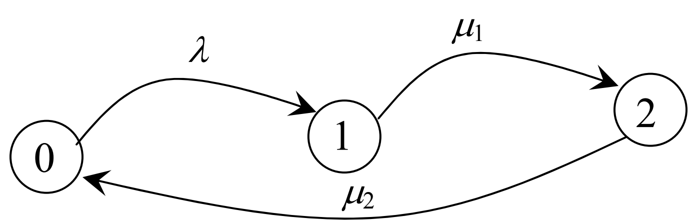
Continuous-Time Markov Chains

- **Definition and Connection to the Exponential Distribution**
  - A continuous-time stochastic process $\{X(t), t \geq 0\}$ taking on positive integers is said to be a continuous-time Markov chain (CTMC) if for all s, $t \geq 0$, i, j, k_u integers, $0 \leq u < s$,
    $P\{X(t+s) = j \mid X(s) = i, X(u) = k_u\} = P\{X(t+s) = j \mid X(s) = i\}$.
  - If, in addition, these transition probabilities are independent of s, the CTMC is said to have stationary or homogenous transition probabilities. We only consider this kind of CTMC.
  - Let T_i be the amount spent in state i before making a transition to another state. Then,
    $$P\{T_i > s + t \mid T_i > s\} = P\{X(t+s) = i \mid X(s) = i\}$$
    $$= P\{X(t) = i \mid X(0) = i\} = P\{T_i > t\} .$$
  - Therefore, T_i has the memoryless property.
  - It follows that T_i is exponentially distributed.
  - Let v_i be the "rate" of transition out of i (i.e., $E[T_i] = 1/v_i$).
  - Define also P_{ij} as the probability that the process enters j after transitioning out of i. By definition,
    $$P_{ii} = 0 ,$$
    $$\sum_{j=0}^{\infty} P_{ij} = 1 .$$
  - The parameters v_i and P_{ij} completely define the CTMC.

- **Example 1**

➢ Consider a shoeshine shop consisting of two chairs. A customer arrives and set in chair 1, where his shoes are cleaned and polish is applied, and then moves to chair 2 where his shoes is buffed. Suppose that customers inter-arrival times are iid exponential rvs with rate λ, and that service times at chair i are iid exponential rvs with rate μ_i, $i = 1$, 2. Suppose that a customer will enter the shop only if both chairs are empty.

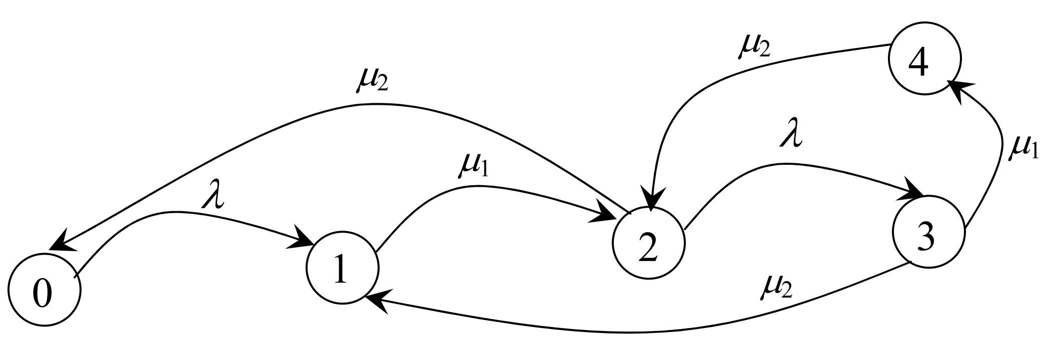
➢ This is a CTMC with three states: 0 (both chairs are empty), 1 (a customer is in chair 1), and 2 (a customer is in chair 2).



➢ In this case, $v_0 = \lambda$, $v_1 = \mu_1$, $v_2 = \mu_2$, $P_{01} = P_{12} = P_{20} = 1$, and $P_{ij} = 0$, otherwise.

➢ What if a customer would enter if only chair 1 is empty?

