Continuous-Time Markov Chains (1)

- **Definition and Connection to the Exponential Distribution**
  - A continuous-time stochastic process $\{X(t), t \geq 0\}$ taking on positive integers is said to be a continuous-time Markov chain (CTMC) if for all s, $t \geq 0$, i, j, k_u integers, $0 \leq u < s$,

$$P\{X(t+s) = j \mid X(s) = i, X(u) = k_u\} = P\{X(t+s) = j \mid X(s) = i\}.$$

  - If, in addition, these transition probabilities are independent of s, the CTMC is said to have stationary or homogenous transition probabilities. We only consider this kind of CTMC.
  - Let T_i be the amount spent in state i before making a transition to another state. Then,

$$P\{T_i > s+t \mid T_i > s\} = P\{X(t+s) = i \mid X(s) = i\}$$
$$= P\{X(t) = i \mid X(0) = i\} = P\{T_i > t\}.$$

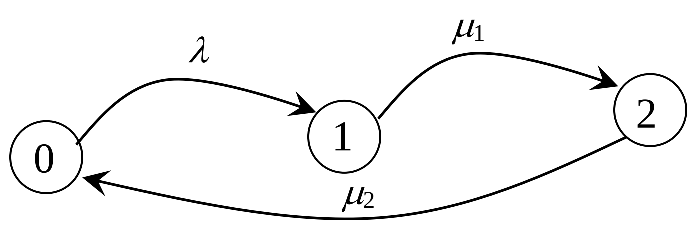
  - Therefore, T_i has the memoryless property.
  - It follows that T_i is exponentially distributed.
  - Let v_i be the "rate" of transition out of i (i.e., $E[T_i] = 1/v_i$).
  - Define also P_{ij} as the probability that the process enters j after transitioning out of i. By definition,

$$P_{ii} = 0,$$
$$\sum_{j=0}^{\infty} P_{ij} = 1.$$

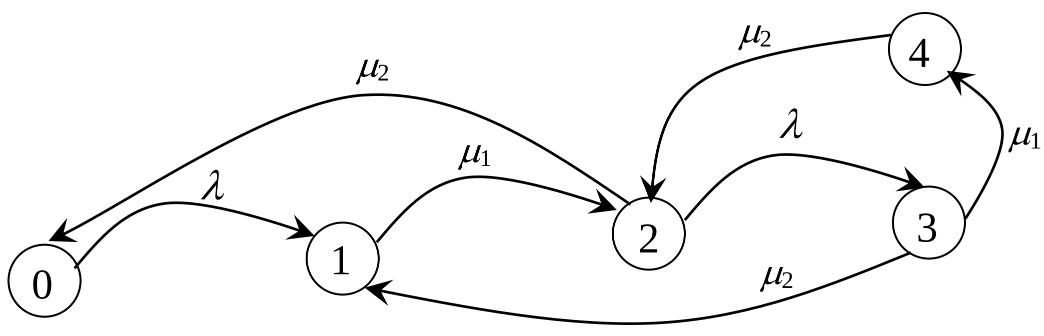
  - The parameters v_i and P_{ij} completely define the CTMC.

- **Example 1**

- Consider a shoeshine shop consisting of two chairs. A customer arrives and set in chair 1, where his shoes are cleaned and polish is applied, and then moves to chair 2 where his shoes is buffed. Suppose that customers inter-arrival times are iid exponential rvs with rate λ, and that service times at chair i are iid exponential rvs with rate μ_i, $i = 1$, 2. Suppose that a customer will enter the shop only if both chairs are empty.
- This is a CTMC with three states: 0 (both chairs are empty), 1 (a customer is in chair 1), and 2 (a customer is in chair 2).



