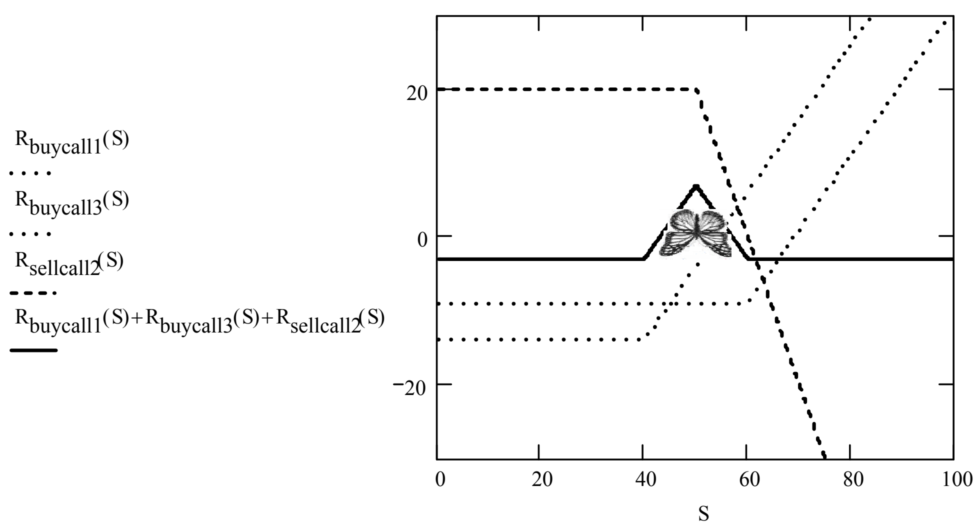
Chapter 12 Basic Option Theory (2)

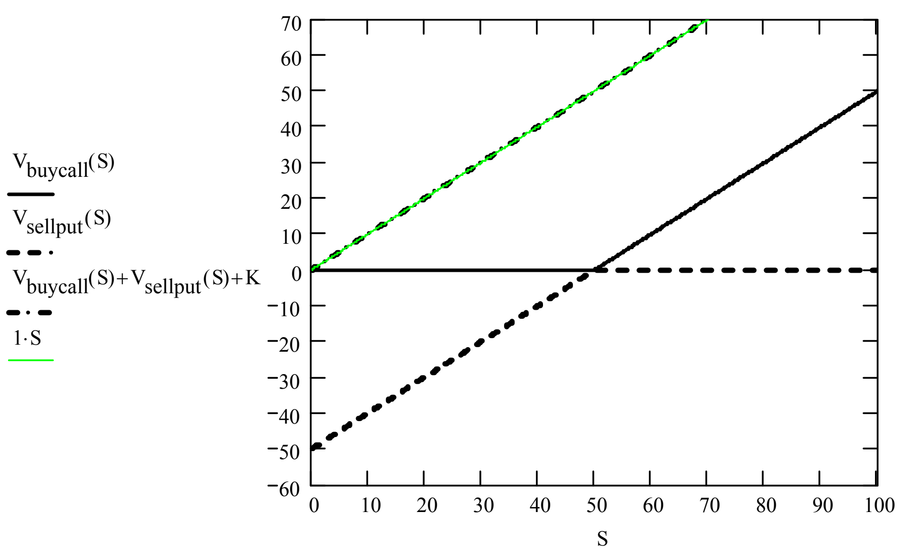
- **Option Combinations**
  - Combinations of options and stock can approximate any payoff function by a piece-wise linear function.
  - A common option combination is the *butterfly spread*. It is constructed by buying two calls with strike prices K_1 and K_3 and by selling two calls with strike price K_2, $K_1 < K_2 < K_3$.
  - The following figure shows the payoff at of a butterfly spread with $K_1 = 40$, $K_2 = 50$, $K_3 = 60$, $C_1 = 14$, $C_2 = 10$, and $C_3 = 9$, where C_i is the price of the option with strike price K_i.[1]



  - The butterfly spread yields a positive profit if the stock price at expiration is close to K_2, and a small loss, otherwise.
  - K_2 is usually close to current stock price. So butterfly spread is used if one believes that the stock price will not vary much.

[1] The payoffs at expiration are $R_{buycall1}(S) = \max(S, K) - C_1$ and $R_{buycall1}(S) = \max(S, K) - C_3$ and $R_{sellcall2}(S) = 2\,[C_2 - \max(S, K)]$.

- **Put-Call Parity**
  - For European options, there is a simple relationship between the price of a call and a put option with the same strike price, *K*, and expiration time, *T*.
  - This relationship is found by noting that a combination of buying a call, selling a put, and lending an amount $d(0, T)K$, where $d(0, T)$ is the discount factor between times 0 and *T*, are equivalent to buying and holding the stock.
  - This fact is best understood graphically. The following figure illustrates this for $K = 50$.[2]



  - Therefore, the put-call parity relation ship is as follows.

$$P - C - d(0,T)K = -S_0 \Leftrightarrow C - P + d(0,T)K = S_0,$$

where S_0 is the current stock price.