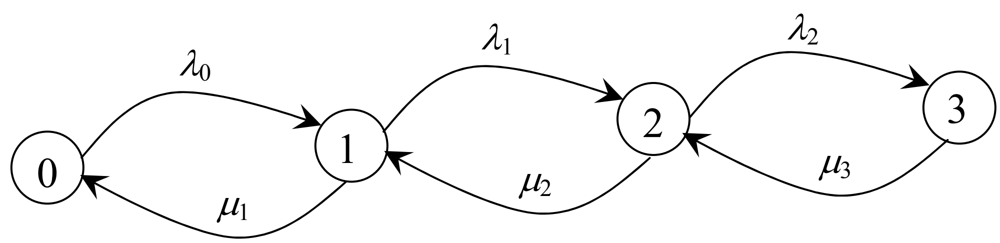
➢ Add two states: 3 (both chairs busy) and 4 (chair 1 waiting).



- **Birth-Death Processes**
  - Consider a stochastic process $\{X(t), t \geq 0\}$ representing the number of people in a population.
  - Suppose that whenever there are n people in a system, the time for the next arrival is exponential with rate λ_n, and the time of the next departure is exponential with rate μ_n.
  - When a departure (arrival) happens in state n, the system moves to state $n-1$ ($n+1$).
  - The process $X(t)$ is called a birth-death process.
  - It is a special case of a CTMC with

    $v_0 = \lambda_0$, $P_{01} = 1$,

    $v_i = \lambda_i + \mu_i$, $P_{i,i+1} = \lambda_i / (\lambda_i + \mu_i)$, $P_{i,i-1} = \mu_i / (\lambda_i + \mu_i)$, $i > 0$.



  - If $\mu_i = 0$, $i = 1,2, \ldots$, X_t is called a *pure birth process*.
  - If $\lambda_i = 0$, $i = 0,1,2, \ldots$, X_t is called a *pure death process*.

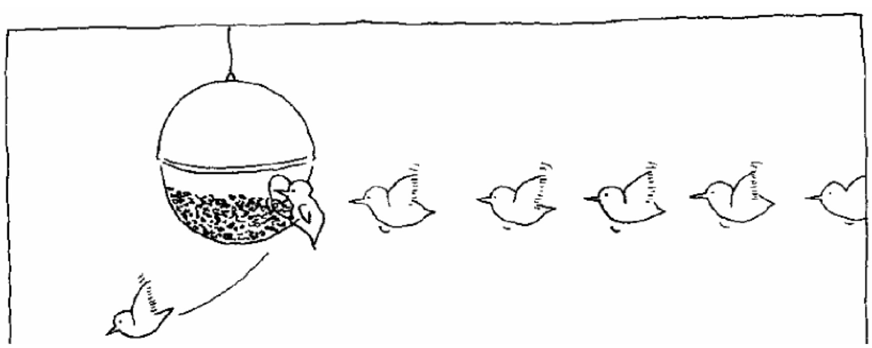
- **Example 2**
  - A pure birth process with $\lambda_i = \lambda$, $i = 0,1,2, \ldots$, is a *Poisson process*. This is the most popular models for arrival processes (where the inter-arrival times are exponential rvs).

- **Example 3**

➢ A pure birth process with $\lambda_i = i\lambda$, $i = 0,1,2, \ldots$, is called a Yule process. This is a model for a population where every person gives birth at a rate λ, independent of others, and no one dies.

- **Example 4**

➢ Consider a system with a single server. Customers arrive to the system according to a Poisson process with rate λ customers per hour. Customers who find the server busy wait in line, and those who find the server idle start service immediately. Service times are iid exponential rvs with rate μ customers per hour.



➢ This is a *queueing* model known as the $M/M/1$ queue.

➢ It is also a birth-death model with $\lambda_i = \lambda$, and $\mu_i = \mu$, $i = 0,1,2, \ldots$

- ***o*(h) Functions**
  - A function $f(h)$ is said to be $o(h)$ if

$$\lim_{h\to 0}\frac{f(h)}{h}=0.$$

  - E.g. $f(h) = h^n$, $n > 1$, is $o(h)$ since $\lim_{h\to 0} f(h)/h=\lim_{h\to 0} h^{n-1}=0.$
  - But $f(h) = h$, is not $o(h)$ since $\lim_{h\to 0} f(h)/h=\lim_{h\to 0} h/h=1.$

- **Properties of Transition Probabilities**
  - Consider a CTMC $\{X(t), t \geq 0\}$. Denote the transition probability from state i to state j within time t by $P_{ij}(t)$. I.e.,

$$P_{ij}(t)=P\{X(t+s)=j\mid X(s)=i\}.$$

  - Let $q_{ij} = v_i P_{ij}$ be the transition rate from state i to state j.