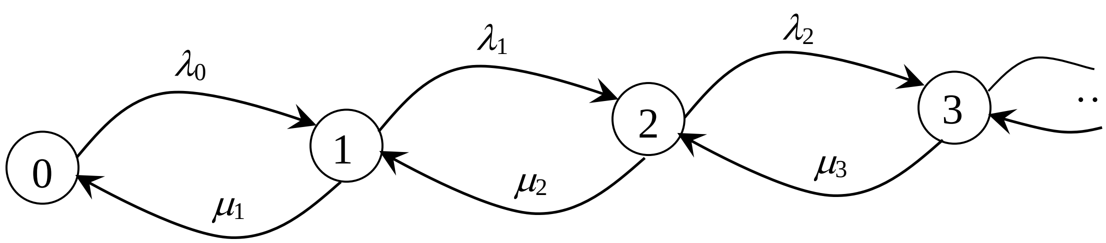
- In this case, $v_0 = \lambda$, $v_1 = \mu_1$, $v_2 = \mu_2$, $P_{01} = P_{12} = P_{20} = 1$, and $P_{ij} = 0$, otherwise.
- What if a customer would enter if only chair 1 is empty?
- Add two states: 3 (both chairs busy) and 4 (chair 1 waiting).



- **Birth-Death Processes**
  - Consider a stochastic process $\{X(t),\ t \geq 0\}$ representing the number of people in a population.
  - Suppose that whenever there are n people in a system, the time for the next arrival is exponential with rate λ_n , and the time of the next departure is exponential with rate μ_n .
  - When a departure (arrival) happens in state n, the system moves to state $n-1$ ($n+1$).
  - The process $X(t)$ is called a birth-death process.
  - It is a special case of a CTMC with

    $v_0 = \lambda_0,\ P_{01} = 1,$

    $v_i = \lambda_i + \mu_i,\ P_{i,i+1} = \lambda_i \,/\, (\lambda_i + \mu_i),\ P_{i,i-1} = \mu_i \,/\, (\lambda_i + \mu_i),\ i > 0$.



  - If $\mu_i = 0,\ i = 1,2,\ \ldots,\ X_t$ is called a *pure birth process*.
  - If $\lambda_i = 0,\ i = 0,1,2,\ \ldots,\ X_t$ is called a *pure death process*.

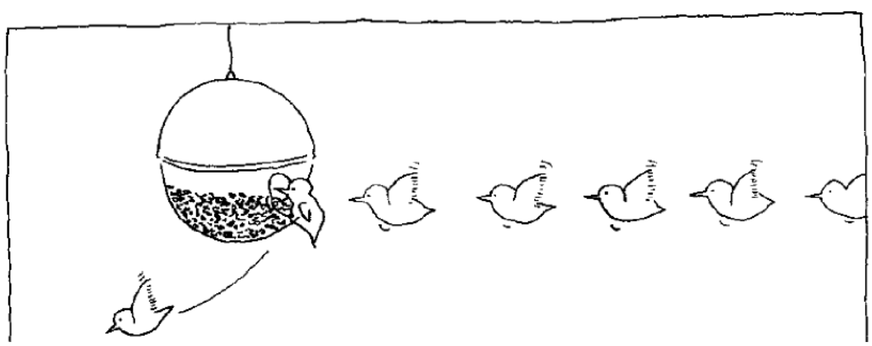
- **Example 2**
  - A pure birth process with $\lambda_i = \lambda,\ i = 0,1,2,\ \ldots,$ is a *Poisson process*. This is the most popular models for arrival processes (where the inter-arrival times are exponential rvs).

- **Example 3**

➢ A pure birth process with $\lambda_i = i\lambda$, i = 0,1,2, …, is called a Yule process. This is a model for a population where every person gives birth at a rate λ, independent of others, and no one dies.

- **Example 4**

➢ Consider a system with a single server. Customers arrive to the system according to a Poisson process with rate λ customers per hour. Customers who find the server busy wait in line, and those who find the server idle start service immediately. Service times are iid exponential rvs with rate μ customers per hour.



➢ This is a *queueing* model known as the *M*/*M*/1 queue.

➢ It is also a birth-death model with $\lambda_i = \lambda$, and $\mu_i = \mu$, i = 0, 1, 2, …

- ***o*(h) Functions**
  - A function $f(h)$ is said to be $o(h)$ if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0 .$$

  - E.g., $f(h) = h^n$, $n > 1$, is $o(h)$ since $\lim_{h \to 0} f(h)/h = \lim_{h \to 0} h^{n-1} = 0.$
  - But $f(h) = h$, is not $o(h)$ since $\lim_{h \to 0} f(h)/h = \lim_{h \to 0} h/h = 1.$

- **Properties of Transition Probabilities**
  - Consider a CTMC $\{X(t), t \geq 0\}$. Denote the transition probability from state i to state j within time t by $P_{ij}(t)$. I.e.,

$$P_{ij}(t) = P\{X(t+s) = j \mid X(s) = i\} .$$

  - Let $q_{ij} = v_i P_{ij}$ be the transition rate from state i to state j.

