Example (using the dual to solve the primal)

(P) Min $z = 2x_1 + 3x_2 + 5x_3 + 2x_4 + 3x_5$

s.t. $x_1 + x_2 + 2x_3 + x_4 + 3x_5 \geq 4 \quad \leftarrow y1$

$2x_1 - 2x_2 + 3x_3 + x_4 + x_5 \geq 3 \quad \leftarrow y2$

$x_i \geq 0$

(D) Max $w = 4y_1 + 3y_2$

s.t. $y_1 + 2y_2 \leq 2 \quad \leftarrow x1$

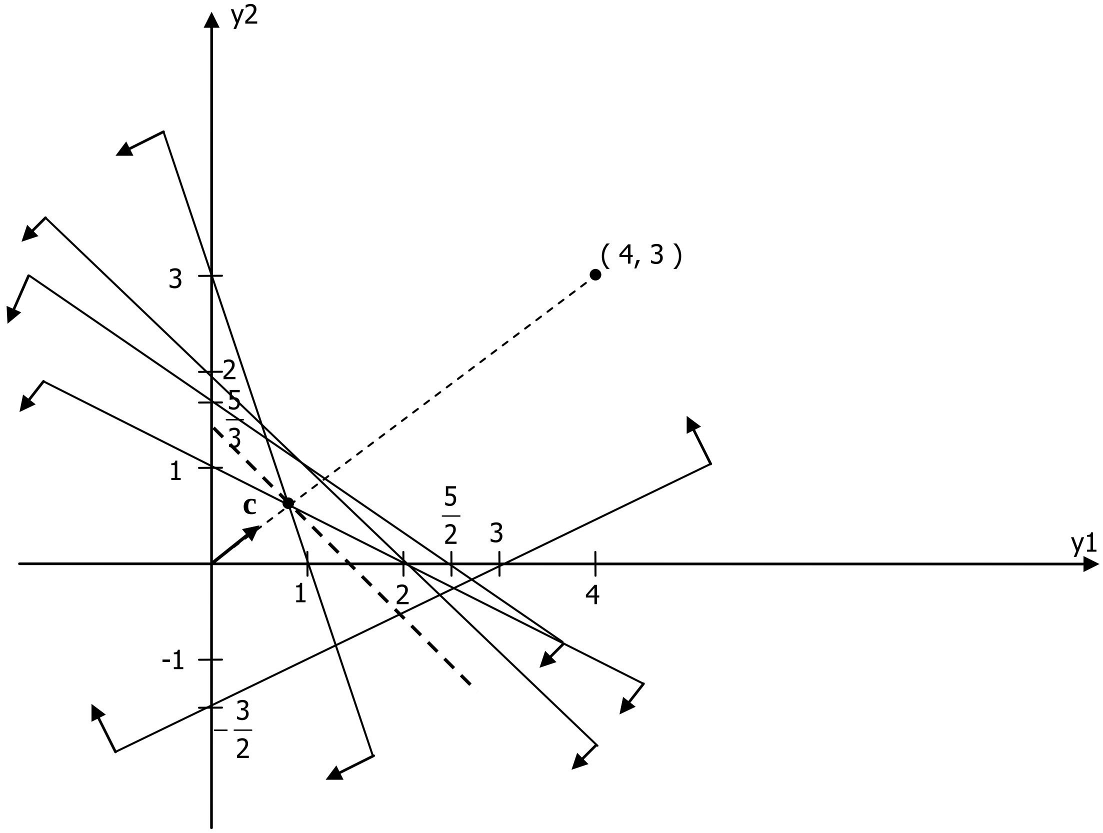
$y_1 - 2y_2 \leq 3 \quad \leftarrow x2$

$2y_1 + 3y_2 \leq 5 \quad \leftarrow x3$

$y_1 + y_2 \leq 2 \quad \leftarrow x4$

$3y_1 + y_2 \leq 3 \quad \leftarrow x5$

$y_i \geq 0$



Graphically, the optimal dual solution is

$y^*_1 = \frac{4}{5}$, $y^*_2 = \frac{3}{5}$, and $w^* = (4 \times \frac{4}{5}) + (3 \times \frac{3}{5}) = 5 = z^*$.

Complementary slackness implies that

$x_2^*(y_1^* - 2y_2^* - 2) = 0$,

$x_3^* (2y_1^* + 3y_2^* - 5) = 0$,

$x_4^* (y_1^* + y_2^* - 2) = 0$,

$\Rightarrow x^*_2 = 0$, $x^*_3 = 0$, and $x^*_4 = 0$ (since dual constraints (2), (3) and (4) are not binding).

