

Probability and Random Variable (1)

- **Sample space and Events**

- Suppose that an experiment with an uncertain outcome is performed (e.g., rolling a die).
- While the outcome of the experiment is not known in advance, the set of all possible outcomes is known. This set is the sample space, Ω .
- For example, when rolling a die $\Omega = \{1, 2, 3, 4, 5, 6\}$. When tossing a coin, $\Omega = \{H, T\}$. When measuring life time of a machine (years), $\Omega = \{1, 2, 3, \dots\}$.
- A subset $E \subset \Omega$ is known as an event.
- E.g., when rolling a die, $E = \{1\}$ is the event that one appears and $F = \{1, 3, 5\}$ is the event that an odd number appears.

- **Probability of an event**

- If an experiment is repeated for a number of times which is large enough, the fraction of time that event E occurs is the probability that event E occurs, $P\{E\}$.
- E.g., when rolling a fair die, $P\{1\} = 1/6$, and $P\{1, 3, 5\} = 3/6 = 1/2$. When tossing a fair coin, $P\{H\} = P\{T\} = 1/2$.
- In some cases, events are not repeated many times.
- For such cases, probabilities can be a measure of belief (subjective probability).

- **Axioms of probability**

(1) For $E \subset \Omega$, $0 \leq P\{E\} \leq 1$;

(2) $P\{\Omega\} = 1$;

(3) For events $E_1, E_2, \dots, E_i, \dots$, with $E_i \subset \Omega$, $E_i \cap E_j = \emptyset$, for all

$$i \text{ and } j, P\left\{\bigcup_{i=1}^{\infty} E_i\right\} = \sum_{i=1}^{\infty} P\{E_i\} .$$

- **Implications**

➤ The axioms of probability imply the following results:

○ For E and $F \subset \Omega$,

$$P\{E \text{ “or” } F\} = P\{E \cup F\} = P\{E\} + P\{F\} - P\{E \cap F\} ;^1$$

○ If E and F are mutually exclusive (i.e., $E \cap F = \emptyset$), then

$$P\{E \cup F\} = P\{E\} + P\{F\};$$

○ For $E \subset \Omega$, let E^c be the complement of E (i.e., $E \cup E^c = \Omega$),

$$P\{E^c\} = 1 - P\{E\};$$

○ $P\{\emptyset\} = 0$.

- **Conditional probability**

➤ The probability that event E occurs given that event F has already occurred is

$$P\{E | F\} = \frac{P\{E \cap F\}}{P\{F\}} .$$

¹ $P\{E \cap F\} = P\{E \text{ “and” } F\} .$

- **Independent events**

- For E and $F \subset \Omega$, $P\{E \cap F\} = P\{E/F\}P\{F\}$.

- Two events are independent if and only if

$$P\{E \cap F\} = P\{E\}P\{F\}. \text{ That is, } P\{E/F\} = P\{E\} .$$

- **Example 1**

- Let $\Omega = \{E_1, \dots, E_{10}\}$, where E_i are mutually exclusive. It is known that $P\{E_i\} = 1/20, i = 1, \dots, 6, P\{E_i\} = 1/5, i = 7, \dots, 9$, and $P\{E_{10}\} = 3/20$,

- Do $P\{E_i\}$ satisfy the axioms of probability?

- No, since $\sum P\{E_i\} > 1$.

- **Example 2**

- Suppose that two fair coins are tossed. What is the probability that either the first or the second coin falls heads?

- In this example, $\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$. Let E (F) be the event that the first (second) coin falls heads, $E = \{(H, H), (H, T)\}$ and $F = \{(H, H), (T, H)\}$, and $E \cap F = \{(H, H)\}$. The desired probability is $P\{E \cup F\} = P\{E\} + P\{F\} - P\{E \cap F\} = 1/2 + 1/2 - 1/4 = 3/4$.

- **Example 3**

- When rolling two fair dice, suppose the first die is 3, what is the probability the sum of the two dice is 7?

- Let E be the event that the sum of the two dice is 7, $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$, and F be the event that the first die is 3, $F = \{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)\}$. Then,

$$\begin{aligned}
 P\{E | F\} &= \frac{P\{E \cap F\}}{P\{F\}} = \frac{P\{(3, 4)\}}{P\{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)\}} \\
 &= \frac{1/36}{6/36} = \frac{1}{6}.
 \end{aligned}$$

• Example 4

- A family has two kids. At least one of them is a boy. What is the probability that both are boys?
- In this example, $\Omega = \{(b, b), (b, g), (g, b), (g, g)\}$. Let $E = \{\text{Both kids are boys}\}$ and $F = \{\text{At least one kid is a boy}\}$. The required probability is then

$$P\{E | F\} = \frac{P\{E \cap F\}}{P\{F\}} = \frac{P\{(b, b)\}}{P\{(b, g), (g, b), (b, b)\}} = \frac{1/4}{3/4} = 1/3.$$

• Example 5

- An urn contains three white balls and four black balls. Two balls are drawn without replacement.
- What is the probability that the two balls are black?
- Let $F = \{\text{First ball is black}\}$, and $E = \{\text{Second ball is black}\}$. The desired probability is

$$P\{E \cap F\} = P\{F\}P\{E | F\} = (4/7)(3/6) = 2/7.$$

- What is the probability that the both balls are of the same color?
- $P\{\text{same color}\} = P\{\text{both black}\} + P\{\text{both white}\}$
 $= 2/7 + (3/7)(2/6) = 3/7.$

- **Finding Probability by Conditioning**

➤ Suppose that we know the probability of event B once event A is realized (or not). We also know $P\{A\}$. That is, we know $P\{B|A\}$, and $P\{B|A^c\}$ and $P\{A\}$. What is $P\{B\}$?

