Sensitivity Analysis

1 Changes in the RHS (b)

Consider an optimal LP solution. Suppose that the original RHS (**b**) is changed from $\mathbf{b}^0$ to $\mathbf{b}^{\text{new}}$. In the following, we study the affect of this change the optimal solution. First, note that changing **b** will not affect optimality. Recall that optimality is based on the condition $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}_j - c_j = z_j - c_j \geq 0,$ for all variables x_j (for max problems). Changing **b** will not affect $z_j - c_j$. Hence, changing **b** will not affect optimality. However; changing **b** may affect feasibility.

To check, whether the LP is still feasible after changing **b** from $\mathbf{b}^0$ to $\mathbf{b}^{\text{new}}$ compute

$$\mathbf{x}^{\text{new}} = \mathbf{B}^{-1}\,\mathbf{b}^{\text{new}}.$$

- If $\mathbf{x}^{\text{new}} \geq 0$ (i.e., $x_i^{new} \geq 0,$ for all i), then the LP is still feasible and the optimal solution is $\mathbf{x}^{\text{new}}$.
- If $x_i^{new} < 0,$ for some i, then the current basic solution is no longer feasible. Then, the dual simplex method is used to restore feasibility (leading to a new optimal basis).

Note that when the LP remains feasible the rate of change of the optimal objective function, Z^*,as a function of a RHS value b_i is y_i^* the corresponding dual variable. This follows by noting that $Z^* = \sum b_i y_i^*$ and $\dfrac{\partial Z^*}{\partial b_i} = y_i^*$. Therefore, changing the b_i by 1 unit will change Z^* by y_i^* (assuming that this change will not affect feasibility)

Example 1.

consider the LP

$$\begin{aligned}
\max\ Z &= 3x_1 + 2x_2 + 5x_3\\
\text{s.t.}\quad & x_1 + 2x_2 + x_3 \leq 430\\
& 3x_1 + 2x_3 \leq 460\\
& x_1 + 4x_2 \leq 420\\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

Let S_1, S_2, S_3 be the slack variables corresponding to the three constraints. Upon solving the LP, the optimal simplex tableau is as follows.

	x_1	x_2	x_3	S_1	S_2	S_3	RHS
Z	4	0	0	1	2	0	1350
x_2	-1/4	1	0	1/2	-1/4	0	100
x_3	3/2	0	1	0	1/2	0	230
S_3	2	0	0	-2	1	1	20

The optimal solution is $x_1^* = 0$, $x_2^* = 100$, $x_3^* = 230$, and $Z^* = 1350$.

- Suppose that **b** is changed from $\begin{pmatrix} 430 \\ 460 \\ 420 \end{pmatrix}$ to $\begin{pmatrix} 500 \\ 460 \\ 600 \end{pmatrix}$. What is the new optimal solution?

$$\mathbf{x}^{new} = \mathbf{B}^{-1}\mathbf{b}^{new} = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 500 \\ 460 \\ 600 \end{pmatrix} = \begin{pmatrix} 135 \\ 230 \\ 60 \end{pmatrix} > 0 \ .$$

Since $\mathbf{x}^{\text{new}}$ is feasible. Then, the new optimal solution is $\mathbf{x}^{*\text{new}} = \mathbf{x}^{\text{new}}$. That is, the new optimal solution is $x_1^{*\text{ new}} = 0$, $x_2^{*\text{ new}} = 135$, $x_3^{*\text{ new}} = 230$, and the new optimal objective value is $Z^{*\text{ new}} = \mathbf{c}_B\, \mathbf{x}^{*\text{new}} = (2 \ \ 5 \ \ 0)\begin{pmatrix} 135 \\ 230 \\ 60 \end{pmatrix} = 1420$. Note that $Z^{*\text{new}}$ can also be obtained using duality. From the simplex tableau, the optimal dual solution is $y_1^* = 1$, $y_2^* = 2$, $y_3^* = 0$. Then,

$$Z^{*\text{new}} = Z^* + (b_1^{\text{new}} - b_1)\, y_1^* + (b_2^{\text{new}} - b_2)\, y_2^* + (b_3^{\text{new}} - b_3)\, y_3^*$$
$$= 1350 + (500 - 430)\times 1 + (460 - 460)\times 2 + (600 - 420)\times 0 = 1420 \ .$$

