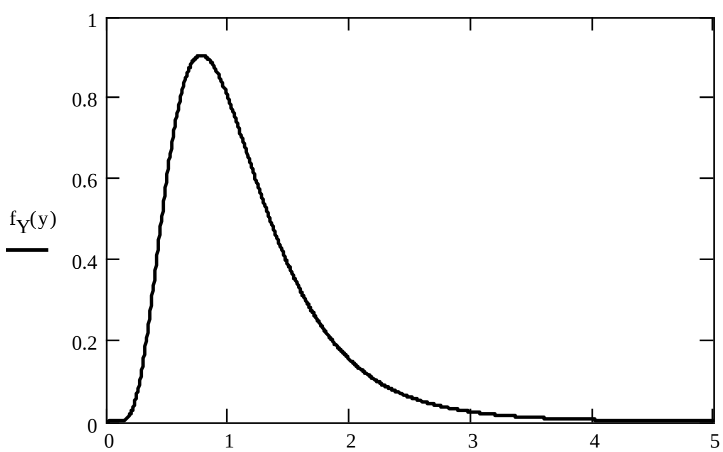
Probability and Random Variable (4)

➢ Lognormal random variable – A stock price model

- A rv Y is said to be *lognormal* if $X = \ln(Y)$ is a normal random variable.
- Alternatively, Y is a lognormal rv if $Y = e^X$, where X is a normal rv.
- If $X = \ln(Y)$ is normal with mean v and variance σ^2, then the density function of Y is

$$f_Y(y) = \frac{1}{y\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln y - v)^2}{2\sigma^2}}, \; y > 0.$$



- The mean and variance of Y are given by

$$E[Y] = e^{v+\sigma^2/2}, \quad \text{var}[Y] = e^{2v+\sigma^2}(e^{\sigma^2} - 1).$$

- Note that $E[Y] \neq e^{E[X]} = e^v$ although $Y = e^X$.

- A popular stock price model based on the lognormal distribution is the *geometric Brownian motion* model, which relates the stock prices at time 0, S_0, and time $t > 0$, S_t by the following relation:

$$\ln(S_t) = \ln(S_0) + (\mu - \sigma^2/2)t + \sigma z(t) \ ,$$

where, μ and $\sigma > 0$ are constants and $z(t)$ is a normal rv with mean 0 and variance t .[1]

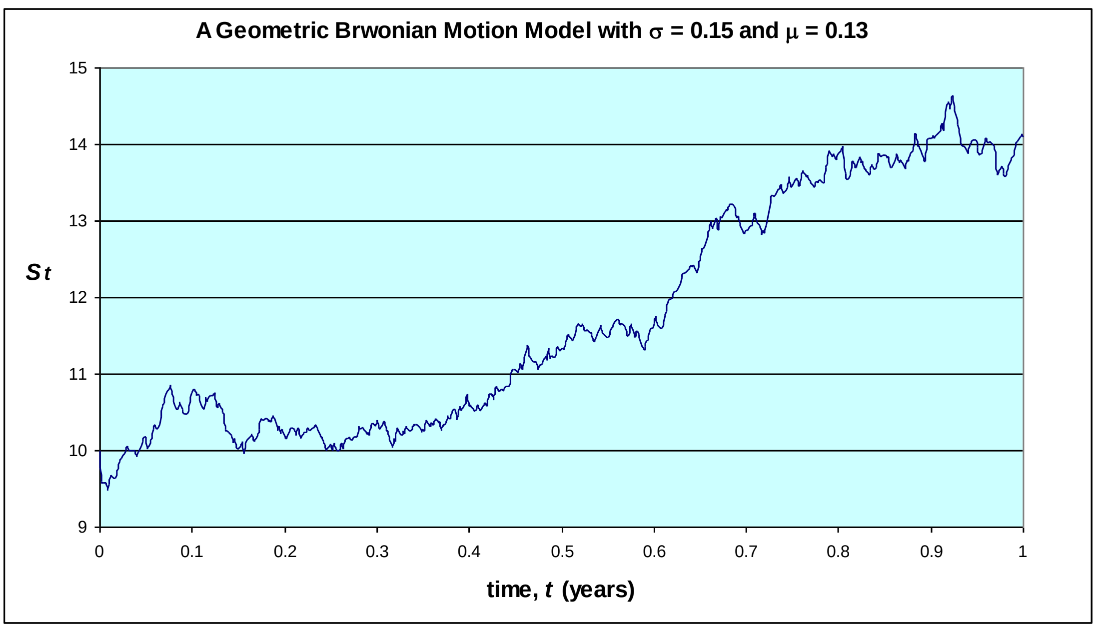
- It follows that $\ln(S_t / S_0)$ is a normal random variable with mean $(\mu - \sigma^2/2)t$ and variance $\sigma^2 t$.
- That is, S_t / S_0 is a lognormal rv with mean and variance

