Simplex method in matrix form (revised simplex method)

- A LP with n decision variables and m constraints can be written as

$$\begin{aligned} \max \quad & Z = \mathbf{cx} \\ \text{s.t.} \quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

where $\mathbf{c} = (c_1, c_2, c_3, \ldots, c_n)$, $\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

- Alternatively, the LP can be written as

$$\begin{aligned} \max \quad & Z = \mathbf{c}_B \mathbf{x}_B + \mathbf{c}_N \mathbf{x}_N \\ \text{s.t.} \quad & \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{1}$$

where the subscripts "B" and "N" denote basic and nonbasic variables respectively.

- For example,

$$\begin{cases} \max \quad Z = 2x_1 + 3x_2 \\ \text{s.t.} \quad x_1 + x_2 \leq 50 \\ \quad 2x_1 + x_2 \leq 30 \\ \quad x_1, x_2 \geq 0 \end{cases} \Leftrightarrow \begin{cases} \max \quad Z = 2x_1 + 3x_2 \\ \text{s.t.} \quad x_1 + x_2 + S_1 = 50 \\ \quad 2x_1 + x_2 + S_2 = 30 \\ \quad x_1, x_2, S_1, S_2 \geq 0 \end{cases}$$

Then, at O(0,0),

$$\mathbf{x}_B = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}, \quad \mathbf{x}_N = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{c}_N = (2,3), \quad \mathbf{c}_B = (0,0), \quad \mathbf{b} = \begin{pmatrix} 50 \\ 30 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

- Solving for $\mathbf{x}_B$ in (1) gives

$$\mathbf{B}\mathbf{x}_B = \mathbf{b} - \mathbf{N}\mathbf{x}_N \Rightarrow \mathbf{x}_B = \mathbf{B}^{-1}\left(\mathbf{b} - \mathbf{N}\mathbf{x}_N\right) = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N$$

- The LP can then be rewritten as

$$\begin{aligned} \max \quad & Z = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} - (\mathbf{c}_B\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N)\mathbf{x}_N \\ \text{s.t.} \quad & \mathbf{x}_B + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = B^{-1}\mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

- Recall that each iteration of the simplex method allows a nonbasic variable (the entering variable) to increase from zero. until one of the basic variables (the leaving variable) hits zero.
- Let N_j be the j^{th} column of N, and V_{i} be the i^{th} component of vector V. Then, an iteration of the simplex method, with x_j being the basic variable and x_i being basic variables, can be represented by the following equations:

$$Z = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} - (c_B\mathbf{B}^{-1}\mathbf{N}_j - \mathbf{c}_j)x_j \quad \Leftrightarrow Z^{new} = Z^0 - \mathbf{c}_j^0 x_j$$

$$x_i = (\mathbf{B}^{-1}\mathbf{b})_i - (\mathbf{B}^{-1}\mathbf{N}_j)_i x_j \quad \Leftrightarrow x_i^{new} = \mathbf{b}_i^0 - \mathbf{A}_{ij}^0 x_j$$

where the superscripts "0" and "new" denote the current solution and the new solution generated by the next iteration.

- For a max problem, we choose the entering variable, x_k, s.t.

$$z_k - c_k = \min\{z_j - c_j \mid j \text{ is nonbasic and } z_j - c_j < 0\}, \qquad (2)$$

where, $z_j \equiv \mathbf{c}_B \mathbf{B}^{-1} \mathbf{N}_j$.[1]

- The leaving variable is x_r s.t.

$$\frac{(\mathbf{B}^{-1}\mathbf{b})_r}{\left(\mathbf{B}^{-1}\mathbf{N}_k\right)_r} = \min\left\{ \frac{(\mathbf{B}^{-1}\mathbf{b})_i}{\left(\mathbf{B}^{-1}\mathbf{N}_k\right)_i} \middle| i \text{ is basic and } \left(\mathbf{B}^{-1}\mathbf{N}_j\right)_i > 0 \right\}. \qquad (3)$$

- The above provides the rational for the revised simplex method which proceeds as follows

Step 0. Determine a starting basic feasible solution with basis Ω.

Step 1. Evaluate $\mathbf{B}^{-1}$.

Step 2. Compute $(z_j - c_j)$ for all nonbasic variables. If $(z_j - c_j) \geq 0$ for a maximization problem (≤ 0 for a minimization), then stop. The optimal solution is $\mathbf{x}_B{}^* = \mathbf{B}^{-1}\mathbf{b}, \quad Z^* = \mathbf{c}_B \mathbf{x}_B{}^*$. Else, determine the entering variable, x_k, using (2), and go to Step 3.

Step 3. Compute $\mathbf{B}^{-1}N_k$. If all elements of $\mathbf{B}^{-1}\mathbf{N}_k \leq 0$, then stop, the solution is unbounded. Else, compute $\mathbf{B}^{-1}\mathbf{b}$ and determine the leaving variable, x_r, using (3).

Step 4. determine the new basis, $\Omega_{new} = \Omega \cup \{x_k\} - \{x_r\}$. Set $\Omega = \Omega_{new}$, and go to Step 1.

[1] This rule for choosing the entering variable is just a rule of thumb. In some cases, there might be better ways to choose the entering variable.