- What if **b** is changed from $\begin{pmatrix} 430 \\ 460 \\ 420 \end{pmatrix}$ to $\begin{pmatrix} 450 \\ 460 \\ 400 \end{pmatrix}$?

$$\mathbf{x}^{\text{new}} = \mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 450 \\ 460 \\ 400 \end{pmatrix} = \begin{pmatrix} 110 \\ 230 \\ -40 \end{pmatrix} .$$

In this case, $\mathbf{x}^{\text{new}}$ is not feasible. The new optimal solution is obtained by applying the dual simplex method. First, change the RHS of the optimal tableau to $\mathbf{x}^{\text{new}}$ and the objective value to $Z(\mathbf{x}^{\text{new}}) = \mathbf{c}_B\, \mathbf{x}^{\text{new}} = (2 \quad 5 \quad 0)\begin{pmatrix} 110 \\ 230 \\ -40 \end{pmatrix} = 1370.$

	x_1	x_2	x_3	S_1	S_2	S_3	RHS
Z	4	0	0	1	2	0	1370
x_2	-1/4	1	0	1/2	-1/4	0	110
x_3	3/2	0	1	0	½	0	230
S_3	2	0	0	-2	1	1	-40
Ratio	--	--	--	1/2	--	--	--

	x_1	x_2	x_3	S_1	S_2	S_3	RHS
Z	5	0	0	0	5/2	1/2	1350
x_2	1/4	1	0	0	0	1/4	100
x_3	3/2	0	1	0	1/2	0	230
S_1	-1	0	0	1	-1/2	-1/2	20

The last tableau is optimal (RHS > 0). The new optimal solution $\mathbf{x}^{*\text{new}}$ is $x_1^{*\text{new}} = 0$, $x_2^{*\text{new}} = 100$, $x_3^{*\text{new}} = 230$, and $Z^{*\text{new}} = 1350$.

<u>Range of b_i where LP remains feasible</u>

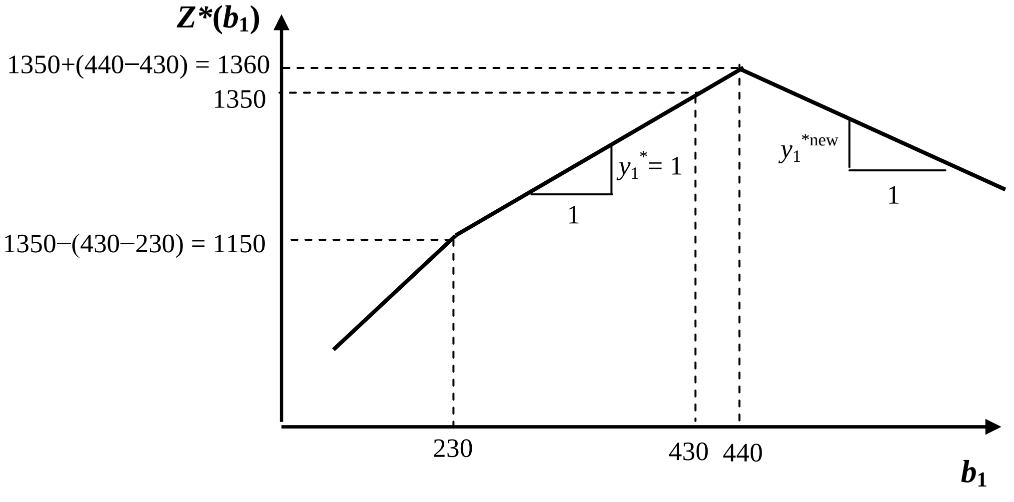
In Example 1, suppose only b_1 is changed to $b_1 + \delta$. In the following, we find the range of δ that keeps the LP feasible (and keeps the current basis (x_2, x_3, S_3) optimal). With this change

$$\mathbf{x}^{\text{new}} = \mathbf{B}^{-1}\mathbf{b}^{\text{new}} = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 430+\delta \\ 460 \\ 420 \end{pmatrix} = \begin{pmatrix} 100+\delta/2 \\ 230 \\ 20-2\delta \end{pmatrix}.$$