[2] In the figure, the values of buying the call and selling the put are $V_{buycall}(S) = \max(0, S-K)$ and $V_{sellput}(S) = -\max(0, K-S)$.

- **Early Exercise (Ross 2003)**

  Proposition *One should never exercise an American style call option before its expiration time T.*

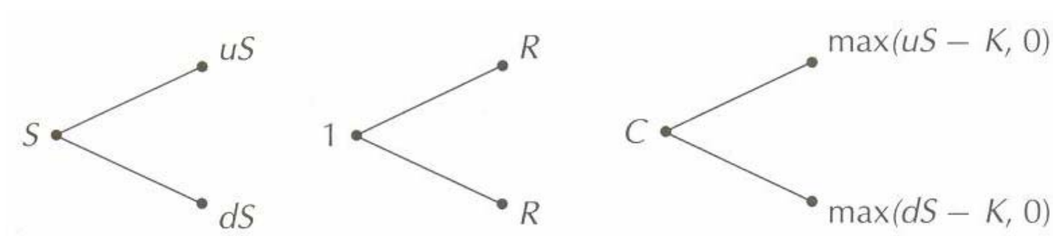
  Proof. Suppose that the stock price at time $t < T$ is S_t, and that you own an American call option on this stock with strike price K which is valid up to T. If you exercise the option at time t, you realize the amount $S_t - K$. If instead of exercising the option at time t, you sell the stock short at time t (and return it at T), and then exercise the option at T (if profitable), then your profit is

$$\begin{cases} S_t / d(t,T) - S_T + S_T - K = S_t / d(t,T) - K > (S_t - K)/d(t,T), & \text{if } S_T > K, \\ S_t / d(t,T) - S_T > (S_t - K)/d(t,T), & \text{if } S_T < K, \end{cases}$$

  where $d(t, T)$ is the discount factor between t and T, and S_T is the stock price at expiration. ■

- **Single-Period Binomial Options Theory**
  - ➢ Consider a stock whose price is governed by a single-period binomial lattice.
  - ➢ The initial stock price is S. At the end of the period the stock price will be uS (dS) w.p. p $(1-p)$, where $0 < d < 1 < u$.
  - ➢ One can also borrow or lend at the risk free rate r per period.
  - ➢ Let $R = 1+r$. To avoid arbitrage, we must have $u > R > d$.
  - ➢ To see this, note that if $R \geq u > d$, then one can short the stock and lend the proceeds and make arbitrage profit.

- If $u > d \geq R$, then one can borrow money and invest in stock, and make arbitrage profit.
- Suppose that we want to determine the price of a call option on this stock with strike price K.
- The option price is determined based on the no-arbitrage principal.
- To do so, we construct a *replicating portfolio* composed of x dollars worth of stock and b dollars invested at the risk-free rate, such that the portfolio return duplicates the option.
- The returns of the stock, the risk-free investment, and the option are as illustrated in the following figure.



- To duplicate the option, the replicating portfolio must satisfy

$$
\begin{aligned}
ux + Rb &= C_u \\
dx + Rb &= C_d \ ,
\end{aligned}
$$

where $C_u = \max(uS - K, 0)$ and $C_u = \max(ud - K, 0)$.

- Solving for x and b we obtain

$$x = \frac{C_u - C_d}{u - d}, \quad b = \frac{uC_u - dC_d}{R(u-d)}.$$

- Therefore, the value of the replicating portfolio is

$$x + b = \frac{1}{R}\left(\frac{R-d}{u-d} C_u + \frac{u-R}{u-d} C_d \right) \ .$$

- In order, to avoid arbitrage, the option price (value) should equal the replicating portfolio value. That is,

$$C = \frac{1}{R}\left(\frac{R-d}{u-d}C_u + \frac{u-R}{u-d}C_d\right) .$$

- Note that the option price is independent of p, the probability of an upward move in the binomial tree.
- This is because the replicating portfolio duplicates the option in both "up" and "down" states regardless of the value of p.

• **Risk-Neutral Pricing**

- The single-period option pricing formula can be written as

$$C = \frac{1}{R}\left[qC_u + (1-q)C_d\right] ,$$

  where $q = (R - d)/(u - d)$.
- The probability q can be seen as the *risk-neutral probability*.
- Interestingly, q can be obtained by applying risk-neutral pricing to the stock

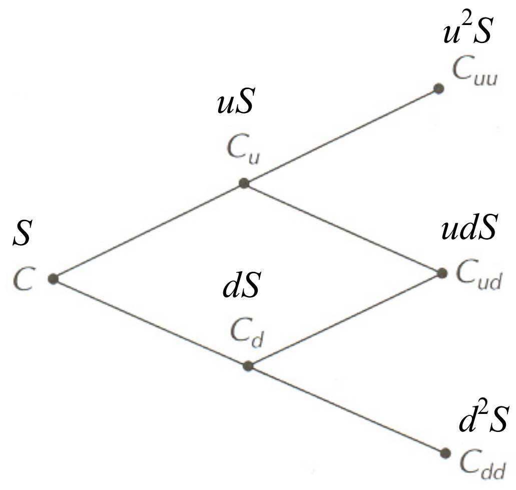
$$S = \frac{1}{R}\left[quS + (1-q)dS\right] \Rightarrow q = \frac{R-d}{u-d} .$$

- Alternatively, the single-period option price can be written as

$$C(T-1) = \hat{E}[C(T)] , \tag{1}$$

  where T is the expiration time, $C(T)$ and $C(T-1)$ are the call values at T and $T-1$, and $\hat{E}$ is the risk-neutral expectation.

