- **Computational Procedure of the Simplex Method**

  - The optimal solution of a general LP problem is obtained in the following steps:

Step 1. Express the problem in tableau form.

Step 2. Select a starting basic feasible (BF) solution. This step involves two cases:

a) If all the constraints in the original problem are "≤" and RHS > 0, the stack variables give the starting BF solution.
b) Otherwise, a technique called the artificial variables technique is used to give a starting BF solution.

Step 3. Generate new BF solutions using both the optimality and feasibility conditions until the optimal solution is reached.

- **Optimality conditions**

  - For max problems, the entering variable is selected as the nonbasic variable having the most negative coefficient in the objective function row (Z-row). <u>For min problems, select the most positive.</u>
  - Break ties at random.
  - If all Z-row coefficients are nonnegative (for max problems) or nonpositive (for min problems), the optimal solution has been reached. Note the optimal solution. Stop.

- **Feasibility conditions**

  - The leaving variable for max and min problems is the basic variable with the smallest ratio of the RHS to the *positive* constraint coefficient of the entering variable (denominator).
  - If denominator ≤ 0, ignore the corresponding variable.
  - Break ties at random.

- **Artificial variables technique (the "Big" M method)**
  - For problems with " $\geq$" and "=" constraints, the slack variables cannot provide a starting feasible solution.
  - Use the Big-M method as follows.
  - Add nonnegative "artificial" variables to the left hand side of each of the equations corresponding to constraints of the " $\geq$" and "=" type.
  - Choose a starting BF solution with these artificial variables as basic variables.
  - To drive the artificial variables out of the basis add a very large unit penalty to these variables in the objective function. (-M for max problems and +M for min problems).
  - Apply simplex iterations. If the LP has a solution, this will lead aa real (non-artificial) optimal solution.

- **Big-M example**

$$
\begin{aligned}
\min \quad & Z = 2x_1 + x_2 \\
\text{s.t.} \quad & 3x_1 + \ \ x_2 = 3 \\
& 4x_1 + 3x_2 \geq 6 \\
& x_1 + 2x_2 \leq 3 \\
& x_i \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{nin} \quad & Z = 2x_1 + x_2 + MR_1 + MR_2 \\
\text{ST} \quad & 3x_1 + \ \ x_2 + R_1 \qquad\qquad\qquad = 3 \\
& 4x_1 + 3x_2 \qquad - S_2 + R_2 \quad = 6 \\
& x_1 + 2x_2 \qquad\qquad\qquad + S_3 = 3 \\
& x_i, S_i, R_i \geq 0
\end{aligned}
$$

  - An initialization step is requires to put the tableau in "the right form". I.e., to have the identity matrix in the artificial variable columns.
  - Then, the simplex method proceeds as usual.

Initialization step,
Set $l_0' = l_0 + Ml_1 + Ml_2$

	x_1	x_2	S_2	R_1	R_2	S_3	RHS	ratio
Z	-2	-1	0	-M	-M	0	0	--
R_1	3	1	0	1	0	0	3	--
R_2	4	3	-1	0	1	0	6	--
S_3	1	2	0	0	0	1	3	--

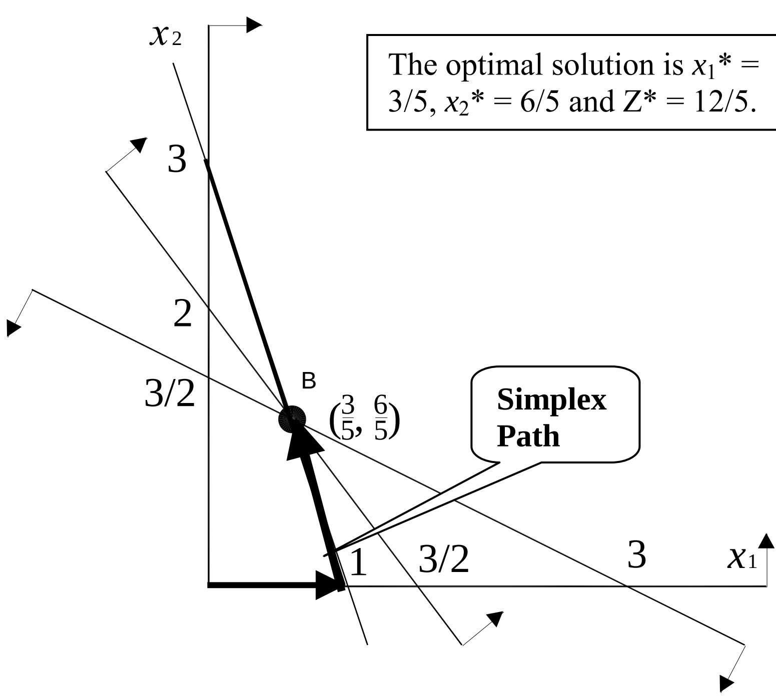
	x_1	x_2	S_2	R_1	R_2	S_3	RHS	ratio
Z	7M-2	4M-1	-M	0	0	0	9M	--
R_1	3	1	0	1	0	0	3	1
R_2	4	3	-1	0	1	0	6	3/2
S_3	1	2	0	0	0	1	3	3

	x_1	x_2	S_2	R_1	R_2	S_3	RHS	
Z	0	(5M-1)/3	-M	(-7M+2)/3	0	0	2M+2	
x_1	1	1/3	0	1/3	0	0	1	3
R_2	0	5/3	-1	-4/3	1	0	2	6/5
S_3	0	5/3	0	-1/3	0	1	2	6/5