LP remains feasible if $\mathbf{x}^{\text{new}} > 0 \;\Rightarrow\; \begin{cases} 100+\dfrac{\delta}{2} \ge 0 \;\Rightarrow\; \delta \ge -200 \\ 20-2\delta \ge 0 \;\Rightarrow\; \delta \le 10 \end{cases}.$

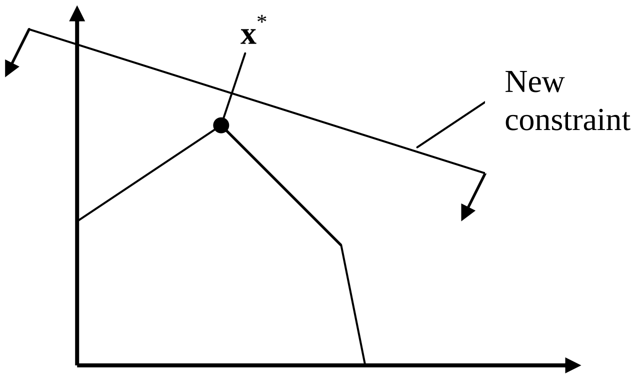
Then, LP remains feasible if $-200 \le \delta \le 10$, or equivalently $230 \le b_1 \le 440$.

Graphically,

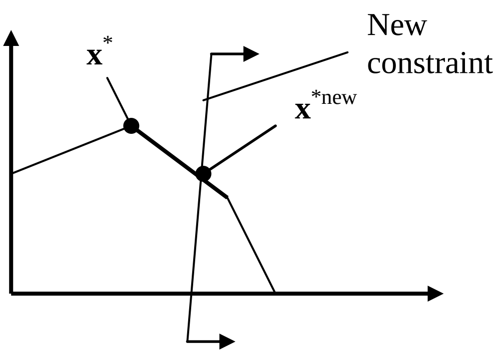


2 Adding a new Constraint

When a new constraint is added, the optimal solution does not improve. If the current optimal solution satisfies the new constraint, the optimal solution does not change. Graphically,



Otherwise, if the optimal solution does not satisfy the constraint, the current optimal solution becomes infeasible, and a new optimal solution could be obtained using the dual simplex method. Graphically,



Example 2.

- Suppose the constraint $3x_1 + x_2 + x_3 \leq 500$ is added to the LP in Example 1. What is the new optimal solution?

The current optimal solution $x_1^* = 0$, $x_2^* = 100$, and $x_3^* = 230$, satisfies the constraint (since $3\times0 + 100 + 230 = 330 < 500$). The optimal solution does not change.

- Suppose the constraint $3x_1 + 3x_2 + x_3 \le 500$ is added to the LP in Example 1. What is the new optimal solution?

The current optimal solution does not satisfy the constraint (since 3×0+3×100 + 230 = 530 > 500). Then, add the constraint to the simplex tableau, rearrange the tableau (to be in the right form) and proceed with the dual simplex method. Let S_4 be the slack variable corresponding to the new constraint.

	x_1	x_2	x_3	S_1	S_2	S_3	S_4	RHS
Z	4	0	0	1	2	0	0	1350
x_2	-1/4	1	0	1/2	-1/4	0	0	100
x_3	3/2	0	1	0	1/2	0	0	230
S_3	2	0	0	-2	1	1	0	20
S_4	3	3	1	0	0	0	1	500

Rearrange the tableau so that the basic variables x_2 and x_3 have zero coefficients in the S_4 row (note that S_4 is now considered a basic variable). Then apply dual simplex.

