- **Writing LP in standard form**
  - In this form all constraints are transformed into equalities.
  - This is achieved by adding *slack* variables for "≤" constraints and surplus variables for "≥" constraints.
  - For example, the LP

$$\begin{aligned}\max\ Z &= 3x_1 + 4x_2\\ \text{subject to}\ \ 3x_1 + 2x_2 &\le 6\\ x_1 + 4x_2 &\le 4\\ x_1 \ge 0, x_2 &\ge 0\end{aligned}$$

is written in standard form as

$$\begin{aligned}\max\ Z &= 3x_1 + 4x_2\\ \text{subject to}\ \ 3x_1 + 2x_2 + S_1 \qquad &= 6\\ x_1 + 4x_2 \qquad + S_2 &= 4\\ S_1 \ge 0, S_2 \ge 0, x_1 \ge 0, x_2 &\ge 0\end{aligned}$$

  - The LP

$$\begin{aligned}\min\ Z &= x_1 + x_2\\ \text{subject to}\ \ x_1 + 2x_2 &\ge 6\\ 2x_1 + x_2 &\ge 4\\ x_1 \qquad &\le 3\\ x_1 \ge 0, x_2 &\ge 0\end{aligned}$$

is written in standard form as

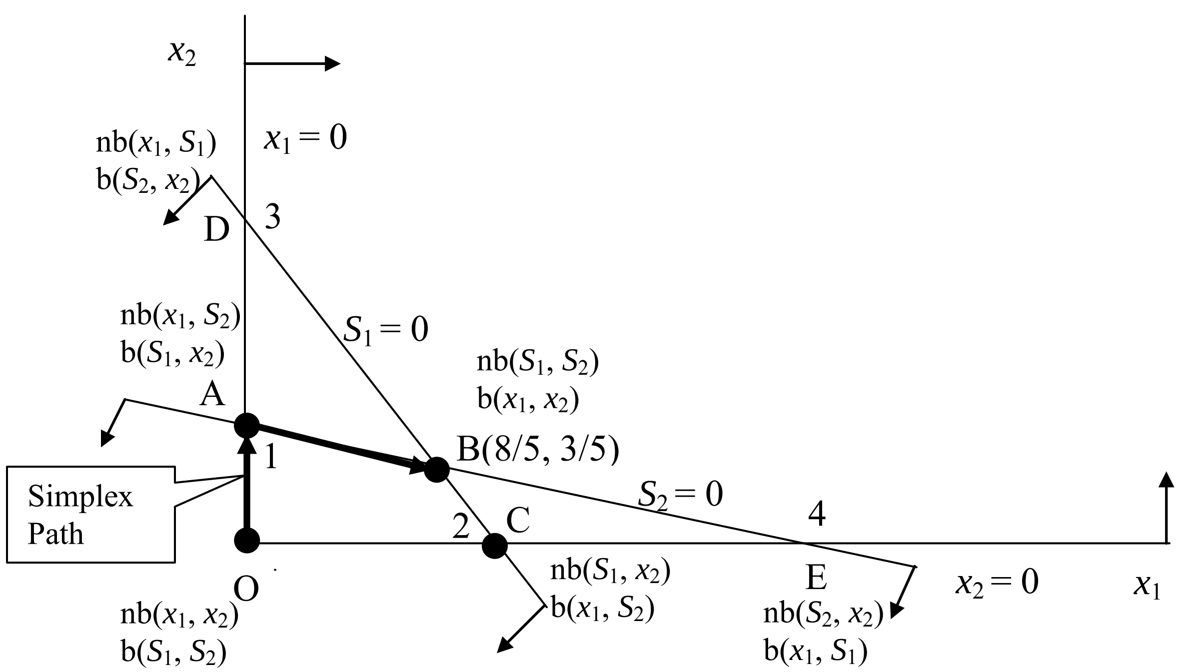
$$\begin{aligned}\min\ Z &= x_1 + x_2\\ \text{subject to}\ \ x_1 + 2x_2 - S_1 \qquad\qquad &= 6\\ 2x_1 + x_2 \qquad - S_2 \qquad &= 4\\ x_1 \qquad\qquad + S_3 &= 3\\ S_i \ge 0, x_i &\ge 0\end{aligned}$$

- **Definitions**
  - A constraint is binding at a given point if this point is on the constraint (i.e., the point's coordinates satisfy the constraint as an equality).
  - In a LP with *n* decision variables, a *corner point* is a point where *n* or more constraints are binding.

