Continuous-Time Markov Chains (2)

- **Alternate definition and Properties of a Poisson Process**

➢ A continuous-time stochastic process $\{N(t),\ t \geq 0\}$ counting the number of events (e.g. arrivals) by time t is said to be a Poisson process with rate $\lambda > 0$, if

(i) $N(0) = 0$.

(ii) $N(t)$ has *independent increments*. The number of events that occurs in disjoint time intervals are independent.

(iii) The number of events that occur in a time interval of length t is Poisson distributed with mean λt.

➢ Obviously, since $N(t)$ counts the number of events, then $N(t)$ takes on nonnegative integer values and $N(t) \geq N(s)$ for $t > s$.

➢ Because a Poisson process is also a pure birth process with birth rates all equal to λ, *the inter-event time is exponentially distributed with mean* $1/\lambda$.

➢ Suppose each event in a Poisson process, $N(t)$, with rate λ can be classified into type I w.p. p and type II w.p. $1-p$.

➢ Then, the number of type I and type II, $N_1(t)$ and $N_2(t)$ are independent Poisson processes with rates λp and $\lambda(1-p)$.

➢ This is the *decomposition property* of the Poisson process.

- **Example 5**

- Suppose cars arrive to a gas station according to a Poisson process with rate 5 per hour.
- What is the probability that 3 cars arrive in an hour?
- Number of cars in an hour, $N(1)$, is Poisson distributed with mean 5. Then,

$$P\{N(1) = 3\} = e^{-5}(5)^3/3! = 0.14 .$$

- What is the probability that 2 cars arrive in 15 minutes?
- Number of cars in an 15 minutes, $N(1/4)$, is Poisson distributed with mean 5/4 =. Then,

$$P\{N(1/4) = 2\} = e^{-5/4}(5/4)^2/2! = 0.224.$$

- What is the expected time before that the third car in an hour arrives?
- Inter-arrival times are exponential with mean 1/5 hours. Then, with the memoryless property, the expected time till third is 3/5 hours = 36 minutes.
- What is the probability that the station, starting empty, has no cars for 30 minutes after opening?
- Let A be the inter-arrival time. A is exponential with mean 1/5 hours. The desired probability is

$$P\{A > 1/2\} = e^{-5/2} = 0.082.$$

- **Example 6**

➢ Customers arrive to a system according to a Poisson process with rate λ and if each customer is a man w.p. 0.5 and a woman w.p. 0.5.

➢ Characterize the arrival process of men into the system.

➢ It's a Poisson process with rate 0.5λ .

- **Example 7**

➢ Cars arrive to an intersection according to a Poisson process with rate λ. A policeman blocks one way and directs the cars to the other way. On average, the policeman directs half of the cars to street A and the other half to street B.

➢ Characterize the arrival process of cars into street A. Is it Poisson?

➢ It's not a Poisson process.

- **Limiting Probabilities**

➢ Similar to the discrete cases, we define limiting probabilities for a CTMC as

$$P_j = \lim_{t \to \infty} P_{ij}(t) .$$

➢ Similar to the discrete case also, these probabilities can be interpreted as the long-run fraction of time spent in state *j*.

- The limiting probabilities exist under the following conditions: (*i*) all states communicate; and (ii) all states are *positive recurrent* meaning that the expected time to return to a state upon leaving it is finite.
- Assuming that the limiting probabilities exist, they can be determined by Kolmogorov's equations (Theorem 1).
- Letting $t \to \infty$ in Theorem 1, implies that

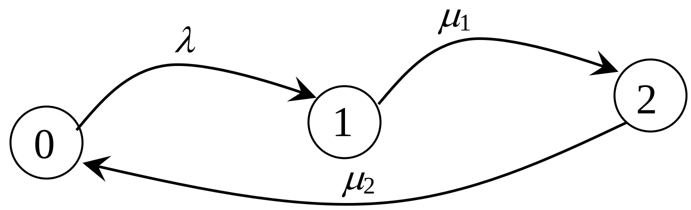
$$\lim_{t\to\infty} \frac{\partial P_{ij}(t)}{\partial t} = \sum_{k\neq j} q_{kj} \lim_{t\to\infty} P_{ik}(t) - v_j \lim_{t\to\infty} P_{ij}(t) \Rightarrow 0 = \sum_{k\neq j} q_{kj} P_k - v_j P_j$$

- Therefore,

$$v_j P_j = \sum_{k\neq j} q_{kj} P_k$$

- This equation has an interesting and useful interpretation. The left hand side, v_jP_j, is the flow out of state j, and the right hand side, $\sum_{k\neq j} q_{kj} P_k$, is the flow into state j.
- This is a *flow balance equation* (flow out = flow in).

