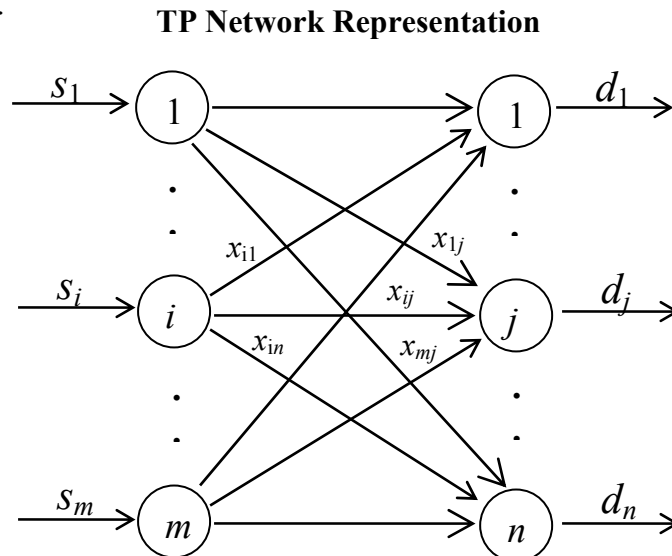


The Transportation Problem (1)

- **Definition**

- The transportation problem (TP) is concerned with shipping a commodity between a set of sources (e.g. manufacturers) and a set of destinations (e.g. retailers).
- Each source has a capacity dictating the amount it supplies.
- Each destination has a demand dictating the amount it receives.



- For *now*, we assume that the system is *balanced*.
- That is, the total amount demanded at all destinations is equal to the total supply available at all sources.
- The objective is to determine the amounts to be shipped between each source and destination in a way as to minimize shipping cost while meeting demand and supply constraints.
- The shipping cost is proportional to the amount shipped.

- **Overview**

- The TP has many practical applications. (The one I like the most is about snow moving in Montreal.)
- The transportation problem can be modeled and solved as a linear program.
- However, this problem has a special *structure*, which allows for specialist solution techniques that are more efficient than the usual simplex algorithm.
- In particular, we will see that starting basic feasible solutions of high quality can be obtained with ease.
- We will then customize the simplex method to produce an efficient *transportation simplex method*.

- **Prototype example and LP Formulation**

- A contractor is hauling gravel to three construction sites. He can purchase 20 tons at a gravel pit in the north of the city and 10 tons at one in the east. He needs 8, 8, and 14 tons at sites A, B, and C, respectively. The hauling costs are as follows:

Hauling cost, c_{ij} (\$/ ton)			
	A (1)	B (2)	C (3)
North (1)	7	5	4
East (2)	6	2	6

- For this problem, the decision variables are x_{ij} the amount of gravel to haul between pit (source) i and site (destination) j , $i = 1, 2$, and $j = 1, 2, 3$.
- The LP formulation is as follows.

$$\begin{aligned}
 \min \quad & Z = 7x_{11} + 8x_{12} + 4x_{13} + 6x_{21} + 2x_{22} + 6x_{23} \\
 \text{subject to} \quad & \\
 & x_{11} + x_{12} + x_{13} = 20 \\
 & x_{21} + x_{22} + x_{23} = 10 \\
 & x_{11} + x_{21} = 8 \\
 & x_{21} + x_{22} = 8 \\
 & x_{13} + x_{23} = 14 \\
 & x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0
 \end{aligned}$$

- Let
 - The number of sources be m ($m = 2$);
 - The number of destination be n ($n = 3$);
 - The shipping cost between source i and destination j be c_{ij} .
 - The supply at source i be s_i .
 - The demand at destination j be d_j .