$$E[S_t / S_0] = e^{\mu t - \sigma^2 t/2 + \sigma^2 t/2} = e^{\mu t} \ ,$$
$$\mathrm{var}[S_t / S_0] = e^{2(\mu t - \sigma^2 t/2) + \sigma^2 t/2}(e^{\sigma^2 t} - 1) = e^{\mu t}(e^{\sigma^2 t} - 1) \ .$$

- Note that μ can be seen as the stock rate of return assuming continuous compounding. So μ is called the *expected return* of the stock.
- In addition, σ measures the variability of the stock price. So σ is called the *volatility* of the stock price.
- Typical values for these parameters are $\mu = 13\%$ and $\sigma = 15\%$ when time t is measured in years.

[1] $z(t)$ is called a Brownian motion.

- In practice the expected return, μ, is too difficult to estimate accurately, while the volatility σ can be estimated reasonably well from historical data.
- The main idea behind the geometric Brownian motion model is that the probability of a certain percentage change in the stock price within a time t is the same at all times.
- This is a memoryless or Markovian behavior indicating that past stock values won't help in predicting future values.
- In addition, the *expected* value and variance of the stock price typically follow an increasing trend, and the as indicated in this figure.



- **Example 20**
- Microsoft stock price (MSFT) is believed to follow a geometric Brownian motion with volatility 37% and expected return 35%.
- If currently MSFT is \$100, what is the probability that MSFT drops to \$95 next week (specifically after 1 week from now)?
- Let t = 1 week = 1/52 years, and let S_t be MSFT at time t. Then, $\Delta = \ln(S_t / S_0)$ is a normal random variable with mean and variance $E[\Delta] = (\mu - \sigma^2/2)t = (0.35 - 0.37^2/2)/52 = 0.005414$, and $\text{var}[\Delta] = \sigma^2 t = 0.37^2 / 52 = 0.002633$. The required probability, is $\text{P}\{\Delta < \ln(95/100)\} = P\{\Delta < -0.051293\}$

$$= P\{Z < (-0.051293 - 0.005414)/\sqrt{0.002633}\}$$
$$\cong P\{Z < -1.11\} = 1 - P\{Z < 1.11\} = 1 - 0.8665 \cong 0.13.$$

- **Conditional probability and conditional expectation**
- Consider two discrete rvs X and Y, we define the *conditional mass function* of X given that $Y = y$ by

$$f_{X|Y}(x \mid y) = P\{X = x \mid Y = y\} = \frac{P\{X = x, Y = y\}}{P\{Y = y\}}.$$

- In many situation, in order to determine the distribution of a rv X it is useful to condition on another rv Y, as follows

$$P\{X = x\} = \sum_{\text{all } y} P\{X = x \mid Y = y\} P\{Y = y\} \Rightarrow f_X(x) = \sum_{\text{all } y} f_{X|Y}(x \mid y) f_Y(y).$$

- For continous rvs, similar definitions can be made

$$P\{X < x\} = \int_{-\infty}^{\infty} P\{X < x \mid Y = y\} f_Y(y) dy \Rightarrow F_X(x) = \int_{-\infty}^{\infty} F_{X|Y}(x \mid y) f_Y(y) dy.$$