Theorm *If a LP has an optimal solution then there exists an optimal solution corresponding to a corner point.*

- **Graphical motivation for the simplex method**
  - Consider the following LP written in standard form

$$
\begin{aligned}
\text{max } Z &= 3x_1 + 4x_2 \\
\text{subject to } \; & 3x_1 + 2x_2 + S_1 = 6 \\
& x_1 + 4x_2 + S_2 = 4 \\
& S_1 \geq 0, S_2 \geq 0, x_1 \geq 0, x_2 \geq 0
\end{aligned}
$$



- Choose an initial corner point feasible solution (i.e., basic feasible solution). In the current example, O(0, 0) can serve as such solution.
- At O define the "zero-variables" x_1 and x_2 as the *nonbasic* variables. Define the remaining nonzero variables S_1 and S_2 as the *basic* variables.
- The problem is of the "max" type. To increase the value of Z, one may increase x_1 or x_2 from their current zero levels at O.
- We select to increase x_2 since it has a better per unit contribution to the objective function. We call x_2 the *entering variable* since it will be > 0 and therefore become basic (x_2 enters the basis).
- Keeping x_1 fixed at 0, increase x_2 (i.e. move along the direction of x_2) while remaining feasible.
- We see that the maximum we can increase x_2 is when A is reached (i.e., x_2 is increased from 0 to 4). Increasing x_2 further would make $S_2 < 0$. We call S_2 the "blocking variable."
- The simplex method requires that *the blocking variable leaves the basis and be replaced by the entering variable*.
- The simplex method then restarts at A with nonbasic variables x_1 and S_2 and basic variables S_1 and x_2.
- The simplex method then moves to other extreme points in a similar fashion until the optimal solution is reached.
- In the current example, it can be verified that the simplex algorithm will move from A to B and stop with B being the optimal solution. This last move involves x_1 entering the basis and S_1 (the blocking variable) leaving the basis.
- The simplex "path" is plotted above.

- **Analytical verification of "blocking"**
  - One can verify which basic variable is blocking analytically without looking at the graph.
  - For example, consider the first simplex move (iteration) from O to A. To verify whether S_1 or S_2 is the blocking variable note that when moving along the x_2 direction (and $x_1 = 0$) the LP constraints give

$$\begin{aligned} 3\times 0 + 2x_2 + S_1 \quad &= 6 \Rightarrow S_1 = 6 - 2x_2 \\ 0 + 4x_2 \quad\quad + S_2 \quad &= 4 \Rightarrow S_2 = 4 - 4x_2 \end{aligned}$$

  - Setting the basic variables, S_1 and S_2, equal to zero gives

$$S_1 = 6 - 2x_2 = 0 \Rightarrow x_2 = 6/2 = 3$$

$$S_2 = 4 - 4x_2 = 0 \Rightarrow x_2 = 4/2 = 1$$

  - Therefore, as x_2 increases from zero (at O) S_2 hits zero first. That is, S_2 is the blocking variable.

- **Introduction to the simplex method in tabular form**
  - Rewrite the LP in standard form as

$$\begin{aligned} Z - 3x_1 - 4x_2 \quad\quad\quad &= 0 \\ 3x_1 + 2x_2 + S_1 \quad &= 6 \\ x_1 + 4x_2 \quad\quad + S_2 \; &= 4 \end{aligned}$$

This gives the simplex *tableau* at O

Entering variable

Basic	Z	x_1	x_2	S_1	S_2	RHS	Ratio
Z	1	-3	-4	0	0	0	-
S_1	0	3	2	1	0	6	6/2=3
S_2	0	1	4	0	1	4	4/4=1

