- **Introduction**
  - A linear program (LP) is a model of an optimization problem with a linear objective function and linear constraints.
  - A LP objective is to determine the values of decision variables that maximize or minimize the objective function.
  - A LP involving maximization with *n* decision variables and *m* constraints is represented as follows

$$
\begin{aligned}
\max\ Z = {} & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{subject to}\ & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \ge b_2 \\
& \ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_n \\
& x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0
\end{aligned}
$$

  - Note that constraints can be of any type "≤", "≥", or "=".
  - The last set of constraints, $x_1 \ge 0$, …., $x_n \ge 0$, are called *nonnegativity* constraints.

- **Example**
  - BM Company produces two products, P_1 and P_2 that are sold at \$3 and \$4 profit margin respectively and require two raw materials, M_1 and M_2. A unit of P_1 requires 3 units of M_1 and 2 units of M_2. A unit of P_2 requires 1 unit of M_1 and 4 units of M_2 . The company has a supply of 6 units of M_1 and 4 units of M_2. The company wants to maximize profit.

- The BM company problem can be modeled with the following LP. Let x_1 and x_2 be the number of units produced of P_1 and P_2. These are the *decision variables*.

$$\begin{aligned} &\max\ Z = 3x_1 + 4x_2 \\ &\text{subject to}\ \ 3x_1 + x_2 \le 6 \\ &\qquad\qquad\quad\ 2x_1 + 4x_2 \le 4 \\ &\qquad\qquad\quad\ x_1 \ge 0, x_2 \ge 0 \end{aligned}$$

- In this LP, the objective function is the profit, $Z = 3x_1 + 4x_2$, and the constraints are based on the availability of raw materials, $3x_1 + x_2 \le 6$ and $2x_1 + 4x_2 \le 4$.

- **Linear programming assumptions**
  - *Proportionality*. The contribution of a decision variable to the objective function and its requirements in the constraints are proportional to the decision variable.
  - *Additivity*. The objective function is the sum of contribution of decision variables to it. A constraint is made up by adding the requirement of each decision variable.
  - *Divisibility*. Decision variables are allowed to assume fractional values.
  - *Certainty*. Parameters values are known with certainty.

- **LP graphical solution**
  - LPs having two decision variables can be solved graphically.
  - The first step of the graphical method is to determine the *feasible region*. This is the set of decision variable values that satisfy all the constraints. This is done using this fact

Fact 1 *The inequality* $a_1x_1 + a_2x_2 \leq b$ *defines a quadrant of the plane bounded from one side by the line* $a_1x_1 + a_2x_2 = b$.

- The second step is to determine the point(s) of the feasible region which correspond(s) to the optimal solution.
- This second step is carried out utilizing the following facts.

Fact 2. *The set of points representing constant values of the objective function,* $Z = c_1x_1 + c_2x_2$, *are parallel straight lines.*

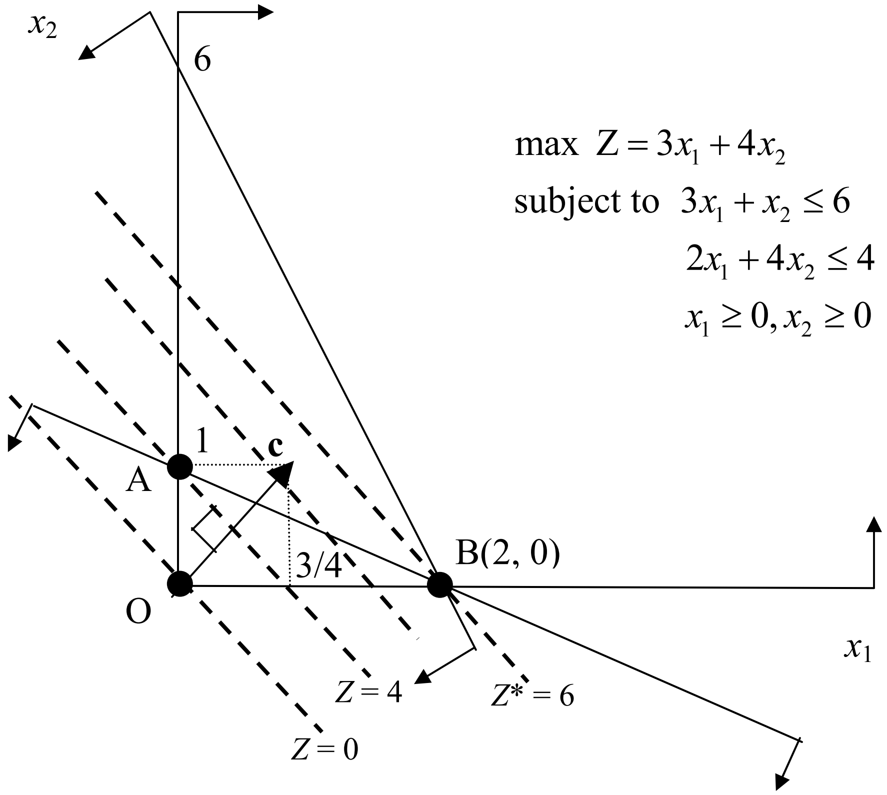
Fact 3. *The lines in Fact* 2 *are perpendicular to the gradient of the objective function defined by the vector* $\mathbf{c} = (c_1, c_2)$.

Fact 4. *Moving in the direction of the gradient increases the objective function.*

Remarks.

1. A point in the feasible region defines a *feasible solution*.
2. The value of the objective function at a feasible solution defines a lower (upper) bound on the optimal objective value for a max (min) problem.
3. The lines in Fact 2 are called *isoprofit* (*isocost*) lines for a max (min) problem.

- E.g., the BM company problem can be solved as follows.



- Step 1. Draw the constraints and define the feasible region. This the region defined by OAB.
- Step 2a. Draw the gradient **c** = (3, 4). Draw an isoprofit line perpendicular to the gradient.
- Step 2b. Move the isoprofit line parallel to itself in the direction indicated by the gradient.
- Step 3c. Stop when the isoprofit line crosses the feasible region at boundary points only. This corresponds to the isoprofit line reaching the point B(2, 0).
- Step 3d. Find the optimal solution. The optimal solution is $x_1^* = 2$, $x_2^* = 0$, and $Z^* = 3\times2 + 4\times0 = 6$.

Remark. For the case of a min problem the isocost line should be moved in the direction *opposite* to that of the gradient in Step 2b.