The Transportation Problem (2)

- **Balance Condition and Number of Basic Variables**
  - Consider the LP formulation for the TP

$$\text{(TP)} \quad \min \quad Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = s_i \ , i = 1,\ldots,m \quad \leftarrow u_i$$

$$\sum_{i=1}^{m} x_{ij} = d_j \ , j = 1,\ldots,n \quad \leftarrow v_j$$

$$x_{ij} \geq 0$$

  - A necessary condition for (TP) to have feasible condition is "balance." That is, $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$.
  - If balance is violated, then an artificial balance could be established by adding "dummy" nodes.
  - For example, if $\sum_{i=1}^{m} s_i > \sum_{j=1}^{n} d_j$, then adding a dummy destination, n+1, with demand $d_{n+1} = \sum_{i=1}^{m} s_i - \sum_{j=1}^{n} d_j$ and $c_{i,n+1} = 0$, $i = 1, \ldots, m$, restores balance.

- The balance condition implies that any of the $m + n$ constraints can be removed without affecting the optimal solution.[1]
- This implies that <u>the number of basic variables in $m + n - 1$</u> .

- **Duality Results**
  - Let u_i , $i = 1, \ldots, m$, and v_j , $j = 1, \ldots, m$, be the dual variables associated with supply and demand constraints respectively.
  - Then, the "Z-row" coefficients (imagine solving with the usual simplex method) corresponding to variable x_{ij} is

$$
\begin{aligned}
z_{ij} - c_{ij} &= \mathbf{c_B B}^{-1}\mathbf{A}_{ij} - c_{ij} \\
&= \mathbf{y}\mathbf{A}_{ij} - c_{ij} \\
&= (u_1, \ldots, u_i, \ldots, u_m, v_1, \ldots, v_j, \ldots, v_n)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - c_{ij} \ ,
\end{aligned}
$$

[1] E.g., the last demand constraint $\sum_{i=1}^{m} x_{ni} = d_n$ could be obtained by adding up supply constraints and then subtracting the sum of the first n -1 demand constraints.

where **y** is the vector of dual variables and $\mathbf{A}_{ij}$ is the column of constraint coefficients corresponding to x_{ij} .

- Observe that all $\mathbf{A}_{ij}$ entries are zeroes except for entries corresponding to rows i and $i + m$.[2]
- It follows that

$$\boxed{z_{ij} - c_{ij} = u_i + v_j - c_{ij}} \qquad \textbf{(1)}$$

- Recall that for basic variables $z_{ij} - c_{ij} = 0$. Then, (1) implies that

$$\boxed{u_i + v_j = c_{ij} \text{ , for } x_{ij} \text{ basic}} \qquad \textbf{(2)}$$

- Solving for the u_i's and v_j's in (2) involves solving m+n−1 linear equations in $m + n$ unknowns. The solution can be obtained by setting one of unknowns to an arbitrary value.
- In particular, set

$$\boxed{u_1 = 0} \qquad \textbf{(3)}$$

- **The Transportation Simplex Method**

- Results (1) – (3) form the theoretical foundation for a modification of the simplex method that benefits from the *structure* of TP to simplify the solution.
- This method is referred to as the *transportation simplex*.
- Like the usual simplex, the transportation simplex starts with a basic feasible (BF) solution, which can be obtained from NW Corner, Least-Cost, or Vogel's method.

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[2] This is so because a particular variable x_{ij} shows up exactly in one demand constraint and one supply constraint.

- Then, utilizing (2) and (3), the dual variables (simplex multipliers) u_i and v_j are obtained.
- Following, an entering variable (if any) is determined from (1). A leaving variable is obtained based on a specialized "chain reaction rule," which benefits from the balance structure.
- Transportation simplex terminates when the optimality conditions, $z_{ij} - c_{ij} = u_i + v_j - c_{ij} \leq 0,$ are satisfied.

- **Transportation Simplex Example**

- An application of the transportation simplex method to our prototype gravel hauling example is given in the following page.
- The starting BF solution is obtained from the NW corner method as done before with a cost of \$172.
- The optimal is obtained after two iterations lowering the cost from \$172 to \$126.
- Notice that the optimal solution is the same one obtained with the Least-Cost and Vogel's methods. (This will not be always the case.)
- Hence, using Least-Cost and Vogel's starting solution has proven to be the most efficient for solving this problem optimally.

		1	2	3	
		$v_1 = 7$	$v_2 = 5$	$v_3 = 4$	Supply
1	$u_1 = 0$	7; 8	5; 8 -	4; 4 +	20
2	$u_2 = 2$	6; 3	2; + ; 5	6; 10 -	10
	Demand	8	8	14	Z = 172

		1	2	3	
		$v_1 = 7$	$v_2 = 0$	$v_3 = 4$	Supply
1	$u_1 = 0$	7; 8 -	5; -5	4; 12 +	20
2	$u_2 = 2$	6; + ; 3	2; 8	6; 2 -	10
	Demand	8	8	14	Z = 132

		1	2	3	
		$v_1 = 7$	$v_2 = 3$	$v_3 = 4$	Supply
1	$u_1 = 0$	7; 6	5; -2	4; 14	20
2	$u_2 = -1$	6; 2	2; 8	6; -3	10
	Demand	8	8	14	Z = 126

Optimal Solution $x_{11} = 6$, $x_{13} = 14$, $x_{21} = 2$, $x_{22} = 8$, $x_{ij} = 0$ (otherwise) and $Z^* = 126$.