- **Multi-Period Options**
  - The single-period pricing method is extended to multi-period setting by working backward one period at a time using (1).
  - For example, the price of a call option expiring in two periods is obtained using the binomial lattice below. (The stock price is shown above the option value at each node.)



  - In this lattice, one first calculates the terminal values $C_{uu} = \max(u^2S - K, 0)$, $C_{ud} = \max(udS - K, 0)$, and $C_{dd} = \max(d^2S - K, 0)$.
  - Then, values in period 1 are found,

$$C_u = \frac{1}{R}\left[qC_{uu} + (1-q)C_{ud}\right], \quad C_d = \frac{1}{R}\left[qC_{uu} + (1-q)C_{dd}\right] .$$

  - Finally, the option price is determined by

$$C = \frac{1}{R}\left[qC_u + (1-q)C_d\right] .$$

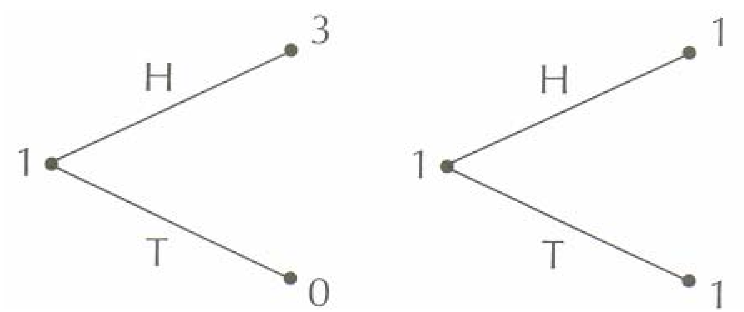
- **Put Options**
  - Finding the price of a European put option using a binomial lattice is similar to finding a call option price. Only terminal values are different.
  - An American put option might be exercised early.[3]
  - Therefore, the value at each node of the lattice is the maximum of the value obtained from immediate exercise and the value obtained by risk-neutral discounting.

- **Option on a Stock with Dividends**
  - If the stock pays a dividend proportional to the value of the stock in period k; i.e., the dividend is δS_k ($0 < \delta < 1$).
  - Then, the factors of the binomial lattice are changed from u and d to $u(1-\delta)$ and $d(1-\delta)$.
  - Exercise 12.5 in the text handles the case of a fixed dividend.

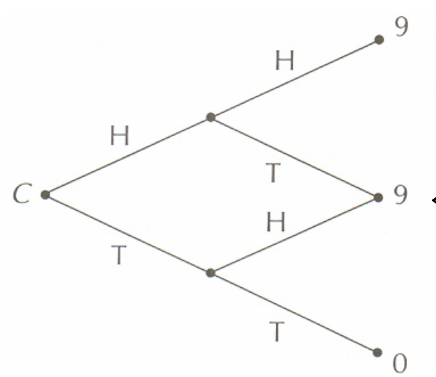
- **Real Options**
  - Sometimes options are associated with investment opportunities that are not financial instruments.
  - E.g., the option of buying a home, hiring new employees, buying new equipment, introduce a new product, etc.
  - It is possible to view any process that allows control as a process with a series of *operational* or *real* options.
  - Option theory can be used to evaluate real options.

[3] This is because there is a limit on a put option profit (equal to the strike price K per share). If the stock price approaches zero at a time before expiration, the holder is better off exercising the option at this time.

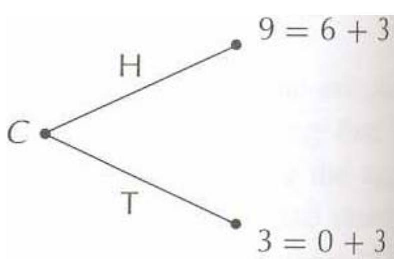
- **Linear Pricing**
  - Risk-neutral pricing is actually linear pricing.
  - I.e., the price of a derivative security is equal to the sum of securities which when bundled together are equivalent to the security. (Remember the replicating portfolio!)
  - Example 12.11 in the text is a nice example of linear pricing.
  - In a simple world, there are two investment opportunities.
  - First, for \$1 flipping a coin pays \$3 if it's heads and \$0, otherwise.
  - Second, keeping the money in your pocket.



  - Consider a third investment where a coin is flipped twice and one gets \$9 if at least one of the flips is heads.
  - How much should you pay for this third opportunity?



$\Leftrightarrow$



$\Leftrightarrow C = 2 + 3 = \$5$