Chapter 13 Additional Option Topics (1)

- **Overview and some History**
  - We saw how binomial lattices can be used to price options.
  - Binomial lattices assume discrete state space and time points where the stock price can change.
  - Now, we will see how to price options based on a continuous geometric Brownian motion model for the stock price.
  - The derivation given here was originated in the seminal papers by Black and Scholes (1973) and Merton (1973).[1]
  - Black, Scholes and Merton work won them the Nobel prize in Economics in 1997 (Black passed away before receiving the award).
  - This work also fueled interest in finance and contributed to the emergence of *Financial Engineering*.
  - Merton and Scholes were also among the founders of the multi-billion dollars company Long Term Capital Management (LTCM) which relied on heavy mathematics and fast computers.
  - However, LTCM collapsed due to unexpected events in the market. (Guess, Brownian motion didn't work then!)

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[1] Black, F. and M. Scholes. The pricing of options and other corporate liabilities. *Journal of Political Economy* **81:** 673-654, 637-654, 1973.
Merton, R. C. Theory of rational option pricing. *Bell Journal of Economics and Management Science* **4:** 141-183, 1973.

- **The Black-Scholes Equation**

  Theorem *Suppose the stock price follows a geometric Brownian motion defined by* $dS(t) = \mu S(t)dt + \sigma S(t)dz$ *and let r be the risk-free interest rate. Then, a derivative security has a (no-arbitrage) price f (S, t) (at time t) which satisfies the partial differential equation*

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf \quad .$$

Proof. We will present the original proof of Black and Sholes (1973) (see also Hull 2006).[2] It is based on (i) Ito's lemma; (ii) forming a portfolio of the option and the stock in a way that eliminates uncertainty; and (iii) equating the portfolio return to the risk-free return to avoid arbitrage. By Ito's lemma, the change in *f* in a small time interval Δt is

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right)\Delta t + \frac{\partial f}{\partial S}\sigma S \ \Delta z \quad . \tag{1}$$

For the stock price,

$$\Delta S = \mu S \ \Delta t + \sigma S \ \Delta z \quad , \tag{2}$$

and for a risk-free asset (a bond) with price *B* under continuous compounding

$$\Delta B = rB \ \Delta t \quad . \tag{3}$$

Consider a portfolio consisting of shorting one unit of the derivative security and buying an amount $\frac{\partial f}{\partial S}$ of stocks.

[2] Our text (Luenberger) adopts a different approach.

The value of this portfolio is $\Pi = -f + \frac{\partial f}{\partial S} S$.

The change of the portfolio value in a small time interval Δt is

$$\begin{aligned}\Delta\Pi &= -\Delta f + \frac{\partial f}{\partial S}\Delta S \\ &= -\frac{\partial f}{\partial S}\mu S\Delta t - \frac{\partial f}{\partial t}\Delta t - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \Delta t - \frac{\partial f}{\partial S}\sigma S\ \Delta z + \frac{\partial f}{\partial S}\mu S\ \Delta t + \frac{\partial f}{\partial S}\sigma S\ \Delta z \\ &= -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t\ .\end{aligned}$$

Note that $\Delta\Pi$ does not involve the random term Δz. That is, the portfolio is risk-free. Therefore, in order to avoid arbitrage the portfolio value must satisfy an equation similar to (3),

$$\Delta\Pi = r\Pi\Delta t \Rightarrow -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t = r\left(-f + \frac{\partial f}{\partial S} S\right)\Delta t\ .$$

The proof follows upon simplification. ∎

- **Simple Black-Scholes Equation Verification**
  - To see that the Black-Scholes equation makes sense, consider some simple cases.
  - If the derivative security is the stock itself, then $f(S, t) = S$,

    $\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = 0 + rS + 0 = rS$ (i.e., equation holds).
  - If the derivative security is a bond with value \$1 at time 0,

    then $f(S, t) = e^{rt}$, $\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = re^{rt} + 0 + 0 = rf$.
  - To avoid arbitrage, the price of any derivative security *must* satisfy the Black-Scholes equation (***if*** assumptions hold).

