The Dual Simplex Method

- **Underlying Theory**
  - Consider the LP (P) and its dual (D)

$$
\begin{array}{llll}
\text{(P)} & \min & Z = \mathbf{cx} & \\
& \text{s.t.} & \mathbf{Ax} \geq \mathbf{b} & \leftarrow \mathbf{y} \\
& & \mathbf{x} \geq \mathbf{0} &
\end{array}
\qquad
\begin{array}{lll}
\text{(D)} & \max & w = \mathbf{by} \\
& \text{s.t.} & \mathbf{yA} \leq \mathbf{c} \\
& & \mathbf{y} \geq \mathbf{0}
\end{array}
$$

  - The primal (P) optimality conditions are

$$z_j - c_j \leq 0, \text{ for all variables } x_j \Rightarrow \mathbf{c_B B^{-1} A} - \mathbf{c} \leq \mathbf{0}.$$

  - Letting $\mathbf{y} = \mathbf{c_B B}^{-1}$, implies that (P) optimality conditions are

$$\mathbf{yA} - \mathbf{c} \leq \mathbf{0} \Rightarrow \mathbf{yA} \leq \mathbf{c}.$$

  - I.e., (P) optimality conditions are (D) feasibility conditions.

Fact *At optimality (and only then) both (P) and (D) are feasible.*

  - Suppose that at the origin O, where $\mathbf{x} = 0$, the optimality conditions for (P) (i.e. (D) feasibility conditions) are satisfied but the feasibility conditions of (P) (i.e. (D) optimality conditions) are not satisfied.
  - This could be the case if $\mathbf{c} \geq \mathbf{0}$, and one of the b_i's is positive.
  - The *dual simplex method* maintains the optimality of (P) (i.e. feasibility of (D)) and iterates, similar to the usual "primal" simplex, until (P) feasibility (i.e. optimality of (D)) is reached.

- **Steps of the Dual Simplex Method**

  (1) Change all ≥ constraints with positive right hand side (rhs) into ≤ constraints. (Multiply both sides of constraints by −1.)

  (2) Add slack variables as usual and construct the simplex tableau with O as starting basic *infeasible* solution (with a basic composed of slack variables).

  (3) Choose a basic variable to leave the basis as the variable having the most negative rhs.

  (4) Choose a nonbasic variable to enter the basis based on a minimum ratio test, where ratios of the elements of the Z-row to corresponding *negative* (and only negative) elements of the leaving variable row are considered.[1]

  (5) Perform a simplex iteration based on the pivot element.

  (6) Repeat steps (3)–(5) until the right hand sides of all rows in the tableau are positive.

- **Interesting Facts**

  1. The basic solutions at all iterations of the dual simplex are *infeasible* except the final solution, which is feasible.
  2. The dual simplex moves from one basic solution into another which is *worse* in terms of optimal objective value (i.e. Z *increases* in each iteration).

[1] This rule for choosing the entering variable will maintain dual feasibility and will change the sign of the rhs corresponding to the entering variable from < 0 to > 0 (i.e. brings the solution closer to primal feasibility).