.
  - Examples of stochastic processes include demand, inventory level, number of people in a waiting line, stock price, etc.
  - The subscript t is often interpreted as time, and X_t is called the *state* of the process at time t .
  - If the index set, T, is countable, then X_t is called a discrete-time stochastic process.
  - If T is continuous, then X_t is called a continuous-time stochastic process.

- **Discrete time Markov Chain**
  - Consider a discrete-time stochastic process $\{X_n, n = 0,1, . . .\}$, that takes on integer values.
  - If $X_n = i$, then X_n is said to be in *state* i .
  - X_n is said to be a Markov chain (MC) if it satisfies the Markovian or memoryless property, i.e.,

    $$P\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} = P\{X_{n+1} = j \mid X_n = i\}.$$

- ➢ This can be interpreted as the future being independent of the past and depending only on the present.
- ➢ A better interpretation is that the future depends on the past only through the present (indirectly).
- ➢ The probabilities $p_{ij}(n) = P\{X_{n+1} = j \mid X_n = i\}$ are called the one-step transition probabilities.
- ➢ A MC is said to have stationary transition probabilities if

$$P\{X_{n+1} = j \mid X_n = i\} = P\{X_n = j \mid X_{n-1} = i\}$$
$$= P\{X_1 = j \mid X_0 = i\} = p_{ij} .$$

- ➢ The one-step transition matrix of X_n is

$$\mathbf{P} = [p_{ij}] = \begin{pmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

- ➢ The sum of a row in P is 1, i.e. $\Sigma_j p_{ij} = 1$.

- **Example 1**
  - ➢ Suppose that the weather tomorrow depends on past weather through today's conditions. If it rains today then it will rain tomorrow w.p.α. If it does not rain today then it will rain tomorrow w.p. β.
  - ➢ This can be modeled with a two-state MC with states 0 and 1 denoting rain and no-rain.

- The transition probability matrix is

$$\mathbf{P} = \begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix} .$$

- **Example 2**
  - A communication system transmits the digits 0 and 1. Each digit transmitted must past through several stages, at each of which there is a probability p that the digit entered will be unchanged.
  - This can be modeled with a MC with states 0 and 1.
  - The transition probability matrix is

$$\mathbf{P} = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} .$$

- **Example 3**
  - Suppose that the weather tomorrow depends on past weather through today and yesterday's conditions. If it rained today and yesterday, then it will rain tomorrow w.p. 0.7. If it rained today but not yesterday, then it will rain tomorrow w.p. 0.5. If it rained yesterday but not today, then it will rain tomorrow w.p. 0.4. If it has not rained either today or yesterday, then it will rain tomorrow w.p 0.2.
  - Can you model the weather condition with a MC?

- Yes. Consider the state of the system as the weather condition on two consecutive days. Then, there are four states:

  State 0, if it rained both today and yesterday,

  State 1, if it rained today but not yesterday,

  State 2, if it rained yesterday but not today,

  State 3, if it did not rain either yesterday or today.

- What is the transition probability matrix of this MC?

$$\mathbf{P} = \begin{pmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{pmatrix}$$