Lemma 1 *The transition probabilities satisfy the following,*

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i \ ,$$

$$\lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij} \ .$$

Proof. Note that

$$P_{ii}(h) = P\{T_i > h\} = e^{-v_i h} = 1 - v_i h + v_i h^2/2 - v_i h^3/6 + \ldots = 1 - v_i h + o(h),$$

which implies that $1 - P_{ii}(h) = v_i h + o(h).$ In addition,

$$P_{ij}(h) = (1 - P_{ii}(h)) P_{ij} = v_i P_{ij} h + o(h) = q_{ij} h + o(h) . \quad \blacksquare$$

Lemma 2 (Chapman-Kolmogorov equations)

$$P_{ij}(t+h)=\sum_{k=0}^{\infty}P_{ik}(t)P_{kj}(h)\,.$$

Proof. Follows by conditioning on the state the process is in at time *t*, similar to the discrete case. $\blacksquare$

Theorem 1 (Kolmogorov's forward equations) *The probability $P_{ij}(t)$ satisfy the following differential equation,*

$$\frac{\partial P_{ij}(t)}{\partial t}=\sum_{k\neq j}q_{kj}P_{ik}(t)-v_jP_{ij}(t)\,.$$

Proof. Lemma 2 implies that

$$P_{ij}(t+h)-P_{ij}(t)=\sum_{k=0}^{\infty}P_{ik}(t)P_{kj}(h)-P_{ij}(t)=\sum_{k\neq j}P_{ik}(t)P_{kj}(h)-(1-P_{jj}(h))P_{ij}(t).$$

Therefore,

$$\lim_{h\to 0}\frac{P_{ij}(t+h)-P_{ij}(t)}{h}=\lim_{h\to 0}\sum_{k\neq j}P_{ik}(t)\frac{P_{kj}(h)}{h}-\frac{(1-P_{jj}(h))}{h}P_{ij}(t).$$

Lemma 1 then completes the proof. $\blacksquare$

- **Pure Birth and Poisson Process Transition Probabilities**
  - For a pure birth process, Kolmogorov's forward equations can be written as

$$\frac{\partial P_{ii}(t)}{\partial t}=-\lambda_iP_{ii}(t),$$

$$\frac{\partial P_{i,i+k}(t)}{\partial t}=\lambda_{i+k-1}P_{i,i+k-1}(t)-\lambda_{i+k}P_{i,i+k}(t).$$

- These differential equations have the following solution, which is obtained sequentially.

$$P_{ii}(t) = e^{-\lambda_i t},$$

$$P_{i,i+k}(t) = \lambda_{i+k-1} e^{-\lambda_{i+k} t} \int_0^t e^{\lambda_{i+k} s} P_{i,i+k-1}(s) ds.$$

- In particular, for a Poisson process, we have $\lambda_i = \lambda$. Then,

$$P_{ii}(t) = e^{-\lambda t},$$

$$P_{i,i+1}(t) = \lambda_i e^{-\lambda_{i+1} t} \int_0^t e^{\lambda_{i+1} s} P_{ii}(s) ds = \lambda e^{-\lambda t} \int_0^t e^{\lambda s} e^{-\lambda s} ds = (\lambda t) e^{-\lambda t},$$

$$P_{i,i+2}(t) = \lambda_{i+1} e^{-\lambda_{i+2} t} \int_0^t e^{\lambda_{i+2} s} P_{i,i+1}(s) ds = \lambda e^{-\lambda t} \int_0^t e^{\lambda s} (\lambda s) e^{-\lambda s} ds = \frac{(\lambda t)^2}{2} e^{-\lambda t},$$

$$\vdots$$

$$P_{i,i+k}(t) = \lambda_{i+k-1} e^{-\lambda_{i+k} t} \int_0^t e^{\lambda_{i+k} s} P_{i,i+k-1}(s) ds = \lambda e^{-\lambda t} \int_0^t e^{\lambda s} \frac{(\lambda s)^{k-1}}{(k-1)!} e^{-\lambda s} ds = \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

- That is, the probability of having k arrival during a time period of length t is a Poisson random variable with mean λt .