- **Pricing Securities by Solving the Black-Scholes Equation**
  - Determining the price of a derivative security can be done using the Black-Scholes equation once appropriate *boundary conditions* are specified.
  - For a European call option maturing at time T and with strike price K and price $C(S, t)$ at t, the boundary condition is
$$C(S,\ T) = \max(S - K,\ 0)\ .$$
  - For a European put option with price $P(S, t)$ at t the boundary condition is
$$P(S,\ T) = \max(K - S,\ 0)\ .$$
  - For an American put option an additional boundary condition related to early exercise is required
$$P(S,\ t) \geq \max(K - S,\ 0)\ .$$

- **Perpetual Call**
  - This is an American call option with expiration time $T = \infty$.
  - Let $f(S, t)$ be the price at time t. Then, $f(S, t)$ satisfies the early exercise condition $f(S, t) \geq \max(0, S-K)$.
  - In addition, the call must cost less than the stock price at t, $f(S, t) \leq S$.
  - The function $f(S, t) = S$, satisfies both boundary conditions and Black-Scholes differential equation.
  - Therefore, $f(S, t) = S$ (i.e. perpetual value equals stock price).

- **The Black-Scholes Call Option Formula**
  - In the case of a European call option, the Back-Scholes differential equation has a closed form solution.

Theorem *The price at time t of a European call option with strike price K and maturity T on an underlying stock with volatility σ is*

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \ ,$$

where S is the stock price at time t, $d_1 = \dfrac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$,

$d_2 = d_1 - \sigma\sqrt{T-t}$ *and* $N(x) = \int_{-\infty}^{x} \dfrac{e^{-x^2/2}}{\sqrt{2\pi}} dx$ *is the standard Normal cdf.*

- To prove the call option formula, Black and Scholes (1973) make the following change of variables

$$f(S,t) = e^{r(t-T)} y\left(u(S,t), v(s,t)\right),$$

$$u(S,t) = \frac{2}{\sigma^2}\left(r - \frac{\sigma^2}{2}\right)\left[\ln\frac{S}{K} - \left(r - \frac{\sigma^2}{2}\right)(t-T)\right],$$

$$v(S,t) = -\frac{2}{\sigma^2}\left(r - \frac{\sigma^2}{2}\right)^2 (t-T).$$

- They then argue that the differential equation reduces to

$$\frac{\partial y}{\partial v} = \frac{\partial^2 y}{\partial u^2} \ ,$$

with boundary conditions

$$y(u,0) = \begin{cases} 0, & \text{if } u < 0 \\ K\left[e^{(u\sigma^2/2)/(r-\sigma^2/2)} - 1\right], & \text{if } u \geq 0 \end{cases}.$$

- Black and Scholes call this differential equation as the "heat-transfer equation of physics", and refer to Churchill (1963) for a solution.

- **Simple Black-Scholes Formula Verification**
  - At expiration, $t = T$

$$\Rightarrow \begin{cases} d_1 = d_2 = \infty, \text{if } S > K \\ d_1 = d_2 = -\infty, \text{if } S < K \end{cases} \Rightarrow \begin{cases} C(S,T) = SN(d_1) - Ke^{-r(T-T)}N(d_2) = S - K, \quad \text{if } S > K \\ C(S,T) = N\Phi(d_1) - Ke^{-r(T-T)}N(d_2) = 0 - 0 = 0, \text{if } S > K \end{cases}$$

  - For a perpetual call, $T = \infty \Rightarrow d_1 = d_2 = \infty$

$$\Rightarrow C(S,t) = SN(d_1) - Ke^{-\infty}(d_2) = S \,.$$

- **Black-Scholes Formula Interpretation (Sharpe 1999)**
  - Similar to the binomial lattice model holding the call option is equivalent to holding a *replicating portfolio* having $N(d_1)$ stocks and borrowing $KN(d_2)$ cash (i.e. shorting $KN(d_2)$ of the risk-free asset).
  - Note that the replicating portfolio composition is continuously changing as the stock price fluctuates and time advances.

- **Risk-Neutral Valuation**
  - The variable μ (stock expected rate of return) does not appear in the Black-Scholes equation. Therefore, the equation is independent of risk preference.
  - The solution to the differential equation is therefore the same in a risk-free world as it is in the real world.
  - This leads to the principle of risk-neutral valuation (similar to the binomial lattice model).
  - For the geometric Brownian motion model, $dS(T) = \mu S(T)dt + \sigma S(T)dz$, the stock price at time $T > t$ is lognormal with $E[S(T)] = S(t)e^{\mu(T-t)}$.
  - In a risk-neutral setting, $E[S(T)] = S(t)e^{r(T-t)}$.
  - Therefore, the risk-neutral "q-distribution" of the stock price is a geometric Brownian motion with an expected rate of return r, i.e., $dS(T) = rS(T)dt + \sigma S(T)d\hat{z}$, where $\hat{z}$ is a Brownian motion.
  - This implies that in the risk-neutral world the stock price distribution at time T is lognormal with parameters $E[\ln S(T)] = \ln S(t) + (r - \sigma^2/2)(T-t)$, $\mathrm{var}[\ln S(T)] = \sigma^2 T(T-t)$.
  - Then, the price of a derivative security with expiration date T under risk-neutral pricing (which is the "right" price) is

$$f(S,t) = e^{-r(T-t)}\hat{E}[f(S,T)],$$

where $\hat{E}$ denotes the expectation with respect to $\hat{E}$.