- Conditioning can be also applied to find expected values.
- For discrete rvs

$$E[X] = \sum_{\text{all } y} E[X \mid Y = y] P\{Y = y\}.$$

- For continuous rvs

$$E[X] = \int_{-\infty}^{\infty} E[X \mid Y = y] f_Y(y) dy.$$

- For both cases we can write

$$E[X] = E[E[X \mid Y]].$$

- **Example 21**
  - Each customer who enters Rebecca's store will buy donuts with probability *p*. Suppose that the number of customers who enter the store in a day is a Poisson rv with mean λ.
  - What is the probability that *k* persons buy donuts on a given day?
  - Let *X* be the number of people who buy donuts, and let *N* be the total number of people in the store. Then, by conditioning on *N*,

$$\begin{aligned}
P\{X = k\} &= \sum_{n=k}^{\infty} P\{X = k \mid N = n\} P\{N = n\} \\
&= \sum_{n=k}^{\infty} \binom{n}{k} (1-p)^{n-k} p^k e^{-\lambda} \frac{\lambda^n}{n!} \\
&= \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} (1-p)^{n-k} p^k e^{-\lambda p} e^{-\lambda(1-p)} \frac{\lambda^{n-k} \lambda^k}{n!} \\
&= \frac{e^{-\lambda p} (\lambda p)^k}{k!} e^{-\lambda(1-p)} \sum_{n=k}^{\infty} \frac{(1-p)^{n-k} \lambda^{n-k}}{(n-k)!} \\
&= \frac{e^{-\lambda p} (\lambda p)^k}{k!} e^{-\lambda(1-p)} \sum_{m=0}^{\infty} \frac{(1-p)^m \lambda^m}{m!} \\
&= \frac{e^{-\lambda p} (\lambda p)^k}{k!} e^{-\lambda(1-p)} e^{\lambda(1-p)} = \frac{e^{-\lambda p} (\lambda p)^k}{k!}.
\end{aligned}$$

➢ Example 22

- You are waiting on the sea side for a taxicab or a bus to give you a ride home. From past experience, you estimates that caps (buses) inter-arrival time is exponentially distributed with rate λ_c (λ_b) .
- What is the probability that you take a taxi back home?
- Let X_c (X_b) be the inter-arrival times of cabs (buses). Then, the desired probability is $P\{X_b > X_c\}$, which can be found by conditioning on X_c ,

$$
\begin{aligned}
P\{X_b > X_c\} &= \int_0^\infty P\{X_b > X_c \mid X_c = x_c\} f_{X_c}(x_c) dx_c \\
&= \int_0^\infty P\{X_b > x_c\} f_{X_c}(x_c) dx_c \\
&= \int_0^\infty e^{-\lambda_b x_c} \lambda_c e^{-\lambda_c x_c} dx_c = \frac{\lambda_c}{\lambda_b + \lambda_c} .
\end{aligned}
$$

➢ Example 23

- The manufacturing of a part is done in two stages. The time to complete the first stage is uniformly distributed between 15 and 20 minutes, while the time to complete the second is uniformly distributed between 10 and 15 minutes.
- What is the probability that the part is completed in less than 30 minutes?
- Let X_1 and X_2 be the time to complete stages 1 and 2. Then, the desired probability is $P\{X_1 + X_2 < 30\}$, which can be found by conditioning as follows.

$$P\{X_1 + X_2 < 30\} = \int_{10}^{15} P\{X_1 < 30 - X_2 \mid X_2 = x_2\} f_{X_2}(x_2) dx_2$$

$$= \int_{10}^{15} P\{X_1 < 30 - x_2\} f_{X_2}(x_2) dx_2$$

$$= \int_{10}^{15} \frac{[(30 - x_2) - 15]}{5} \frac{1}{5} dx_2 = \frac{1}{25} \int_{10}^{15} (15 - x_2) dx_2 = \frac{1}{25} \frac{(15 - 10)^2}{2} = \frac{1}{2}.$$

➢ Example 24

- Find the expectation of a geometric random variable, X, with parameter p using conditioning.
- Conditioning on "the first thing that happens." Let $Y = 1$ if the first trial is a success and $Y = 0$, otherwise. Then,

$$E[X] = E[E[X \mid Y]] = E[X \mid Y = 1] P\{Y = 1\} + E[X \mid Y = 0] P\{Y = 0\}$$
$$\Rightarrow E[X] = 1 \times p + (1 + E[X])(1 - p)$$
$$\Rightarrow pE[X] = 1 \Rightarrow E[X] = 1/p.$$