- Then, the compact formulation, which applies to any TP, is

$$\min \quad Z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^n x_{ij} = s_i, i = 1, \dots, m$$

$$\sum_{i=1}^m x_{ij} = d_j, j = 1, \dots, n$$

$$x_{ij} \geq 0$$

- **Tableau Representation**

- It is convenient to represent TP in a tableau format.
- For our prototype example, the *transportation tableau* is

	1	2	3	Supply
1	7	5	4	20
2	6	2	6	10
Demand	8	8	14	

- **Obtaining a Starting Basic Feasible Solution**

- This is relatively easy and can be done in a number of ways.
- We shall discuss the *Northwest corner*, the *least-cost*, and *Vogel's approximation* methods.
- These methods give “good” solutions, which can be directly implemented, or used as starting solutions for procedures that give optimal solutions. These methods are called *heuristics*.

- **The NW Corner Method**

➤ This method operates as follows.

1. Start with all cells in the transportation tableau empty.
2. Assign the highest possible value to the NW corner.
3. Cross out NW cell column (row), if demand < (>) supply.
4. If NW cell supply = demand cross out row *or* column.
5. Repeat until all demand and supply have been assigned.

➤ For our prototype example, the NW corner method solution is as follows.

	1	2	3	Supply
1	7 ⓪	5 ⓪	4 ⓪	2 0 1 2 4 0
2	6 ⓪	2 ⓪	6 ⓪	1 0 0
Demand	8 0	8 0	4 1 0 0	

➤ Therefore, the NW solution is $x_{11} = 8$, $x_{12} = 8$, $x_{13} = 4$, $x_{21} = 0$, $x_{22} = 0$, $x_{23} = 10$, and $Z = 7 \times 8 + 5 \times 8 + 4 \times 4 + 10 \times 6 = 172$.

- **The Least-Cost Method**

➤ This method operates as follows.

1. Start with all cells in the transportation tableau empty.
2. Assign the highest possible value to the cell with least cost.
3. Cross out least cost cell column or row and break ties as in the NW method.
4. Repeat until all demand and supply have been assigned.

➤ For our prototype example, the Least-Cost method solution is

	1	2	3	Supply
1	7 (6)	5	4 (14)	20 6
2	6 (2)	2 (8)	6	10 2
Demand	8 6	8 0	14 0	

➤ Therefore, the Least-Cost method solution is $x_{11} = 6$, $x_{12} = 0$, $x_{13} = 14$, $x_{21} = 2$, $x_{22} = 8$, $x_{23} = 0$, and $Z = 7 \times 6 + 4 \times 14 + 6 \times 2 + 2 \times 8 = 126$.

• Vogel's Approximation Method

➤ This method operates as follows.

1. Start with all cells in the transportation tableau empty.
2. Compute a “penalty” for each row and column as the difference of the least cost and the next smallest cost.
3. Identify the row or column with highest penalty.
4. Assign the highest possible value to the cell with least cost in the row or column identified in (3).
5. Cross out corresponding row or column and break ties as in the NW and least-cost methods.
6. Repeat until one row or column remains uncrossed.
7. Assign values to last row or column according to least cost.

- The penalty of a row or column in (2) is the minimum extra cost incurred by failing to make an assignment to the cell with the least cost in that row or column.
- By operating on rows and columns with high penalties, Vogel's method avoids large costs.
- For our prototype example, Vogel's method solution is

	1	2	3	Supply	Penalty
1	7 ⑥	5	4 ⑭	20 6 0	1 3
2	6 ②	2 ⑧	6	10 2 0	4 0
Demand	8 6 0	8 0	14 0		
Penalty	1	3	2		

- Therefore, the Vogel's method solution is the same as the Least-Cost method solution. (Later we shall see that this solution is in fact optimal.)

- **Comparison of NW Corner, Least-Cost and Vogel's Methods**

- In terms of computational effort, NW Corner requires the least effort followed by Least-Cost and then Vogel's method.
- In terms of quality of solutions, research has shown that Vogel's method gives lowest cost followed by Least-Cost and then NW Corner method in most cases.
- Moreover, when used as a starting solution to the transportation simplex method, Vogel's method solution leads to the optimal solution in the least number of iterations.
- The most efficient approach which leads to an optimal solution is believed to be one that starts with Vogel's method in most cases.¹

¹ However, if you are away from your computer talking about transportation costs at a cocktail party, then the NW Corner and Least-Cost Method are definitely enough to impress your audience.