Inventory Theory (2)

- **The single-period newsvendor model**
  - Consider a newsvendor, who, at the start of each day, must decide the amount of newspapers to stock, S.
  - Placing an order has a negligible cost.
  - Daily demand for the newspaper is D (a random variable).
  - If demand during the day is less than S, than a holding cost at a rate h per unit is charged for each unit remaining in inventory.
  - If demand during the days is greater than S, than a shortage cost b is charged on each unit remaining in inventory.
  - By conditioning on daily demand, the expected daily cost is

$$EC(S) = h\int_0^S (S-x)f_D(x)dx + b\int_S^\infty (x-S)f_D(x)dx$$

where $f_D(.)$ is the pdf of D. (Denote by $F_D(.)$ the cdf of D.)

  - Differentiating with respect to S (using Leibniz rule, see below) gives

$$\frac{\partial EC(S)}{\partial S} = h\int_0^S f_D(x)dx - b\int_S^\infty f_D(x)dx = hF_D(S) - b(1-F_D(S))$$

$$\frac{\partial^2 EC(S)}{\partial S^2} = (h+b)f_D(S) > 0$$

- It follows that $EC(S)$ is convex in S with an optimal order quantity given by

$$F_D(S^*) = P\{D < S^*\} = \frac{b}{b+h}.$$

- **Leibniz rule**

$$\frac{d}{ds}\int_{a_1(s)}^{a_2(s)} f(x,s)dx = \int_{a_1(s)}^{a_2(s)} \frac{\partial f(x,s)}{\partial s}dx + f(a_2(s),s)\frac{\partial a_2(s)}{\partial s} - f(a_1(s),s)\frac{\partial a_1(s)}{\partial s}$$

- **Newsvendor model facts**
  - The newsvendor has many applications beyond the newspaper case. E.g., it applies to perishable goods (e.g., produce, bread, etc.) and to style clothing.
  - The model can be even applied in manufacturing when deciding on how much to produce in a single batch.
  - The ratio $b/(b+h)$ is known as the *critical fractile*. This ratio can be written as $c_u/(c_u+c_o)$, where c_u and c_o are the unit cost of underage and overage.
  - There are different derivations of the newsvendor model that consider other parameters such as unit variable cost c, unit salvage value, v, and selling price r.
  - These parameters can be included into the model easily. E.g., setting $h' = h + c - v$ and $b' = b + (r - c)$, allows deriving S^* via a critical fractile $b' / (b' + h')$.

- If the period starts with an initial on hand, inventory, x, then the optimal policy is to order $S^* - x$ if $x < S^*$ and not order otherwise, where S^* is as given above. (This is called an *order-up-to* policy.)
- This follows because $EC(S)$ is convex in S.
- If the initial inventory is $x < S^*$, then the optimal order quantity is $Q^* = S^* - x$, which minimizes expected cost.
- If $x \geq S^*$, then any order quantity $Q > 0$ increases cost.

- **Example 9.**
  - The owner of a newsstand wants to determine the number of *USA now* newspapers that must be ordered at the beginning of each day. The owner pays ¢30 per copy and sells it for ¢75. Newspapers left at the end of the day are sold for recycling purposes at a price of ¢5. Daily demand is assumed to be normally distributed with mean 300 and standard deviation 20.
  - In this case, the order quantity can be found by defining equivalent holding and penalty costs, $h' = 30 - 5 = ¢25$ and $b' = 75 - 30 = ¢45$.
  - Let μ and σ be the mean and standard deviation of daily demand. The optimal order quantity is given by

$$P\{D < S^*\} = \frac{b'}{b'+h'} \Rightarrow P\{Z < \frac{S^*-\mu}{\sigma}\} = \frac{b'}{b'+h'}$$
$$\Rightarrow S^* = \mu + \sigma\Phi^{-1}\left(\frac{b'}{b'+h'}\right),$$

where $\Phi^{-1}(.)$ is the inverse of the standard normal cdf.

- Therefore, the optimal order quantity is 307, derived as

$$S^* = 300 + 20\Phi^{-1}\left(\frac{45}{45+30}\right) = 300 + 20\Phi^{-1}(0.643)$$
$$\approx 300 + 20 \times 0.37 \approx 307$$

- Note that $\Phi^{-1}(0.643) \approx 0.37$ was found from Table B.1.