$l_4' = l_4 - l_2 - 3l_1$

	x_1	x_2	x_3	S_1	S_2	S_3	S_4	RHS
Z	4	0	0	1	2	0	0	1350
x_2	-1/4	1	0	1/2	-1/4	0	0	100
x_3	3/2	0	1	0	1/2	0	0	230
S_3	2	0	0	-2	1	1	0	20
S_4	9/4	0	0	-3/2	1/4	0	1	-30
Ratio	--	--	--	2/3	--	--	--	

	x_1	x_2	x_3	S_1	S_2	S_3	S_4	RHS
Z	11/2	0	0	0	13/6	0	2/3	1330
x_2	1/2	1	0	0	-1/6	0	1/3	90
x_3	3/2	0	1	0	1/2	0	0	230
S_3	-1	0	0	0	2/3	1	-4/3	60
S_1	-3/2	0	0	1	-1/6	0	-2/3	20

The last tableau is optimal. The new optimal solution is $x_1^{*new} = 0$, $x_2^{*new} = 90$, $x_3^{*new} = 230$, and $Z^{*new} = 1330$.

3 Changes in the objective function coefficients (c)

Consider an optimal LP tableau. Recall that the RHS is given by $\mathbf{B}^{-1}\mathbf{b}$. Therefore, changing **c** will not affect feasibility. It will only affect the optimality conditions given by $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}_j - c_j = z_j - c_j \geq 0,$ (for max problems). When **c** is changed from $\mathbf{c}^0$ to $\mathbf{c}^{\text{new}}$, two situations could result:

i) The optimality conditions continue to hold (if $z_j^{new} - c_j \geq 0,$ for all j, for max problems). Then, the optimal solution does not change (but the optimal objective value $Z^{*\text{new}} = \mathbf{c}_B^{\text{new}}\,\mathbf{x}^*$ could change).

ii) The optimality conditions do not hold anymore ($z_j^{new} - c_j < 0,$ for some j, for max problems). Then, the optimal solution changes. The (usual "primal") simplex method is used to obtain a new optimal solution.

Example 3.

- In the LP of Example 1, suppose the objective function is changed from $Z = 3x_1 + 2x_2 + 5x_3$ to $Z = 2x_1 + 3x_2 + 4x_3$. That is, **c** is changed from $\mathbf{c}^0 = (3\ \ 2\ \ 5\ )$ to $\mathbf{c}^{\text{new}} = (2\ \ 3\ \ 4)$. What is the new optimal solution?

In terms of the current optimal solution basis (x_2, x_3, S_3), $\mathbf{c}_B$ is changed from $\mathbf{c}_B^0 = (2\ \ 5\ \ 0)$ to $\mathbf{c}_B^{\text{new}} = (3\ \ 4\ \ 0)$. Then $z_j^{\text{new}} - c_j^{\text{new}}$ for nonbasic variables are

$$\mathbf{c}_B^{\text{new}}\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N^{\text{new}} = \begin{pmatrix}3 & 4 & 0\end{pmatrix}\begin{pmatrix}1/2 & -1/4 & 0\\ 0 & 1/2 & 0\\ -2 & 1 & 1\end{pmatrix}\begin{pmatrix}\overbrace{\begin{matrix}1 & 1 & 0\end{matrix}}^{x_1\ \ S_1\ \ S_2}\\ \begin{matrix}3 & 0 & 1\end{matrix}\\ \begin{matrix}1 & 0 & 0\end{matrix}\end{pmatrix} - \begin{pmatrix}2 & 0 & 0\end{pmatrix}$$

$$= \begin{pmatrix}3/2 & 5/4 & 0\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\ 3 & 0 & 1\\ 1 & 0 & 0\end{pmatrix} - \begin{pmatrix}2 & 0 & 0\end{pmatrix}$$

$$= (21/4 \quad 3/2 \quad 5/4) - (2 \quad 0 \quad 0)$$

$$= \begin{pmatrix}\overbrace{\begin{matrix}13/4 & 3/2 & 5/4\end{matrix}}^{x_1\ \ \ S_1\ \ \ S_2}\end{pmatrix} \geq \mathbf{0}\ .$$