- **Example 8**
- For the shoeshine shop of Example 1, what is the fraction of time that the shop is busy?



- The desired probability is $P_1 + P_2$ or $1 - P_0$.
- The limiting probabilities are given by the following flow balance equations.

  State 0. $\lambda P_0 = \mu_2 P_2$.

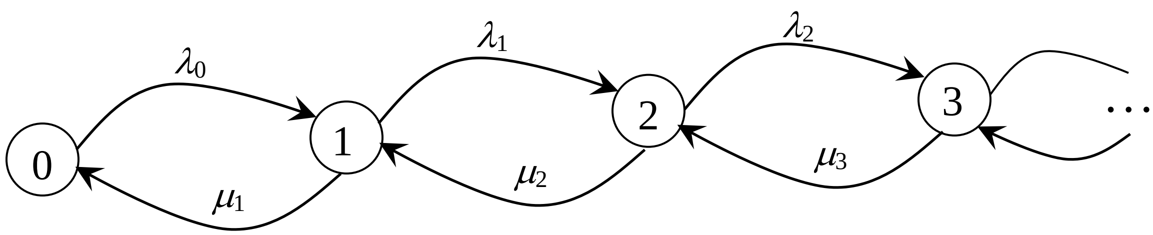
  State 1. $\mu_1 P_1 = \lambda P_0$.

  State 2. $\mu_2 P_2 = \mu_1 P_1$.

- Therefore, $P_1 = (\lambda/\mu_1)P_0$, $P_2 = (\lambda/\mu_2)P_0$. Noting that $P_0+P_1+P_2 = 1$ implies that $P_0 = 1/[1 + (\lambda/\mu_1) + (\lambda/\mu_2)]$.
- The desired probability is

$$1 - P_0 = [(\lambda/\mu_1) + (\lambda/\mu_2)] / [1 + (\lambda/\mu_1) + (\lambda/\mu_2)] .$$

- **Limiting Probabilities for a Birth-Death Process**



- The flow balance equations are

  State 0. $\lambda_0 P_0 = \mu_1 P_1$.

  State 1. $\mu_1 P_1 + \lambda_1 P_1 = \lambda_0 P_0 + \mu_2 P_2$.

  State 2. $\mu_2 P_2 + \lambda_2 P_2 = \lambda_1 P_1 + \mu_3 P_3$.

  State $n > 2$. $\mu_n P_n + \lambda_n P_n = \lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1}$.

- Replacing the first equation in the second gives $\lambda_1 P_1 = \mu_2 P_2$.
- Replacing this in the third equation gives $\lambda_2 P_2 = \mu_3 P_3$.

- Continuing in this manner we find that

$$\lambda_n P_n = \mu_{n+1} P_{n+1} \ , \ n \geq 0$$

- Then,

$$P_1 = \frac{\lambda_0}{\mu_1} P_0 \ , \quad P_2 = \frac{\lambda_1}{\mu_2} P_1 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} P_0 , \quad P_3 = \frac{\lambda_2}{\mu_3} P_2 = \frac{\lambda_2 \lambda_1 \lambda_0}{\mu_3 \mu_2 \mu_1} P_0 \ .$$

- In general,

$$P_n = \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_2 \mu_1} P_0 \ .$$

- Since $\sum_{n=0}^{\infty} P_n = 1$, it follows that

$$P_0 = \left(1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_2 \mu_1} \right)^{-1} ,$$

$$P_n = \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_2 \mu_1} \left(1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_2 \mu_1} \right)^{-1} , \ n = 1, 2, \ldots$$

- A necessary condition for the existence of P_ns is

$$\sum_{n=1}^{\infty} \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_2 \mu_1} < \infty.$$