Probability and Random Variable (1)

- **Sample space and Events**
  - Suppose that an experiment with an uncertain outcome is performed (e.g., rolling a die).
  - While the outcome of the experiment is not known in advance, the set of all possible outcomes is known. This set is the sample space, Ω.
  - For example, when rolling a die $\Omega = \{1, 2, 3, 4, 5, 6\}$. When tossing a coin, $\Omega = \{H, T\}$. When measuring life time of a machine (years), $\Omega = \{1, 2, 3, \ldots\}$.
  - A subset $E \subset \Omega$ is known as an event.
  - E.g., when rolling a die, $E = \{1\}$ is the event that one appears and $F = \{1, 3, 5\}$ is the event that an odd number appears.

- **Probability of an event**
  - If an experiment is repeated for a number of times which is large enough, the fraction of time that event E occurs is the probability that event E occurs, $P\{E\}$.
  - E.g., when rolling a fair die, $P\{1\} = 1/6$, and $P\{1, 3, 5\} = 3/6 = 1/2$. When tossing a fair coin, $P\{H\} = P\{T\} = 1/2$.
  - In some cases, events are not repeated many times.
  - For such cases, probabilities can be a measure of belief (subjective probability).

- **Axioms of probability**

(1) For $E \subset \Omega$, $0 \le P\{E\} \le 1$;

(2) $P\{\Omega\} = 1$;

(3) For events $E_1, E_2, \ldots, E_i, \ldots$, with $E_i \subset \Omega$, $E_i \cap E_j = \varnothing$, for all i and j, $P\left\{\bigcup_{i=1}^{\infty} E_i\right\} = \sum_{i=1}^{\infty} P\{E_i\}$.

- **Implications**
  - The axioms of probability imply the following results:
    - For E and $F \subset \Omega$,

      $P\{E \text{ “or” } F\} = P\{E \cup F\} = P\{E\} + P\{F\} - P\{E \cap F\}$;[1]
    - If E and F are mutually exclusive (i.e., $E \cap F = \varnothing$), then

      $P\{E \cup F\} = P\{E\} + P\{F\}$;
    - For $E \subset \Omega$, let E^c be the complement of E (i.e., $E \cup E^c = \Omega$),

      $P\{E^c\} = 1 - P\{E\}$;
    - $P\{\varnothing\} = 0$.

- **Conditional probability**
  - The probability that event E occurs given that event F has already occurred is

$$P\{E \mid F\} = \frac{P\{E \cap F\}}{P\{F\}} .$$

[1] $\mathrm{P}\{E \cap F\} = \mathrm{P}\{E \text{ “and” } F\}$.

- **Independent events**
  - For E and $F \subset \Omega$, $P\{E \cap \text{F}\} = P\{E|F\}P\{F\}$.
  - Two events are independent if an only if $P\{E \cap F\} = P\{E\}P\{F\}$. That is, $P\{E|F\} = P\{E\}$.

- **Example 1**
  - Let $\Omega = \{\text{E}_1, \ldots, \text{E}_{10}\}$, where E_i are mutually exclusive. It is known that $P\{E_i\} = 1/20$, $i = 1, \ldots, 6$, $P\{E_i\} = 1/5$, $i = 7, \ldots, 9$, and $P\{E_{10}\} = 3/20$,
  - Do $P\{E_i\}$ satisfy the axioms of probability?
  - No, since $\Sigma P\{E_i\} > 1$.

- **Example 2**
  - Suppose that two fair coins are tossed. What is the probability that either the first or the second coin falls heads?
  - In this example, $\Omega = \{(\text{H, H}), (\text{H, T}), (\text{T, H}), (\text{T,T})\}$. Let E (F) be the event that the first (second) coin falls heads, $E = \{(\text{H, H}), (\text{H, T})\}$ and $F = \{(\text{H, H}), (\text{T, H})\}$, and $\text{E} \cap \text{F} = \{\text{H, H}\}$. The desired probability is $P\{E \cup F\} = P\{E\} + P\{F\} - P\{E \cap F\} = 1/2 + 1/2 - 1/4 = 3/4$.

- **Example 3**
  - When rolling two fair dice, suppose the first die is 3, what is the probability the sum of the two dice is 7?
  - Let E be the event that the sum of the two dice is 7, $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$, and F be the event that the first die is 3, $F = \{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)\}$. Then,

$$P\{E \mid F\} = \frac{P\{E \cap F\}}{P\{F\}} = \frac{P\{(3,4)\}}{P\{(3,\ 1),\ (3,\ 2),\ (3,\ 3),\ (3,\ 4),\ \ (3,\ 5),\ (3,\ 6)\}}$$
$$= \frac{1/36}{6/36} = \frac{1}{6}.$$

- **Example 4**
  - A family has two kids. At least one of them is a boy. What is the probability that both are boys?
  - In this example, $\Omega = \{(\text{b, b}), (\text{b, g}), (\text{g, b}), (\text{g, g})\}$. Let E ={Both kids are boys} and F = {At least one kid is a boy}. The required probability is then

$$P\{E \mid F\} = \frac{P\{E \cap F\}}{P\{F\}} = \frac{P\{(b,b)\}}{P\{(b,g),(g,b),(b,b)\}} = \frac{1/4}{3/4} = 1/3.$$