Therefore, the optimal solution remains the same, $x_1^* = 0$, $x_2^* = 100$, $x_3^* = 230$, and

$$Z^{*\,new} = \mathbf{c}_B^{new}\,\mathbf{x}^* = (3 \quad 4 \quad 0)\begin{pmatrix}100\\230\\20\end{pmatrix} = 1220\ .$$

- Suppose now that $\mathbf{c}$ is changed from $\mathbf{c}^0 = (3 \quad 2 \quad 5)$ to $\mathbf{c}^{new} = (6 \quad 3 \quad 4)$. What is the new optimal solution?

In this case, $\mathbf{c}_B$ is changed from $\mathbf{c}_B^0 = (2 \quad 5 \quad 0)$ to $\mathbf{c}_B^{new} = (3 \quad 4 \quad 0)$, and $z_j^{new} - c_j^{new}$ (j nonbasic) is given by

$$\mathbf{c}_B^{new}\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N^{new} = (21/4 \quad 3/2 \quad 5/4) - (6 \quad 0 \quad 0) \quad = \begin{array}{c} \overbrace{\begin{array}{ccc} x_1 & S_1 & S_2 \end{array}} \\ \left(\begin{array}{ccc} -3/4 & 3/2 & 5/4 \end{array}\right) \end{array}.$$ The

current solution is no longer optimal. Modify the Z-row of the current simplex tableau is as follow.

$Z^{new} = \mathbf{c}_B^{new}\,\mathbf{x}^0 = 1220$

	x_1	x_2	x_3	S_1	S_2	S_3	RHS	Ratio
Z	-3/4	0	0	3/2	5/4	0	1220	--
x_2	-1/4	1	0	1/2	-1/4	0	100	--
x_3	3/2	0	1	0	1/2	0	230	460/3
S_3	2	0	0	-2	1	1	20	10

Then, proceed with the simplex method.

	x_1	x_2	x_3	S_1	S_2	S_3	RHS
Z	0	0	0	3/4	13/8	3/8	2455/2
x_2	0	1	0	1/4	-1/8	1/8	205/2
x_3	0	0	1	3/2	-1/4	-3/4	215
x_1	1	0	0	-1	1/2	1/2	10

The last tableau is optimal. The new optimal solution is $x_1^{*new} = 10$, $x_2^{*new} = 205/2$, $x_3^{*new} = 215$, and $Z^{*new} = 2425/2$.

<u>Optimality Range of c_j</u>

Suppose, in the LP of Example 1, c_1 is changed to $c_1 + \delta$. In the following, we find the range of δ values that keep the LP optimal with the current basis. In this case, $\mathbf{c}^{new} = (3+\delta \quad 2 \quad 5)$, and $\mathbf{c}_B^{new} = (2 \quad 5 \quad 0)$. Then, $z_j - c_j$ (j nonbasic) is given by

$$\mathbf{c}_B^{new}\mathbf{B}^{-1}\mathbf{N}-\mathbf{c}_N^{new} = (2 \quad 5 \quad 0)\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{array}{c} \begin{array}{ccc} x_1 & S_1 & S_2 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{array} - (3+\delta \quad 0 \quad 0)$$

$$= (4-\delta \quad 1 \quad 2)$$

Then, LP is optimal if $4-\delta \geq 0 \Leftrightarrow \delta \leq 4 \Leftrightarrow c_1 \leq 7$.