Leaving (blocking) variable

- Once the tableau is constructed, we determine the entering variable. This is the variable with the most negative value in the Z row. In our example, it's x_2. This corresponds to choosing to increase x_2 in the graphical method.
- We then determine the leaving or the blocking variable. This is based on a "minimum ration test." The blocking variable is the basic variable having the minimum ratio between the right hand side (RHS) and the corresponding cell in the entering variable column. See the Ratio column in the above tableau. This implies that S_2 leaves the basis.
- With the new basic and nonbasic variables we need to develop a new simplex tableau (which corresponds to the new corner point the simplex moved to; in our case the new corner point is A).
- The new tableau is filled by utilizing the element at the intersection of the entering variable column and the leaving variable row as pivot element in *Gauss-Jordan raw operations*.
- If a_{rk} is the pivot element (in our example, $a_{rk} = a_{33} = 4$), then rows a_i, are rewritten in the new tableau as

$$a'_r = a_r / a_{rk} \Rightarrow a'_{rj} = a_{rj} / a_{rk}, \forall\ j$$
$$a'_i = a_i - a_{ik} a'_r \Rightarrow a'_{ij} = a_{ij} - a_{ik} a'_{rj}, \forall i \neq r,\ j$$

- Applying this procedure to the above tableau gives the following simplex tableau (at A)

Entering variable ↓ (x_1)

Leaving (blocking) variable ← (S_1)

Basic	Z	x_1	x_2	S_1	S_2	RHS	Ratio
Z	1	-2	0	0	1	4	-
S_1	0	5/2	0	1	-1/2	4	8/5
x_2	0	1/4	1	0	1/4	1	4

- *Optimality criteria*: The optimal solution is reached when all the values in the Z-row are nonnegative.
- This is not the case in the above tableau.
- Therefore, we perform another "simplex iteration." (This iteration corresponds to moving from A to B in the graphical method.)
- Applying similar steps as with the first table we see that x_1 enters the basis and S_1 leaves the basis yielding the following simplex tableau (at B)

Basic	Z	x_1	x_2	S_1	S_2	RHS	Ratio
Z	1	0	0	4/5	3/5	36/5	-
x_1	0	1	0	2/5	-1/5	8/5	-
x_2	0	0	1	-1/10	3/10	3/5	-

- In this tableau, all the values in the Z-row are ≥ 0. This indicates that the simplex method has "converged" to the optimal solution. The optimal solution is

$$x_1{*} = 8/5, \quad x_2{*} = 3/5.$$

  The corresponding Z* is 36/5.
- Note that this indeed corresponds to corner point B.

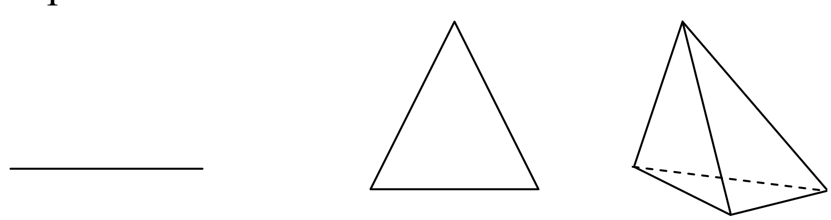
- **Basic, feasible, basic feasible, and basic infeasible solutions**

  - A basic solution corresponds to a corner point. For example, in the above (refer to the figure on page 2), O, A, B, C, D, and E all correspond to basic solutions.
  - A feasible solution is a solution that satisfies all constraints.
  - A basic solution can either be feasible (hence called basic feasible solution) or infeasible (called basic infeasible solution).
  - In the above, O, A, B, C are basic feasible solutions, while E and F are basic infeasible solutions.

- The simplex method starts at a basic feasible solution and iterates through a sequence of basic feasible solutions until it reaches optimality.

- **Local optimal and global optimal**
  - In a LP, a local optimal is a global optimal.
  - If in a maximization (minimization) problem the Z-value of a corner point is larger (smaller) than that of the adjacent corner points, then this corner point is an optimal solution.
  - For example, knowing that the Z-value at A, B, and C are 4, 36/5, and 6 respectively in the above example, allows us to conclude that B is optimal.

- **Accurate definition of "simplex" and how it comes into play**
  - A corner point and its adjacent corner points define a "simplex"
  - Loosely defined, a simplex in $\mathbb{R}^n$ is the geometric shape which defines an object in $\mathbb{R}^n$ with the minimum number of corner points.
  - For example, in $\mathbb{R}^1$, a simplex is a line segment. In $\mathbb{R}^2$ (the plane) a simplex is a triangle and in $\mathbb{R}^3$ (the 3D space) a simplex is a tetrahedron.



  - At each iteration, the simplex method examines whether the solution can be improved by moving along the edges of a simplex having the current solution as on one of its vertices.
  - This procedure gives an optimal solution because a local optimal is a global optimal as explained above.