- **Example 5**
  - An urn contains three white balls and four black balls. Two balls are drawn without replacement.
  - What is the probability that the two balls are black?
  - Let F = {First ball is black}, and E = {Second ball is black}. The desired probability is

$$P\{E \cap F\} = P\{F\}P\{E \mid F\} = (4/7)(3/6) = 2/7.$$

  - What is the probability that the both balls are of the same color?
  - P{same color} = P{both black} + P{both white}

$$= 2/7 + (3/7)(2/6) \ = 3/7\ .$$

- **Finding Probability by Conditioning**
  - Suppose that we know the probability of event B once event A is realized (or not). We also know $P\{A\}$. That is, we know $P\{B|A\}$, and $P\{B|A^c\}$ and $P\{A\}$. What is $P\{B\}$?
  - Note that

    $B = (A \cap B) \cup (A^c \cap B) \Rightarrow P\{B\} = P\{A \cap B\} + P\{A^c \cap B\}.$
  - Therefore,

$$\begin{aligned} P\{B\} &= P\{B|A\}P\{A\} + P\{B|A^c\}P\{A^c\} \\ &= P\{B|A\}P\{A\} + P\{B|A^c\}(1 - P\{A\}) . \end{aligned}$$

  - Here we are finding $P\{B\}$ by "conditioning" on A .
  - In general, if the realization of B depends on a *partition* A_i of Ω, $A_1 \cup A_2 \cup \ldots \cup A_n = \Omega$, $A_i \cap A_j = \varnothing$, $(i, j) \in \{1, \ldots, n\}^2, i \neq j$,

$$P\{B\} = \sum_{i=1}^{n} P\{B \mid A_i\} P\{A_i\} .$$

- **Bayes' Formula**
  - This follows from conditional probabilities. For two events,

$$P\{A \mid B\} = \frac{P\{A \cap B\}}{P\{B\}} = \frac{P\{B \mid A\}P\{A\}}{P\{B \mid A\}P\{A\} + P\{B \mid A^c\}P\{A^c\}}.$$

  - With a partition A_i,

$$P\{A_j \mid B\} = \frac{P\{A_j \cap B\}}{P\{B\}} = \frac{P\{B \mid A_j\}P\{A_j\}}{\sum_{i=1}^{n} P\{B \mid A_i\}P\{A_i\}}.$$

- **Example 6**
  - Consider two urns. The first urn contains three white and seven black balls, and the second contains five white and five black balls. We flip a coin and then draw a ball from the first urn or the second urn depending on whether the outcome was heads or tails.
  - What is the probability that a white ball is selected?

    $P\{W\} = \mathrm{P}\{W|H\}\mathrm{P}\{H\} + \mathrm{P}\{W|T\}P\{T\} = (3/10)(1/2) + (5/10)(1/2)$

    $= 2/5$.
  - What is the probability that a black ball is selected?

    $P\{B\} = 1 - P\{W\} = 3/5$.
  - What is the probability that the coin has landed heads given that a white ball is selected?

    From Bayes' formula, $P\{H \mid W\} = \frac{P\{W \mid H\}P\{H\}}{P\{W\}} = \frac{(3/10)(1/2)}{2/5} = \frac{3}{8}$.

- **Example 7**
  - In a n assembly plant, three machines, M_1, M_2, and M_3, make 30%, 45%, and 25%, respectively of the products. It is known that 2%, 3%, and 2% of the products made by each machine, respectively, are defective.
  - What is the probability that a randomly selected product is defective?

    $P\{D\} = P\{D|M_1\}P\{M_1\} + P\{D|M_2\}P\{M_2\} + P\{D|M_3\}P\{M_3\}$

    $= (0.02)(0.3) + (0.03)(0.45) + (0.02)(0.25) = 0.0245$.
  - If a product is found to be defective, what is the probability that it's made by M_2?

$$P\{M_2 \mid D\} = \frac{P\{D \mid M_2\}P\{M_2\}}{P\{D\}} = \frac{(0.03)(0.45)}{0.025} = 0.551\,.$$

- **Example 8**
  - In answering a question on a multiple-choice test, a student either knows the answer or guesses. Let p be the probability that she knows the answer. Assume that the student who guesses the answer will answer correctly with probability $1/m$, where m is the number of multiple-choice alternatives.
  - What is the probability that a student answers a question correctly?

    $P\{\text{Correct}\}=P\{\text{Correct}|\text{Know}\}P\{\text{Know}\}+P\{\text{Correct}|\text{Guess}\}P\{\text{Guess}\}$

    $= (1)(p) + (1/m)(1-p)$

    $= p + (1-p)/m$
  - E.g., if you know the answers to half of EEE questions ($m=4$, $p = 1/2$), then the probability of a correct answer is 1/2 + 1/8 = 5/8, and your likely score is 500/800 .
  - What is the conditional probability that a student guessed the question given that she answered correctly?

    $\text{P}\{\text{Guess}|\text{Correct}\} = P\{\text{Correct}|\text{Guess}\}P\{\text{Guess}\}/P\{\text{Correct}\}$

    $= (1/m)\,(1-p)/[p + (1-p)/m]$

    $= (1-p)/[mp + (1-p)].$
  - In the EEE case, the conditional probability of guessing given a right answer is (1/2)/(2+1/2) = 1/5.