➢ Example 25

- A miner is trapped in a mine containing three doors. The first door leads to a tunnel that takes him to safety after two hours travel. The second door leads to a tunnel that returns him to the mine after three hours of travel. The third door leads to a tunnel that returns him to the mine after five hours.
- Assuming that the miner is at all times equally likely to choose one of the doors, what is the expected length of time until the miner reaches safety?
- Conditioning on "the first thing that happens." Let $Y = i$ if the miner chooses door i, $i = 1, 2, 3$, on the first trial. Let X be the amount of time the miner stays trapped in the mine. Then,

$$E[X] = E[E[X \mid Y]] = E[X \mid Y=1]P\{Y=1\} + E[X \mid Y=2]P\{Y=2\}$$
$$+ E[X \mid Y=3]P\{Y=3\}$$
$$\Rightarrow E[X] = 2(1/3) + (3 + E[X])(1/3) + (5 + E[X])(1/3)$$
$$\Rightarrow (1/3)E[X] = 10/3 \Rightarrow E[X] = 3 \text{ hours}.$$

- **Example 26**
  - Suppose that the number of accidents per week in Beirut is a rv N. The number of people injured in accident i, Y_i, are iid random variables with mean $E[Y]$.
  - What is the expected number of car accident injuries in Beirut in a week?
  - Let X be the number of injuries, then

$$X = \sum_{i=1}^{N} Y_i .$$

By conditioning on N,

$$E[X] = E[E[X \mid N]] = E\left[E\left[\sum_{i=1}^{N} Y_i \mid N\right]\right] = E[NE[Y]] = E[N]E[Y].$$

- **Example 27**
  - Let Y be a lognormal random variable with parameters $E[\ln(Y)] = \lambda$ and $\text{var}[\ln(Y)] = \sigma^2$, a pdf $f_Y(y)$ and let $N(z) = \int_{-\infty}^{z} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz$ be the standard normal CDF.

- Show that

$$E[\max(Y-K,0)] = e^{\lambda+\sigma^2/2}N(d_1) - KN(d_2)\ ,$$

where

$$d_1 = \frac{\ln[e^{\lambda+\sigma^2/2}/K] + \sigma^2/2}{\sigma},$$

$$d_2 = \frac{\ln[e^{\lambda+\sigma^2/2}/K] - \sigma^2/2}{\sigma} = d_1 - \sigma.$$

- This is proven by conditioning on $Y > K$,

$$E[\max(Y-K,0)] = E[Y-K \mid X > K] = \int_K^\infty (y-K)\, f_Y(y)dy.$$

- Using the standard change of variable $z = (\ln(y) - \lambda)/\sigma$, implies that $y = e^{\lambda+\sigma z}$, $dz = dy/(\sigma y)$, and

$$f_Y(y)dy = \frac{1}{y\sqrt{2\pi\sigma^2}}e^{-\frac{(\ln y-\lambda)^2}{2\sigma^2}}dy = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz = n(z)dz,$$

where $n(z)$ is the standard normal pdf.

- Then,

$$\int_K^\infty (y-K)\, f_Y(y)dy = \int_{(\ln(K)-\lambda)/\sigma}^\infty (e^{\lambda+\sigma z} - K)n(z)dz = \int_{(\ln(K)-\lambda)/\sigma}^\infty e^{\lambda+\sigma z}n(z)dz - K\int_{(\ln(K)-\lambda)/\sigma}^\infty n(z)dz.$$

- The second integral gives

$$K\int_{(\ln(K)-\lambda)/\sigma}^\infty n(z)dz = K\left[1 - N[(\ln(K)-\lambda)/\sigma]\right] = KN[(-\ln(K)+\lambda)/\sigma]$$