- Example

$$\begin{cases} \max \quad Z = 2x_1 + 3x_2 \\ \text{s.t.} \qquad x_1 + x_2 \le 50 \\ \qquad\quad 2x_1 + x_2 \le 30 \\ \qquad\quad x_1, x_2 \ge 0 \end{cases} \Leftrightarrow \begin{cases} \max \quad Z = 2x_1 + 3x_2 \\ \text{s.t.} \qquad x_1 + x_2 + S_1 \qquad = 50 \\ \qquad\quad 2x_1 + x_2 \qquad + S_2 = 30 \\ \qquad\quad x_1, x_2, S_1, S_2 \ge 0 \end{cases}$$

- **Iteration 1.**
  - **Step 0.** Starting basic feasible solution at O(0,0),

$$\Omega = \{S_1, S_2\}, \mathbf{x}_B = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}, \quad \mathbf{x}_N = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{c}_N = (2,3), \quad \mathbf{c}_B = (0,0),$$

$$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 50 \\ 30 \end{pmatrix}$$

  - **Step 1.**

$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

  - **Step 2.**

$$z_{x_1} - c_{x_1} = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{N}_{x_1} - c_{x_1} = (0,0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2 = -2$$

$$z_{x_2} - c_{x_2} = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{N}_{x_2} - c_{x_2} = (0,0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 3 = -3$$

  The entering variable is x_2 .
  - **Step 3.**

$$\mathbf{B}^{-1} \mathbf{N}_{x_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{matrix} (S_1) \\ (S_2) \end{matrix}$$

$$\mathbf{B}^{-1} \mathbf{b} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 50 \\ 30 \end{pmatrix} = \begin{pmatrix} 50 \\ 30 \end{pmatrix} \quad \begin{matrix} (S_1) \\ (S_2) \end{matrix}$$

  Then, $\dfrac{(\mathbf{B}^{-1}\mathbf{b})_{S_1}}{\left(\mathbf{B}^{-1}\mathbf{N}_{x_2}\right)_{S_1}} = \dfrac{50}{1} = 50$, $\dfrac{(\mathbf{B}^{-1}\mathbf{b})_{S_2}}{\left(\mathbf{B}^{-1}\mathbf{N}_{x_2}\right)_{S_2}} = \dfrac{30}{1} = 30$.

  The leaving variable is S_2 .
  - **Step 4.** The new basis is $\Omega = \{S_1, S_2\} \cup \{x_2\} - \{S_2\} = \{S_1, x_2\}$.

- **Iteration 2.**
  - **Step 1.**

$$\Omega = \{S_1, x_2\},\ \mathbf{x}_B = \begin{pmatrix} S_1 \\ x_2 \end{pmatrix},\ \mathbf{x}_N = \begin{pmatrix} x_1 \\ S_2 \end{pmatrix},\ \mathbf{c}_B = (0,3),\ \mathbf{c}_N = (2,0),$$

$$\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \mathbf{N} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \mathbf{B}^{-1} = \frac{1}{1}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$ [2]

  - **Step 2.**

$$z_{x_1} - c_{x_1} = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{N}_{x_1} - c_{x_1} = (0,3)\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2 = (0,3)\begin{pmatrix} -1 \\ 2 \end{pmatrix} - 2 = 4$$

$$z_{S_2} - c_{S_2} = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{N}_{S_2} - c_{S_2} = (0,3)\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - 0 = (0,3)\begin{pmatrix} -1 \\ 1 \end{pmatrix} - 0 = 3$$

All $z_j - c_j > 0$, for all j nb. Stop. The optimal solution is reached. The optimal solution is

$$\mathbf{x}_B{}^* = \begin{pmatrix} S_1{}^* \\ x_2{}^* \end{pmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 50 \\ 30 \end{pmatrix} = \begin{pmatrix} 20 \\ 30 \end{pmatrix}$$

$$Z^* = \mathbf{c}_B \mathbf{x}_B{}^* = \begin{pmatrix} 0 & 3 \end{pmatrix}\begin{pmatrix} 20 \\ 30 \end{pmatrix} = 90$$

Therefore, the optimal solution is $x_1^* = 0$, $x_2^* = 30$, and $Z^* = 90$.

Remark. The solution to the same problem in tabular form is presented on the next page. It is instructive to compare the two solution methods.

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[2] The inverse of $\mathbf{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\mathbf{B}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. (This inverse exists only if $ad - bc \neq 0$.)

Iteration 1.

Entering variable

$z_j - c_j$

Basic	x_1	x_2	S_1	S_2	RHS	Ratio
Z	−2	−3	0	0	0	-
S_1	1	1	1	0	50	50/1=50
S_2	2	1	0	1	30	30/1=30

$\frac{(\mathbf{B}^{-1}\mathbf{b})_i}{\left(\mathbf{B}^{-1}\mathbf{N}_k\right)_i}$

Leaving (blocking) variable

B^{-1}

Iteration 2.

$z_j - c_j$

$\mathbf{c_B B}^{-1}\mathbf{b}$

$\mathbf{B}^{-1}\mathbf{b}$

Basic	x_1	x_2	S_1	S_2	RHS
Z	4	0	0	3	90
S_1	−1	0	1	−1	20
x_2	2	1	0	1	30

$\mathbf{B}^{-1}$