Discrete Time Markov Chains (2)

- **Chapman-Kolmogorov (C-K) Equations**

  - Consider a MC, X_n with state space $\{0,1, 2, \ldots\}$. Let $p_{ij}^{(n)}$ be the n-step transition probability from state i to state j. I.e.,

$$p_{ij}^{(n)} = P\{X_{n+k} = j \mid X_k = i\}.$$

  - Let $\mathbf{P}^{(n)} = [p_{ij}^{(n)}]$ be the n-step transition probability matrix.
  - Obviously, $\mathbf{P}^{(1)} = \mathbf{P}$.
  - By conditioning on the state the process is in at time $n+k$, we derive

$$\begin{aligned} p_{ij}^{(n+m)} &= P\{X_{n+m+k} = j \mid X_k = i\} \\ &= \sum_{r=0}^{\infty} P\{X_{n+m+k} = j \mid X_{n+k} = r, X_k = i\} P\{X_{n+k} = r \mid X_k = i\} \\ &= \sum_{r=0}^{\infty} p_{rj}^{(m)} p_{ir}^{(n)}. \end{aligned}$$

  - These are called the Chapman-Kolmogorov equations, which can be written in matrix form as

$$\mathbf{P}^{(n+m)} = \mathbf{P}^{(n)}\mathbf{P}^{(m)}.$$

  - The C-K equations imply that $\mathbf{P}^{(2)} = \mathbf{P}^{(1)}\mathbf{P}^{(1)} = \mathbf{P}^2$. Then, it can be shown by induction that

$$\mathbf{P}^{(n)} = \mathbf{P}^n.$$

- **Example 4**
  - Consider the two-state MC for the weather condition in Example 1. Suppose $\alpha = 0.7$ and $\beta = 0.4$.
  - If it rains Monday, what is the probability that it will rain on Friday?
  - We find the four-step transition matrix

$$\mathbf{P}^{(4)} = \mathbf{P}^4 = \mathbf{P}^2\mathbf{P}^2,$$

where

$$\mathbf{P}^2 = \mathbf{P}\mathbf{P} = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}\begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix}.$$

Then,

$$\mathbf{P}^4 = \mathbf{P}^2\mathbf{P}^2 = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix}\begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix} = \begin{pmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{pmatrix}.$$

The desired probability is then $p_{00}^{(4)} = 0.5749$.

- **Example 5**
  - Consider the four-state MC for the weather condition in Example 3.
  - If it rains Monday and Tuesday, what is the probability that it will rain on Thursday?
  - We find the two-step transition matrix

$$\mathbf{P}^2 = \begin{pmatrix} 0.7 & 0.3 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{pmatrix} \begin{pmatrix} 0.7 & 0.3 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.49 & 0.12 & 0.21 & 0.18 \\ 0.35 & 0.2 & 0.15 & 0.30 \\ 0.2 & 0.12 & 0.2 & 0.48 \\ 0.10 & 0.16 & 0.10 & 0.64 \end{pmatrix}$$

Then, the desired probability is

P{RR on MT→ RR on WTh, or RR on MT→ NRR on WTh}

$= p_{00}^{(2)} + p_{01}^{(2)} = 0.49 + 0.12 = 0.61$.

(Recall that state 0 = RR, state 1 = NRR, state 2 = RNR, state 3 = NRNR.)

- **Transient Solution**
  - Let $\pi_j^n = P\{X_n = j\}$ be the unconditional probability that the process is in state *j* at time *n*, and let $\pi^n = (\pi_0^n, \pi_1^n, \ldots, \pi_j^n, \ldots)$ be the unconditional distribution at time *n* .
  - By conditioning on the state the process is in at time *n*−1,

$$P\{X_n = j\} = \sum_{i=0}^{\infty} P\{X_n = j \mid X_{n-1} = i\} P\{X_{n-1} = i\}$$

$$\Rightarrow \pi_j^n = \sum_{i=0}^{\infty} p_{ij} \pi_i^{n-1} .$$

  - It follows that

$$\pi^n = \pi^{n-1}\mathbf{P} .$$

  - And by induction

$$\pi^n = \pi^0 \mathbf{P}^n$$

- **Example 6**
  - Suppose that a manufacturer of a product (Brand 1) is competing with only one other similar product (Brand 2). Both manufacturers have been engaged in aggressive advertising programs. A survey is taken to find out the rates at which consumers are switching brands or staying loyal to brands. Responses to the survey are summarized in the following table which shows the number of customers who buy Brand i last week and Brand j this week, $i = 1, 2$, $j = 1, 2$.