- **Simple Risk-Neutral Verification**
  - If the derivative security is the stock itself, $f(S,T) = S(T)$,

$$f(S,t) = e^{-r(T-t)}\hat{E}[f(S,T)] = e^{-r(T-t)}\int_0^\infty s_T g_{S_T}(s_T)ds_T = e^{-r(T-t)}E[S(T)]$$
$$= e^{-r(T-t)}S(t)e^{r(T-t)} = S(t)\,,$$

    where $g_{S_T}(.)$ is the lognormal density function of $S(T)$.

  - For a derivative security consisting of \$1 of cash (risk-free asset) at time 0, $f(S,T) = e^{rT}$,

$$f(S,0) = e^{-rT}\hat{E}[f(S,T)] = e^{-rT}\int_0^\infty e^{rT} g_{S_T}(s_T)ds_T = 1\,.$$

- **Risk-Neutral Derivation of the Black-Scholes Formula**
  - We first present the following supporting lemma. This is from Hull (2006).

    Lemma *If X is a lognormal random variable with parameters E[ln(X)] =* λ *and var[ln(X)] =* σ^2*, then*

$$E[\max(X-K,0)] = e^{\lambda+\sigma^2/2}N(d_1) - KN(d_2),$$

    where $d_1 = \dfrac{\ln[e^{\lambda+\sigma^2/2}/K] + \sigma^2/2}{\sigma}$ *and* $d_2 = d_1 - \sigma$.

  - The proof of this lemma is obtained by using the standard change of variable $Y = (\ln X - \lambda)/\sigma$, and noting that

$$E[\max(S-K,0)] = \int_K^\infty (x-K)g_X(x)dx = \int_{(\ln K-\lambda)/\sigma}^\infty (e^{y\sigma+\lambda} - K)\varphi(y)dy\,,$$

where $g_X(.)$ and $\varphi(.)$ are the density functions of Y,

$\varphi(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}}$, and X.

- Then, for a European call option risk-neutral pricing gives

$$C(S,t) = e^{-r(T-t)}\hat{E}[f(S,T)] = e^{-r(T-t)}\hat{E}[\max(S(T)-K,0)] \ ,$$

where $S(T)$ is a lognormal rv with

$$E[\ln S(T)] = \ln S(t) + (r - \sigma^2/2)(T-t) \ , \ \mathrm{var}[\ln S(T)] = \sigma^2(T-t).$$

- Applying the lemma

$$E[\max(S(T)-K,0)] = e^{\ln S(t)+(r-\sigma^2/2)(T-t)+(\sigma^2/2)(T-t)}N(d_1) - KN(d_2)$$
$$= S(t)e^{r(T-t)}N(d_1) - KN(d_2), \quad \text{where}$$

$$d_1 = \frac{\ln[e^{\ln S(t)+(r-\sigma^2/2)(T-t)+(\sigma^2/2)(T-t)}/K] + (\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$
$$= \frac{\ln[S(t)/K] + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

- The Black-Scholes formula follows.

- **Convenient form of Black-Scholes Formula (Ross 2003)**
  - A convenient way for writing the Black-Sholes formula follows by recalling that

$$\ln S(T) = \ln(S(0)) + (r - \sigma^2/2)(T-t) + \sigma\hat{z}(T-t) .$$

  - This implies that

$$C(S,t) = e^{-r(T-t)}E[\max(Se^W - K,0)] \ ,$$

where W is a normal random variable with mean $(r-\sigma^2/2)(T-t)$ and variance $\sigma^2\,(T-t)$.

- **Properties of the Black-Scholes Formula (Ross 2003)**

  Lemma *The call price at time t* $C(S, T, K, \sigma, t)$ *is*

  (i) Increasing and convex in the stock price at time t, S;

  (ii) Increasing and convex in the strike price, K;

  (iii) Increasing in the expiration time, T;

  (iv) Increasing in the stock price volatility, σ;

  (v) Increasing in the risk-free interest rate, r.

  - The proof of (i) follows easily from the convenient form, $C(S,t) = e^{-r(T-t)} E[\max(Se^{W} - K, 0)]$, since the function $e^{-r(T-t)} \max(se^{W} - K, 0)$ is increasing convex in s for all W.
  - The proof of (ii) is similar.