$$= KN[(\lambda + \sigma^2/2 - \sigma^2/2 - \ln(K))/\sigma]$$

$$= KN[(\ln(e^{\lambda+\sigma^2/2}) - \ln(K) - \sigma^2/2)/\sigma]$$

$$= KN[(\ln(e^{\lambda+\sigma^2/2}/K) - \sigma^2/2)/\sigma] = KN(d_2).$$

➢ The first integral gives

$$\int_{(\ln(K)-\lambda)/\sigma}^{\infty} e^{\lambda+\sigma z} n(z)dz = \frac{1}{\sqrt{2\pi}} \int_{(\ln(K)-\lambda)/\sigma}^{\infty} e^{\lambda+\sigma z - z^2/2} dz = \frac{1}{\sqrt{2\pi}} \int_{(\ln(K)-\lambda)/\sigma}^{\infty} e^{-(z^2-2\sigma z)/2+\lambda} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{(\ln(K)-\lambda)/\sigma}^{\infty} e^{-(z^2-2\sigma z+\sigma^2)/2+\sigma^2/2+\lambda} dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{\lambda+\sigma^2/2} \int_{(\ln(K)-\lambda)/\sigma}^{\infty} e^{-(z-\sigma)^2/2} dz$$

➢ Using the change variable, $u = z - s$, this simplifies to

$$\int_{(\ln(K)-\lambda)/\sigma}^{\infty} e^{\lambda+\sigma z} n(z)dz = e^{\lambda+\sigma^2/2} \int_{(\ln(K)-\lambda)/\sigma-\sigma}^{\infty} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du$$

$$= e^{\lambda+\sigma^2/2} \left[1 - N\left((\ln(K)-\lambda)/\sigma - \sigma\right)\right]$$

$$= e^{\lambda+\sigma^2/2} \left[N\left((\lambda-\ln(K))/\sigma + \sigma\right)\right]$$

$$= e^{\lambda+\sigma^2/2} \left[N\left((\lambda-\ln(K))/\sigma + \sigma^2/\sigma\right)\right]$$

$$= e^{\lambda+\sigma^2/2} \left[N\left(\left(\lambda+\sigma^2/2-\ln(K)+\sigma^2/2\right)/\sigma\right)\right]$$

$$= e^{\lambda+\sigma^2/2} \left[N\left(\left(\ln(e^{\lambda+\sigma^2/2})-\ln(K)+\sigma^2/2\right)/\sigma\right)\right]$$

$$= e^{\lambda+\sigma^2/2} \left[N\left(\left(\ln(e^{\lambda+\sigma^2/2}/K)+\sigma^2/2\right)/\sigma\right)\right] = e^{\lambda+\sigma^2/2} N(d_1).$$

- **Example 28**

➢ A *European call option* is a financial instrument that gives its holder the right, but not the obligation, to *buy* one (or more) share(s) of stock price for a *strike price K* per share at a *maturity time T* in the future.

➢ The buyer of the option pays a price or a *premium*, *C*, in exchange for it.

- To derive C a “no-arbitrage” assumption is adopted, which guarantees that no market player can “have a free lunch” and make a profit with no risk.
- The no-arbitrage assumption allows determining C based on “risk-neutral pricing” under a “change of measure,” where μ is replaced by r in the GBM model,

$$\ln(S_t) = \ln(S_0) + (r - \sigma^2/2)t + \sigma z(t) \ .$$

- The option premium is then determined as

$$C = e^{-rT} E[f(S,T)] = e^{-rT} E[\max(S_T - K, 0)] \ .$$

- This is a “discounted expected cash flow” (DECF) pricing.
- Show that C is given as a function of S_0, K, σ, T, and r by the *Black-Scholes formula*,

$$C = S_0 N(d_1) - K e^{-rT} N(d_2),$$

where $d_1 = \dfrac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$, and $d_2 = d_1 - \sigma\sqrt{T}$.

- Note that S_T is a lognormal random variable with $\ln(S_T)$ having mean and variance $\ln(S_0) + (r - \sigma^2/2)T$ and $\sigma^2 T$.
- Applying the result in Example 27 gives

$$E[\max(S_T - K, 0)] = e^{\ln S_0 + (r-\sigma^2/2)T + (\sigma^2/2)T} N(d_1) - KN(d_2) = S_0 e^{rT} N(d_1) - KN(d_2) \ ,$$

where

$$d_1 = \frac{\ln[e^{\ln(S_0)+(r-\sigma^2/2)T+(\sigma^2/2)T} / K] + (\sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln[e^{\ln(S_0)} e^{rT} / K] + (\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$= \frac{\ln[e^{\ln(S_0)} / K] + \ln[e^{rT}] + (\sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln[S_0/K] + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \ .$$