Degeneracy

	x_1	x_2	S_2	R_1	R_2	S_3	RHS
Z	0	0	-1/5	2/5-M	1/5-M	0	12/5
x_1	1	0	1/5	3/5	-1/5	0	3/5
x_2	0	1	-3/5	-4/5	3/5	0	6/5
S_3	0	0	1	1	-1	1	$\boxed{0}$

The optimal solution is $x_1^* = 3/5$, $x_2^* = 6/5$ and $Z^* = 12/5$.



- **Variants of the Simplex Method**
  - **Degeneracy**. A BF solution is said to be degenerate if more than *m* constraints are binding at it, where *m* is the number of decision variables.
  - The point B above represents a degenerate solution.
  - Degeneracy can be detected from the simplex tableau when two nonnegative "ratios" are equal.
  - At a degenerate solution at least one basic variable equal to zero. E.g., In the optimal tableau above (corresponding to B), $S_3 = 0$, although S_3 is a basic variable.
  - In some *rare* cases, degeneracy leads to *cycling*, where the simplex algorithm gets "stuck" at a degenerate corner point for ever (the simplex cycles indefinitely between bases corresponding to this point). See the cycling example in this set.
  - In other equally *rare* cases, degeneracy can lead to *stalling* where the simplex method spends a long time moving between degenerate bases at a point before moving to another point.

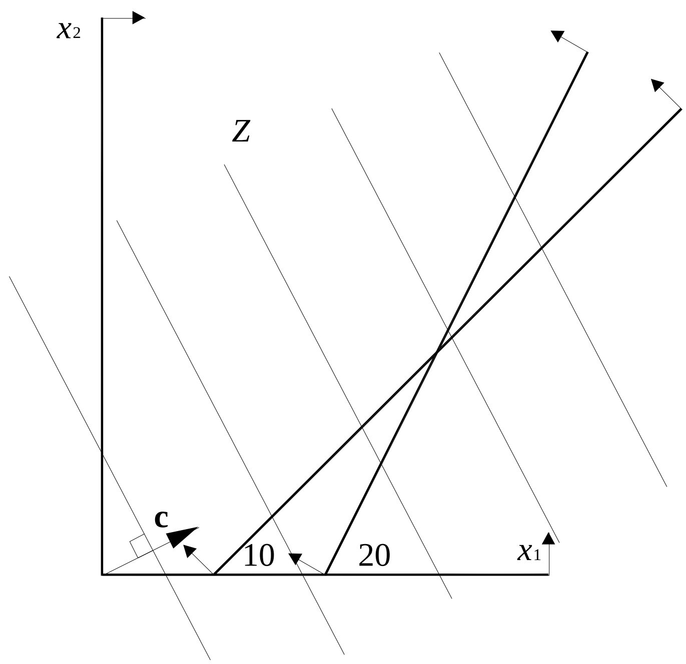
  - **Unboundedness**. An LP feasible region is said to be unbounded if one decision variable can be increased indefinitely without violating the constraints. If, in addition, the objective function increases (decreases) indefinitely with the decision variable increase, then the optimal solution is unbounded.
  - In the simplex tableau, if all the constraint coefficients corresponding to a nonbasic variable are ≤ 0, then the feasible region is unbounded. If, in addition, the coefficient of the nonbasic variable in the Z-row is negative (for a max problem) or positive (for a min problem), then the optimal solution is unbounded. E.g.,

$$\begin{aligned} \max \quad & Z = 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 - x_2 \le 10 \\ & 2x_1 - x_2 \le 40 \\ & x_1, x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \max \quad & Z = 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 - x_2 + S_1 \qquad = 10 \\ & 2x_1 - x_2 \qquad + S_2 = 40 \\ & x_1, x_2, S_1, S_2 \ge 0 \end{aligned}$$

Unbounded Solution

	x_1	x_2	S_1	S_2	RHS	ratio
Z	-2	-1	0	0	0	--
S_1	1	-1	1	0	10	
S_2	2	-1	0	1	40	



- **Alternate Optima**. This happens when the objective function is "parallel" to one of the constraints which are binding at an optimal solution. In this case, there exist infinitely many optima all having the same (optimal) objective value.

