Duality

- **Overview**
  - Every LP problem has a twin problem associated with it. One problem is known as the *primal*, and the other as the *dual*.
  - The properties of the two problems are closely related, and they have the same optimal solution.
  - The reasons we care about the dual of an LP are
    - Utilizing the dual solution, we can do a sensitivity analysis of the LP. That is, we assess the effect of changing the problem parameters on the optimal solution.[1]
    - It may be easier to solve the dual first and then deduce the primal solution than to solve the primal directly.

- **Construction of the dual problem**
  - If the primal is a max (min) problem then the dual is a min (max) problem.
  - To each primal constraint corresponds a dual variable.
  - The objective function coefficients of the dual are the RHS values of the primal.
  - The sign of dual variables is based on the type of the corresponding primal constraint ("≤", "≥", or "=") as indicated in the table below.
  - To each primal variable corresponds a dual constraint.
  - The type of a dual constraint is defined based on the sign of the corresponding primal variable as indicated in the table below.

[1] This is very important in practice where accurate data is seldom available.

- The coefficients of a dual constraint are the constraint coefficients of the corresponding primal variable.

Primal	**Dual**
“max” problem constraint	**“min” problem variable**
$\geq$	≤ 0
$\leq$	≥ 0
$=$	unrestricted
“max” problem variable	**“min” problem constraint**
≥ 0	$\geq$
≤ 0	$\leq$
unrestricted	$=$

- **Examples**

  - Example 1
    - Primal

$$\begin{array}{llll}
 & \overset{\text{Const. 1}}{\downarrow}\quad\overset{\text{Const. 2}}{\downarrow} & & \\
\max & Z = 5x_1 + 2x_2 & & \\
\text{s.t.} & x_1 + x_2 & \leq 100 & \longleftarrow y_1 \\
 & 4x_1 - x_2 & \geq 20 & \longleftarrow y_2 \\
 & 4x_1 + x_2 & = 80 & \longleftarrow y_3 \\
 & x_1 \text{ unrestricted}, \ x_2 \leq 0 & &
\end{array}$$

    - Dual

$$\begin{array}{lll}
\min & w = 100y_1 + 20y_2 + 80y_3 & \\
\text{s.t.} & y_1 + 4y_2 + 4y_3 & = 5 \\
 & y_1 - y_2 + y_3 & \leq 2 \\
 & y_1 \geq 0, \ y_2 \leq 0, \ y_3 \text{ unrestricted} &
\end{array}$$

- Example 2
  - Primal

$$\max\ Z = 5x_1 + 12x_2 + 4x_3$$
$$\text{s.t.}\quad x_1 + 2x_2 + x_3 \le 5$$
$$2x_1 - x_2 + 3x_3 = 2$$
$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$$

  - Dual

$$\min\ w = 5y_1 + 2y_2$$
$$\text{s.t.}\quad y_1 + 2y_2 \ge 5$$
$$2y_1 - y_2 \ge 12$$
$$y_1 + 3y_2 \ge 4$$
$$y_1 \ge 0,\ y_2 \text{ unrestricted}$$

- Example 3
  - Primal

$$\min\ Z = 5x_1 + 2x_2$$
$$\text{s.t.}\quad x_1 + 2x_2 \ge 5$$
$$2x_1 - x_2 \ge 12$$
$$x_1 + 3x_2 \ge 4$$
$$x_1 \ge 0,\ x_2 \text{ unrestricted}$$

  - Dual

$$\max\ w = 5y_1 + 12y_2 + 4y_3$$
$$\text{s.t.}\quad y_1 + 2y_2 + y_3 \le 5$$
$$2y_1 - y_2 + 3y_3 = 2$$
$$y_1 \ge 0,\ y_2 \ge 0,\ y_3 \ge 0$$

- The "dual of the dual" is the primal.

- **Canonical form of duality**
  - A LP can be written as

$$\begin{aligned} \text{(P)} \quad \max \quad & Z = \mathbf{cx} \\ \text{s.t.} \quad & \mathbf{Ax} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{aligned}$$

  - The dual is then given by

$$\begin{aligned} \text{(D)} \quad \max \quad & \mathbf{w} = \mathbf{yb} \\ \text{s.t.} \quad & \mathbf{yA} \ge \mathbf{c} \\ & \mathbf{y} \ge \mathbf{0} \end{aligned}$$

  - This form of duality allows us to prove the following important properties on primal-dual relationships.