Lemma 1 *The transition probabilities satisfy the following:*

$$\lim_{h\to 0}\frac{1-P_{ii}(h)}{h}=v_i\ ,$$

$$\lim_{h\to 0}\frac{P_{ij}(h)}{h}=q_{ij}\ .$$

Proof. Note that

$$P_{ii}(h)=P\{T_i>h\}=e^{-v_i h}=1-v_i h+v_i h^2/2-v_i h^3/6+\ldots=1-v_i h+o(h),$$

which implies that $1-P_{ii}(h)=v_i h+o(h)$. In addition,

$$P_{ij}(h)=(1-P_{ii}(h))P_{ij}=v_i P_{ij} h+o(h)=q_{ij} h+o(h). \quad \blacksquare$$

Lemma 2 (Chapman-Kolmogorov equations)

$$P_{ij}(t+h)=\sum_{k=0}^{\infty}P_{ik}(t)P_{kj}(h)\,.$$

Proof. Follows by conditioning on the state the process is in at time *t*, similar to the discrete case. $\blacksquare$

Theorem 1 (Kolmogorov's forward equations) *$P_{ij}(t)$ satisfy the following differential equation:*

$$\frac{\partial P_{ij}(t)}{\partial t}=\sum_{k\neq j}q_{kj}P_{ik}(t)-v_jP_{ij}(t)\,.$$

Proof. Lemma 2 implies that

$$P_{ij}(t+h)-P_{ij}(t)=\sum_{k=0}^{\infty}P_{ik}(t)P_{kj}(h)-P_{ij}(t)=\sum_{k\neq j}P_{ik}(t)P_{kj}(h)-(1-P_{jj}(h))P_{ij}(t)$$

$$\Rightarrow\lim_{h\to 0}\frac{P_{ij}(t+h)-P_{ij}(t)}{h}=\lim_{h\to 0}\sum_{k\neq j}P_{ik}(t)\frac{P_{kj}(h)}{h}-\frac{(1-P_{jj}(h))}{h}P_{ij}(t)\,.$$

Lemma 1 then completes the proof. $\blacksquare$

- **Pure Birth and Poisson Process Transition Probabilities**
  - For a pure birth process, Kolmogorov's forward equations can be written as

$$\frac{\partial P_{ii}(t)}{\partial t}=-\lambda_iP_{ii}(t)\,,$$

$$\frac{\partial P_{ij}(t)}{\partial t}=\lambda_{j-1}P_{i,j-1}(t)-\lambda_jP_{ij}(t)\,,\ j>i\,.$$

➢ These differential equations have the following solution.

$$P_{ii}(t) = e^{-\lambda_i t},$$

$$P_{ij}(t) = \lambda_{j-1} e^{-\lambda_j t} \int_0^t e^{\lambda_j s} P_{i,j-1}(s) ds \text{ , } j > i \text{ .}$$

➢ In particular, for a Poisson process, we have $\lambda_i = \lambda$. Then,

$$P_{ii}(t) = e^{-\lambda t}$$

$$P_{i,i+1}(t) = \lambda e^{-\lambda t} \int_0^t e^{\lambda s} e^{-\lambda s} ds = (\lambda t) e^{-\lambda t}$$

$$P_{i,i+2}(t) = \lambda e^{-\lambda t} \int_0^t e^{\lambda s} (\lambda s) e^{-\lambda s} ds = \frac{(\lambda t)^2}{2} e^{-\lambda t}$$

$$\vdots$$

$$P_{i,i+k}(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

➢ That is, the probability of having k arrival during a time period of length t is a Poisson random variable with mean λt .

- **Alternate definition and Properties of a Poisson Process**

➢ A continuous-time stochastic process $\{N(t), t \geq 0\}$ counting the number of certain events (e.g. arrivals) by time t is said to be a Poisson process with rate $\lambda > 0$, if

(i) $N(0) = 0$.

