Duality (Continuous)

- **Economic Interpretation of duality**

  - A resource allocation example. Dakota furniture company manufactures desks, tables and chairs. The manufacturing of each type of furniture requires three resources: Lumber, finishing labor, and carpentry labor. The amount of each resource needed to make one unit of a certain type of furniture is as follows.

	Desk	Table	Chair
Lumber	8 lft	6 lft	1 lft
Finishing hours	4 hrs	2 hrs	1.5 hrs
Carpentry hours	2 hrs	1.5 hrs	0.5 hrs

At present, 48 lft of lumber, 20 hours of finishing hours, and eight hours of carpentry hours are available. A desk sells for \$60, a table for \$30, and a chair for \$20. How many of each type of furniture should Dakota produce?

  - Let x_1, x_2, and x_3 respectively denote the number of desks, tables and chairs produced. Dakota problem is solved with the following LP:

$$
\begin{aligned}
\max \quad & 60x_1 + 30x_2 + 20x_3 \\
\text{s.t.} \quad & 8x_1 + 6x_2 + x_3 \le 48 \\
& 4x_1 + 2x_2 + 1.5\,x_3 \le 20 \\
& 2x_1 + 1.5\,x_2 + 0.5\,x_3 \le 8 \\
& x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0
\end{aligned}
$$

  - Suppose that an entrepreneur wants to buy Dakota's resources. What are the *fair* prices, y_1, y_2, y_3, that the entrepreneur should pay for a lft of lumber, one hour of finishing and one hour of carpentry?

- The entrepreneur wants to minimize buying cost. Then, the entrepreneur objective is

$$\min \quad 48y_1 + 20y_2 + 8y_3$$

- In exchange for the resources that could make one desk, the entrepreneur is offering ($8y_1 + 4y_2 + 3y_3$) dollars. This amount should be larger than what Dakota could make out of manufacturing one desk (\$60). Therefore,

$$8y_1 + 4y_2 + 2y_3 \geq 60$$

- Similarly, by considering the "fair" amounts that the entrepreneur should pay for the combination of resources that are required to make one table and one chair, we conclude

$$6y_1 + 2\ y_2 + 1.5y_3 \geq 30$$
$$y_1 + 1.5y_2 + \ 0.5y_3 \geq 20$$

- Consequently, the entrepreneur should pay the prices y_1, y_2, y_3, which are the solution to the following LP:

$$\begin{aligned} \min \quad & 48y_1 + 20y_2 + 8y_3 \\ \text{s.t.} \quad & 8y_1 + \ 4y_2 + \ 2y_3 \geq 60 \\ & 6y_1 + 2\ y_2 + 1.5y_3 \geq 30 \\ & y_1 + 1.5y_2 + 0.5y_3 \geq 20 \\ & y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0 \end{aligned}$$

- The above LP is the dual to the manufacturer's LP.
- For this reason, the dual variables are often referred to as **shadow prices**, the fair market prices, or the dual prices.
- Shadow prices are used by utilities, such as electricity companies, to set and justify prices.
- Airlines and other service industries use the concept of shadow prices to decide which fare classes to "close" at a certain time period. This is *network revenue management*.