Remark. In cases like the above where c_j is changed for j nonbasic .The optimality range for c_j can be deduced directly from the current optimal tableau .The corresponding δ is the coefficient of x_j in the Z-row of the current optimal tableau. This coefficient is called ***reduced cost*** of x_j. It represents the minimum amount by which the objective function coefficient of x_j should be improved in order for x_j to become basic. E.g., in the above, increasing c_1 by 4 will make x_1 basic.

4 Addition of a new variable x_j

The new variable, x_j, can be thought of as a nonbasic variable. Consider an optimal tableau. To investigate the effect of adding the new variable x_j find

$$z_j - c_j = \mathbf{c}_B\, \mathbf{B}^{-1}\mathbf{A}_j - c_j = \mathbf{y}^*\mathbf{A}_j - c_j\ ,$$

where $\mathbf{y}^*$ is the current dual variables vector ($\mathbf{y}^*$ is given by the Z-row coefficients of the starting basic variables).

- If $z_j - c_j \geq 0$ (for max problem), the optimal solution is unchanged with $x^*_j = 0$ (since x_j is nonbasic).
- Otherwise, if $z_j - c_j < 0$, then use the simplex method to find a new optimal solution with possibly $x^*_j > 0$ (since x_j enters the basis).

Example 4.

- In the LP of Example 1, suppose the variable x_4 with $c_4 = 2$, and $\mathbf{A}_4 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ is added. Then,

$$z_4 - c_4 = \mathbf{y}^*\mathbf{A}_4 - c_4 = (1 \quad 2 \quad 0)\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - 2 = 3 - 2 = 1 > 0\ .$$

Therefore, the optimal solution is unchanged with $x^*_1 = 0$, $x^*_2 = 0$, $x^*_3 = 230$, $x^*_4 = 0$, and $Z^* = 1350$.

- Suppose now that x_4 with $c_4 = 4$ and $\mathbf{A}_4 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ is added. Then,

$$z_4 - c_4 = \mathbf{y}^*\mathbf{A}_4 - c_4 = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - 4 = 3 - 4 = -1 < 0 .$$

Therefore, the current solution is no longer optimal. Add x_4 to the simplex tableau and proceed with the simplex method. The coefficient of x_4 in the Z-row is $z_4 - c_4 = -1$. The constraint coefficients corresponding to x_4 are given by

$$B^{-1}\mathbf{A}_4 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{pmatrix} .$$

The new simplex tableau is given below with subsequent simplex iterations.

	x_1	x_2	x_3	x_4	S_1	S_2	S_3	RHS	Ratio
Z	4	0	0	-1	1	2	0	1350	--
x_2	-1/4	1	0	1/4	1/2	-1/4	0	100	400
x_3	3/2	0	1	1/2	0	1/2	0	230	460
S_3	2	0	0	1	-2	1	1	20	20

	x_1	x_2	x_3	x_4	S_1	S_2	S_3	RHS	Ratio
Z	6	0	0	0	-1	3	1	1370	--
x_2	-3/4	1	0	0	1	-1/2	-1/4	95	95
x_3	1/2	0	1	0	1	0	-1/2	220	220
x_4	2	0	0	1	-2	1	1	20	--

	x_1	x_2	x_3	x_4	S_1	S_2	S_3	RHS
Z	21/4	1	0	0	0	5/2	3/4	1465
S_1	-3/4	1	0	0	1	-1/2	-1/4	95
x_3	5/4	-1	1	0	0	1/2	-1/4	125
x_4	1/2	2	0	1	0	0	1/2	210

The last tableau is optimal. The new optimal solution (with x_4 basic) is $x_1 = 0$, $x_2^* = 0$, $x_3^* = 125$, $x_4^* = 210$, and $Z^* = 1465$.

Remarks.

- Determining the effect of changing the constraint coefficients of a nonbasic variable is similar to the above.
- Determining the effect of changing the constraint coefficients of a basic variable is a bit more involved. But it can be done by applying similar principles (see, for example, Bazaraa et al., *Linear Programming and Network Flows*).