- It means that the objective can be achieved in several ways.
- In the simplex tableau, an alternate optimal situation is detected when one of the coefficients corresponding to a nonbasic var iable in the Z-row is zero. E.g.,

$$\begin{aligned} \max \quad & Z = 2x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \le 5 \\ & x_1 + x_2 \le 4 \\ & x_1, x_2 \ge 0 \end{aligned}$$

	x_1	x_2	S_1	S_2	RHS	ratio
Z	-2	-4	0	0	0	--
S_1	1	2	1	0	5	5/2
S_2	1	1	0	1	4	4

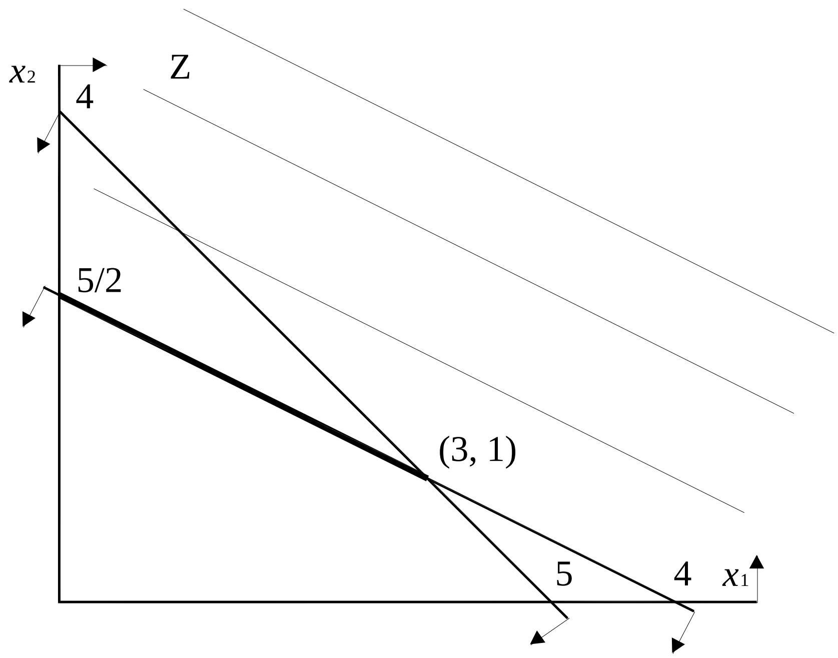
Alternate Optima

	x_1	x_2	S_1	S_2	RHS	ratio
Z	$\boxed{0}$	0	2	0	**10**	--
x_2	1/2	1	1/2	0	5/2	5
S_2	1/2	0	-1/2	1	3/2	3

Alternate Optima

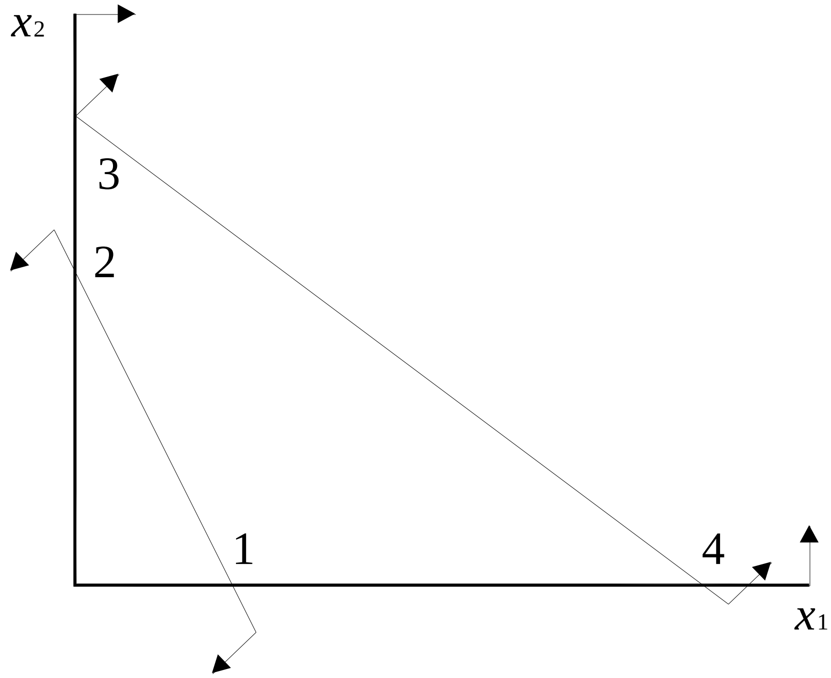
	x_1	x_2	S_1	S_2	RHS
Z	0	0	2	$\boxed{0}$	**10**
x_2	0	1	1	-1	1
x_1	1	0	-1	2	3

- We conclude that the segment joining the two points (0, 5/2) and (3, 1) defines the set of alternate optima with a Z* = 10.



- **Infeasible Solution**. This means the constraints of the LP cannot be satisfied by any solution (i.e., the constraints are conflicting).
  - With nonnegative decision variables, infeasible solutions are encountered with "=" or "≥" constraints.
  - In this case, the Big-M method terminates with an artificial variable being basic (see text example 3.5-4, p. 121, text).

$$
\begin{aligned}
\text{max} \quad & Z = 3x_1 + 2x_2 \\
\text{s.t.} \quad & 2x_1 + x_2 \le 2 \\
& 3x_1 + 4x_2 \ge 12 \\
& x_1, x_2 \ge 0
\end{aligned}
$$


x_2
3
2
1
4
x_1