(ii) $N(t)$ has *independent increments*: The number of events that occurs in disjoint time intervals are independent.

(iii) The number of events that occur in a time interval of length t is Poisson distributed with mean λt .

- Obviously, since $N(t)$ counts the number of events, then $N(t)$ takes on nonnegative integer values and $N(t) \geq N(s)$ for $t > s$.
- Because a Poisson process is also a pure birth process with birth rates all equal to λ, *the inter-event time is exponentially distributed with mean 1/λ* .
- Suppose each event in a Poisson process, $N(t)$, with rate λ can be classified into type I w.p. p and type II w.p. $1-p$.
- Then, the number of type I and type II, $N_1(t)$ and $N_2(t)$ are independent Poisson processes with rates λp and $\lambda(1-p)$.
- This is the *decomposition property* of the Poisson process.

- **Example 5**
- Suppose cars arrive to gas station according to a Poisson process with rate 5 per hour.
- What is the probability that 3 cars arrive in an hour?
- Number of cars in an hour, $N(1)$, is Poisson distributed with mean 5. Then,

$$P\{N(1) = 3\} = e^{-5}(5)^3/3! = 0.14 .$$

- What is the probability that 2 cars arrive in 15 minutes?
- Number of cars in an 15 minutes, $N(1/4)$, is Poisson distributed with mean 5/4 =. Then,

$$P\{N(1/4) = 2\} = e^{-5/4}(5/4)^2/2! = 0.224.$$

- What is the expected time before that the third car in an hour arrives?

- Inter-arrival times are exponential with mean 1/5 hours. Then, the expected time till third is 3/5 hours = 36 minutes.
- What is the probability that the station, starting empty, has no cars for 30 minutes after opening?
- Let A be the inter-arrival time. A is exponential with mean 1/5 hours. The desired probability is

$$P\{A > 1/2\} = e^{-5/2} = 0.082.$$

- **Example 6**

- Customers arrive to a system according to a Poisson process with rate λ and if each customer is a man w.p. 0.5 and a woman w.p. 0.5.
- Characterize the arrival process of men into the system.
- It's a Poisson process with rate 0.5λ.

- **Example 7**

- Cars arrive to an intersection according to a Poisson process with rate λ. A policeman blocks one way and directs the cars to the other way. On average, the policeman directs half of the cars to street A and the other half to street B.
- Characterize the arrival process of cars into street A. Is it Poisson?
- It's not a Poisson process.

- **Limiting Probabilities**

➢ Similar to the discrete cases, we define limiting probabilities for a CTMC as

$$P_j = \lim_{t\to\infty} P_{ij}(t).$$

➢ Similar to the discrete case also, these probabilities can be interpreted as the long-run fraction of time spent in state j .

➢ The limiting probabilities exist under the following conditions: (i) all states communicate; and (ii) all states are *positive recurrent* meaning that the expected time to return to a state upon leaving it is finite.

➢ Assuming that the limiting probabilities exist, they can be determined by Kolmogorov's equations (Theorem 1).

➢ Letting $t \to \infty$ in Theorem 1, implies that

$$\lim_{t\to\infty} \frac{\partial P_{ij}(t)}{\partial t} = \sum_{k\neq j} q_{kj} \lim_{t\to\infty} P_{ik}(t) - v_j \lim_{t\to\infty} P_{ij}(t) \Rightarrow 0 = \sum_{k\neq j} q_{kj} P_k - v_j P_j$$

➢ Therefore,

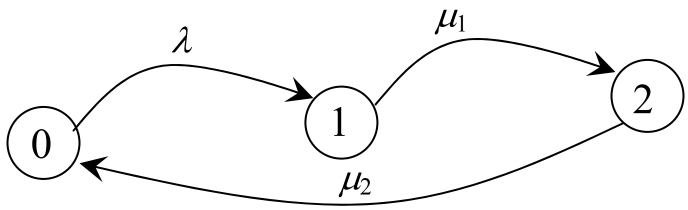
$$\boxed{v_j P_j = \sum_{k\neq j} q_{kj} P_k}$$

➢ This equation has an interesting and useful interpretation. The left hand side, $v_j P_j$, is the flow out of state j, and the right hand side, $\sum_{k\neq j} q_{kj} P_k$, is the flow into state j .