➤ Note that

$$B = (A \cap B) \cup (A^c \cap B) \Rightarrow P\{B\} = P\{A \cap B\} + P\{A^c \cap B\}.$$

➤ Therefore,

$$\begin{aligned} P\{B\} &= P\{B|A\}P\{A\} + P\{B|A^c\}P\{A^c\} \\ &= P\{B|A\}P\{A\} + P\{B|A^c\}(1 - P\{A\}). \end{aligned}$$

➤ Here we are finding $P\{B\}$ by “conditioning” on A .

➤ In general, if the realization of B depends on a *partition* A_i of Ω , $A_1 \cup A_2 \cup \dots \cup A_n = \Omega$, $A_i \cap A_j = \emptyset$, $(i, j) \in \{1, \dots, n\}^2, i \neq j$,

$$P\{B\} = \sum_{i=1}^n P\{B | A_i\}P\{A_i\}.$$

- **Bayes’ Formula**

➤ This follows from conditional probabilities. For two events,

$$P\{A | B\} = \frac{P\{A \cap B\}}{P\{B\}} = \frac{P\{B | A\}P\{A\}}{P\{B | A\}P\{A\} + P\{B | A^c\}P\{A^c\}}.$$

➤ With a partition A_i ,

$$P\{A_j | B\} = \frac{P\{A_j \cap B\}}{P\{B\}} = \frac{P\{B | A_j\}P\{A_j\}}{\sum_{i=1}^n P\{B | A_i\}P\{A_i\}}.$$

• Example 6

- Consider two urns. The first urn contains three white and seven black balls, and the second contains five white and five black balls. We flip a coin and then draw a ball from the first urn or the second urn depending on whether the outcome was heads or tails.

- What is the probability that a white ball is selected?

$$P\{W\} = P\{W|H\}P\{H\} + P\{W|T\}P\{T\} = (3/10)(1/2) + (5/10)(1/2) \\ = 2/5 .$$

- What is the probability that a black ball is selected?

$$P\{B\} = 1 - P\{W\} = 3/5 .$$

- What is the probability that the coin has landed heads given that a white ball is selected?

$$\text{From Bayes' formula, } P\{H|W\} = \frac{P\{W|H\}P\{H\}}{P\{W\}} = \frac{(3/10)(1/2)}{2/5} = \frac{3}{8} .$$

• Example 7

- In an assembly plant, three machines, M_1 , M_2 , and M_3 , make 30%, 45%, and 25%, respectively of the products. It is known that 2%, 3%, and 2% of the products made by each machine, respectively, are defective.

- What is the probability that a randomly selected product is defective?

$$P\{D\} = P\{D|M_1\}P\{M_1\} + P\{D|M_2\}P\{M_2\} + P\{D|M_3\}P\{M_3\} \\ = (0.02)(0.3) + (0.03)(0.45) + (0.02)(0.25) = 0.0245 .$$

- If a product is found to be defective, what is the probability that it's made by M_2 ?

$$P\{M_2|D\} = \frac{P\{D|M_2\}P\{M_2\}}{P\{D\}} = \frac{(0.03)(0.45)}{0.0245} = 0.551 .$$

- **Example 8**

➤ In answering a question on a multiple-choice test, a student either knows the answer or guesses. Let p be the probability that she knows the answer. Assume that the student who guesses the answer will answer correctly with probability $1/m$, where m is the number of multiple-choice alternatives.

➤ What is the probability that a student answers a question correctly?

$$\begin{aligned} P\{\text{Correct}\} &= P\{\text{Correct}|\text{Know}\}P\{\text{Know}\} + P\{\text{Correct}|\text{Guess}\}P\{\text{Guess}\} \\ &= (1)(p) + (1/m)(1-p) \\ &= p + (1-p)/m \end{aligned}$$

➤ E.g., if you know the answers to half of EEE questions ($m=4$, $p = 1/2$), then the probability of a correct answer is $1/2 + 1/8 = 5/8$, and your likely score is $500/800$.

➤ What is the conditional probability that a student guessed the question given that she answered correctly?

$$\begin{aligned} P\{\text{Guess}|\text{Correct}\} &= P\{\text{Correct}|\text{Guess}\}P\{\text{Guess}\}/P\{\text{Correct}\} \\ &= (1/m)(1-p)/[p + (1-p)/m] \\ &= (1-p)/[mp + (1-p)]. \end{aligned}$$

➤ In the EEE case, the conditional probability of guessing given a right answer is $(1/2)/(2+1/2) = 1/5$.