	This week		
Last Week	Brand 1	Brand 2	Total
Brand 1	90	10	100
Brand 2	40	160	200

  - Can this situation be modeled by a MC? Under what assumption?
  - Assuming that the consumer switching behavior is not changed over time. The, the situation can be modeled with a MC, where $X_n = 0$, if a customer buys Brand 1 and $X_n = 1$, if a customer buys Brand 2. The one-step transition probability matrix is

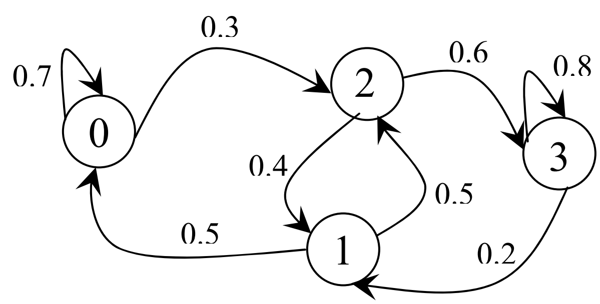
$$\mathbf{P} = \begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix}.$$

  - Suppose that 1/3 of the customers buy Brand 1 this week.

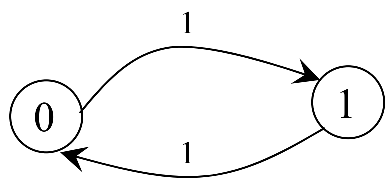
- What percentage of customers will buy Brand 1 and Brand 2 next week?
- We have $\pi^0 = (1/3, 2/3)$. Then, the required probabilities are $\pi^1 = \pi^0 \mathbf{P} = (1/3, 2/3)\begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} = (0.433, 0.567)$.
- What percentage of customers will buy Brand 1 and Brand 2 after two weeks from now?

$$\pi^2 = \pi^1 \mathbf{P} = (0.43, 0.57)\begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} = (0.503, 0.497).$$

- **Classification of States**
  - State *j* is accessible (reachable) from state *i* ($i \to j$) if there exists some sequence of possible transitions which would take the process from state *i* to state *j*.
  - Formally, $i \to j$ if there exists $n \geq 0$, such that $p_{ij}^{(n)} > 0$.
  - Two states *i* and *j* *communicate* ($i \leftrightarrow j$) if $i \to j$ and $j \to i$.
  - A MC is said to be *irreducible* if all its states communicate.
  - Two states that communicate are said to be in the same class.
  - E.g., the MC in Example 3 is irreducible as indicated in the following *transition flow diagram*.



- The period of a state k is the grand common divider (GCD) of all integers n s.t. $p_{kk}^{(n)} > 0$.
- If the period of all states is 1, then the MC is called *aperiodic*.
- E.g., the MC in Example 3 is aperiodic.
- A state is said to be *periodic* if its period is greater than 1.
- Each state has a period of 2 in the following two-state MC chain:



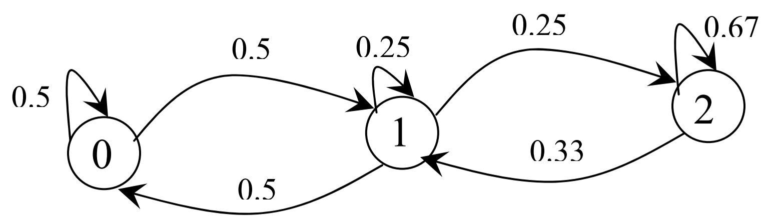
- A state i is said to be *recurrent* if starting in state i, the MC will reenter state i infinitely often.
- A state i is said to be *transient* if the number times the MC reenters state i is finite.
- A state i is said to be *absorbing* if the MC "stays" in i, once i is reached (i.e., $p_{ii} = 1$).

Fact. *If state i is recurrent (transient), and state j communicates with state j, then state j is recurrent (transient).*

- That is, *recurrence is a class property*. So, we can classify the MC states into transient, recurrent and absorbing classes.
- Note finally that a MC with a finite number of states cannot have all transient states. If in such a MC all states communicate, then the chain has a single recurrent class.

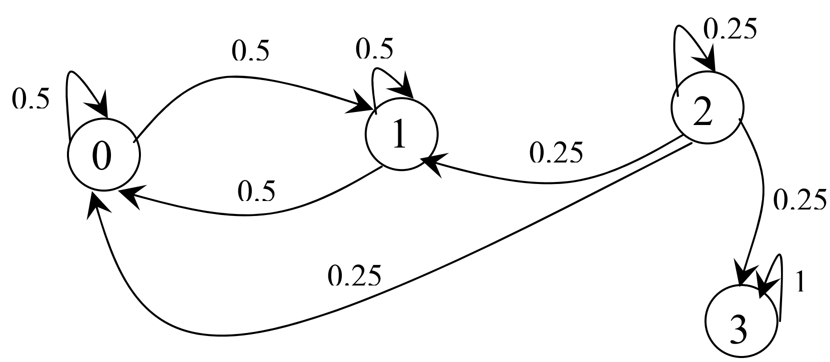
- **Example 7**
  - ➢ Classify the states of the MCs having the following transition Matrices.

$$\mathbf{P} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.25 & 0.25 \\ 0 & 0.33 & 0.67 \end{pmatrix}$$



Single recurrent class {0,1,2} since all states communicate.

$$\mathbf{P} = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Three classes: {0,1}, recurrent, {2}, transient, and {3} absorbing.