➢ This is a *flow balance* equation (flow out = flow in).

- **Example 8**

➢ For the shoeshine shop of Example 1. What is the fraction of time that the shop is busy?



➢ The desired probability is $P_1 + P_2$ or $1 - P_0$, where the limiting probabilities are given by the following flow balance equations:

State 0: $\lambda P_0 = \mu_2 P_2$

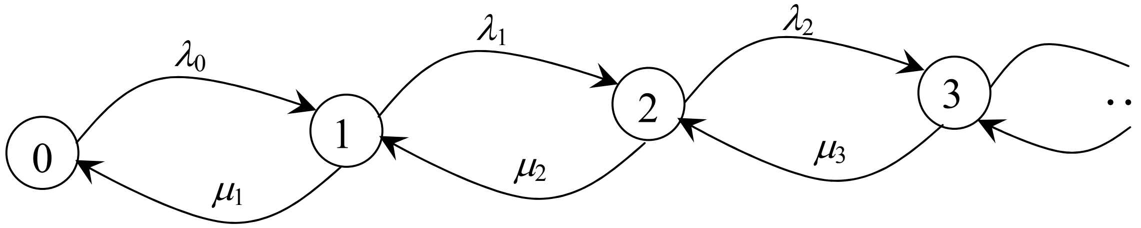
State 1: $\mu_1 P_1 = \lambda P_0$

State 2: $\mu_2 P_2 = \mu_1 P_1$

Therefore, $P_1 = (\lambda/\mu_1)P_0$, $P_2 = (\lambda/\mu_2)P_0$. Noting that $P_0+P_1+P_2 = 1$ implies that $P_0 = 1/[1 + (\lambda/\mu_1) + (\lambda/\mu_2)]$, and the desired probability is

$$1 - P_0 = [(\lambda/\mu_1) + (\lambda/\mu_2)] / [1 + (\lambda/\mu_1) + (\lambda/\mu_2)] .$$

- **Limiting Probabilities for a Birth-Death Process**



- The flow balance equations are

  State 0: $\lambda_0 P_0 = \mu_1 P_1$

  State 1: $\mu_1 P_1 + \lambda_1 P_1 = \lambda_0 P_0 + \mu_2 P_2$

  State 2: $\mu_2 P_2 + \lambda_2 P_2 = \lambda_1 P_1 + \mu_3 P_3$

  State $n > 2$: $\mu_n P_n + \lambda_n P_n = \lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1}$
- Replacing the first equation in the second gives $\lambda_1 P_1 = \mu_2 P_2$.
- Replacing this in the third equation gives $\lambda_2 P_2 = \mu_3 P_3$.
- Continuing in this manner we find that

$$\lambda_n P_n = \mu_{n+1} P_{n+1} \text{ , } n \geq 0$$

- Then,

$$P_1 = \frac{\lambda_0}{\mu_1} P_0 \text{ ,} \quad P_2 = \frac{\lambda_1}{\mu_2} P_1 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} P_0, \quad P_3 = \frac{\lambda_2}{\mu_3} P_2 = \frac{\lambda_2 \lambda_1 \lambda_0}{\mu_3 \mu_2 \mu_1} P_0 \text{ .}$$

- And in general,

$$P_n = \frac{\lambda_{n-1} \lambda_{n-2} \ldots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \ldots \mu_2 \mu_1} P_0 \text{ .}$$

- Since $\sum_{n=0}^{\infty} P_n = 1$, it follows that

$$P_0 = \left(1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \lambda_{n-2} \ldots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \ldots \mu_2 \mu_1} \right)^{-1} \text{ ,}$$

$$P_n = \frac{\lambda_{n-1} \lambda_{n-2} \ldots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \ldots \mu_2 \mu_1} \left(1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \lambda_{n-2} \ldots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \ldots \mu_2 \mu_1} \right)^{-1} \text{ , } n = 1, 2, \ldots$$

- A necessary condition for the existence of P_ns is

$$\sum_{n=1}^{\infty} \frac{\lambda_{n-1} \lambda_{n-2} \ldots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \ldots \mu_2 \mu_1} < \infty$$