In addition,

$y^*_1\,(\,x^*_1 + x^*_2 + 2x^*_3 + x^*_4 + 3x^*_5 - 4\,) = 0\,,$

$y^*_2\,(\,2x^*_1 - 2x^*_2 + 3x^*_3 + x^*_4 + x^*_5 - 3\,) = 0\,,$

$$\Leftrightarrow \quad \begin{aligned} x^*_1 + 3x^*_5 &= 4\,, \\ 2x^*_1 + x^*_5 &= 3\,, \end{aligned}$$

$\Rightarrow$ $x^*_1 = 1$ and $x^*_5 = 1$.

Then, the optimal Solution is

$x^*_1 = 1,\ x^*_2 = 0,\ x^*_3 = 0,\ x^*_4 = 0,\ x^*_5 = 1$, and $Z^* = 5$.

In tabular form

$$\begin{array}{lllr}
\text{Max } w = & 4y_1 + 3y_2 & & \\
\text{s.t.} & y_1 + 2y_2 + S_1 & = 2 & \leftarrow x1 \\
 & y_1 - 2y_2 + S_2 & = 3 & \leftarrow x2 \\
 & 2y_1 + 3y_2 + S_3 & = 5 & \leftarrow x3 \\
 & y_1 + y_2 + S_4 & = 2 & \leftarrow x4 \\
 & 3y_1 + y_2 + S_5 & = 3 & \leftarrow x5 \\
 & y_i \ge 0 & &
\end{array}$$

	y_1	y_2	S_1	S_2	S_3	S_4	S_5	RHS	Ratio
w	-4	-3	0	0	0	0	0	0	-
S_1	1	2	1	0	0	0	0	2	2
S_2	1	-2	0	1	0	0	0	3	3
S_3	2	3	0	0	1	0	0	5	5/2
S_4	1	1	0	0	0	1	0	2	2
S_5	3	1	0	0	0	0	1	3	1

	y_1	y_2	S_1	S_2	S_3	S_4	S_5	RHS	Ratio
w	0	-5/3	0	0	0	0	4/3	4	-
S_1	0	5/3	1	0	0	0	-1/3	1	3/5
S_2	0	-7/3	0	1	0	0	-1/3	2	-
S_3	0	7/3	0	0	1	0	-2/3	3	9/4
S_4	0	2/3	0	0	0	1	-1/3	1	3/2
y_1	1	1/3	0	0	0	0	1/3	1	3

	Y_1	y_2	S_1	S_2	S_3	S_4	S_5	RHS	Ratio
w	0	0	1	0	0	0	1	5	-
y_2	0	1	3/5	0	0	0	-1/5	3/5	-
S_2	0	0	7/5	1	0	0	-4/5	7/5	-
S_3	0	0	-7/5	0	1	0	-1/5	5/3	-
S_4	0	0	-2/5	0	0	1	-1/5	3/5	-
y_1	1	0	-1/5	0	0	0	2/5	4/5	-

B^{-1}

The optimal solution is the same as before.

$y^*_1 = \frac{4}{5}$, $y^*_2 = \frac{3}{5}$, and $w^* = 5$.

The primal optimal solution can then be obtained in two ways.

(i)

$$B^{-1} = \begin{pmatrix} 3/5 & 0 & 0 & 0 & -1/5 \\ 7/5 & 1 & 0 & 0 & -4/5 \\ -7/5 & 0 & 1 & 0 & -1/5 \\ -2/5 & 0 & 0 & 1 & -1/5 \\ -1/5 & 0 & 0 & 0 & 2/5 \end{pmatrix}$$

$$\begin{aligned} \Rightarrow x^* &= \begin{pmatrix} x^*_1 & x^*_2 & x^*_3 & x^*_4 & x^*_5 \end{pmatrix} \\ &= c_B B^{-1} \\ &= \begin{pmatrix} 3 & 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 3/5 & 0 & 0 & 0 & -1/5 \\ 7/5 & 1 & 0 & 0 & -4/5 \\ -7/5 & 0 & 1 & 0 & -1/5 \\ -2/5 & 0 & 0 & 1 & -1/5 \\ -1/5 & 0 & 0 & 0 & 2/5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

(ii) Simply look at Z-row of the optimal dual tableau. Under the slack variable S_j it has,

$$c_B B^{-1} A_{Sj} - c_{Sj} = (c_B B^{-1})_j,$$

where A_{Sj} is the column under S_j in the starting tableau, $A_{Sj} = (0, \ldots, 0, 1, 0, \ldots, 0)^T$,

$\Rightarrow x^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \end{pmatrix}$ (by Fact 1, see duality notes).