- **Primal-dual relationships**
  - <u>Weak duality property</u>

    Theorem 1 *Let $\boldsymbol{x}^0$ and $\boldsymbol{y}^0$ be feasible solutions to (P) and (D), respectively. Then, $Z(\boldsymbol{x}^0) \le w(\boldsymbol{y}^0)$. That is, $\boldsymbol{cx^0} \le \boldsymbol{y^0 b}$.*

    Proof. Suppose w.l.g. that the LP is in canonical form. The feasibility of $\mathbf{x}^0$ and $\mathbf{y}^0$ imply that $\mathbf{Ax}^0 \le \mathbf{b}$ and $\mathbf{y}^0\mathbf{A} \ge \mathbf{c}$. Then, $\mathbf{cx^0} \le \mathbf{y}^0\mathbf{Ax^0} \le \mathbf{y^0b}$. ■

  - <u>Unboundedness/Infeasibility</u>

    Lemma 1 (i) *If (P) is unbounded $\Rightarrow$ (D) is infeasible.*
    (ii) *If (D) is unbounded $\Rightarrow$ (P) is infeasible.*

    Proof. Suppose (D) has a feasible solution $\mathbf{y}^0$. If (P) is unbounded then there exists $\mathbf{x}^0$ such that $Z(\mathbf{x}^0) > w(\mathbf{y}^0)$. This contradicts weak duality (Theorem 1). Similar proof for (ii). ■

    Remark. The converse of Lemma 1 is not always true.

- Strong duality

Lemma 2 *If (P) has an optimal solution, then (D) also has an optimal solution, and vice versa. In addition, when the two problems have optimal solutions, they both have the same optimal (finite) objective value.*

Proof. Let $\mathbf{x}^*$ be an optimal solution to (P). Recall that this solution is obtained from $\mathbf{x_B}^* = \mathbf{B}^{-1}\mathbf{b}$ and $\mathbf{x_N}^* = \mathbf{0}$ (see notes on revised simplex method). Recall also that at optimality, $\mathbf{c_B B^{-1} N}_j - c_j \geq 0$ for all nonbasic variables, x_j, and $\mathbf{c_B B^{-1} B}_i - c_i = 0$, for all basic variables x_i. Let $\pi = \mathbf{c_B B^{-1}}$, then it follows that $\pi\mathbf{A} - \mathbf{c} \geq \mathbf{0}$. This implies that $\pi\mathbf{A} \geq \mathbf{c}$, and π is feasible for (D). Now, observe that $w(\pi) = \mathbf{c_B B^{-1} b} = Z(\mathbf{x}^*)$. In addition, weak duality (Theorem 1) implies that $w(\pi) = Z(\mathbf{x}^*) \leq w(\mathbf{y}^0)$, which proves that π is an optimal solution to (D). Similar proof for the "vice versa" part. ■

Fact 1 The solution to (D) is given by $\mathbf{y}^* = \pi = \mathbf{c_B B^{-1}}$ and $w(\mathbf{y}^*) = \mathbf{c_B B^{-1} b}$.

- Complementary slackness

Lemma 3 *Optimal solutions* $\mathbf{x}^*$ *for the primal and* $\mathbf{y}^*$ *for the dual are related by*

$$x_j^*(\mathbf{y}^*\mathbf{A}_j - \mathbf{c}_j) = 0, \text{ for all primal variable } x_j^*$$

and

$$y_i^*(\mathbf{A}^i\,\mathbf{x}^* - b_i) = 0, \text{ for all dual variables } y_i^*$$ [2]

Proof. If $x_j^* \neq 0$, then x_j is basic, and $\mathbf{y}^*\mathbf{A}_j - \mathbf{c}_j = \mathbf{c_B B^{-1} B}_j - c_j = 0$. In addition, if $\mathbf{y}^*\mathbf{A}_j - \mathbf{c}_j > 0$, then x_j^* is nonbasic, and consequently, $x_j^* = 0$. This proves that $x_j^*(\mathbf{y}^*\mathbf{A}_j - \mathbf{c}_j) = 0$. The identity $y_i^*(\mathbf{A}^i\,\mathbf{x}^* - b_i) = 0$ can be proven in a similar fashion.

[2] $\mathbf{A}_j$ denotes the j^{th} column of $\mathbf{A}$ and $\mathbf{A}^i$ denotes the i^{th} row of $\mathbf{A}$.