Chapter 2 Factors: How Time and Interest Affect Money

- **Overview**
  - Factors are formulas that simplify cash flow equivalence calculations.
  - They also present a useful and intuitive notation.
  - They are particularly useful as some cash flow patterns are common in practice, e.g. single payments, uniform, arithmetic and geometric series.

- **Single payment factors**
  - Recall that P dollars now are equivalent to F dollars after n time periods at an interest rate of i per time period[1], where

$$F = P(1+i)^n .$$

  - Rewrite this as $F = P \times (F/P, i, n)$,

    where $(F/P, i, n) = (1+i)^n$ is the "*F*/*P* factor."

  - In addition, this implies that

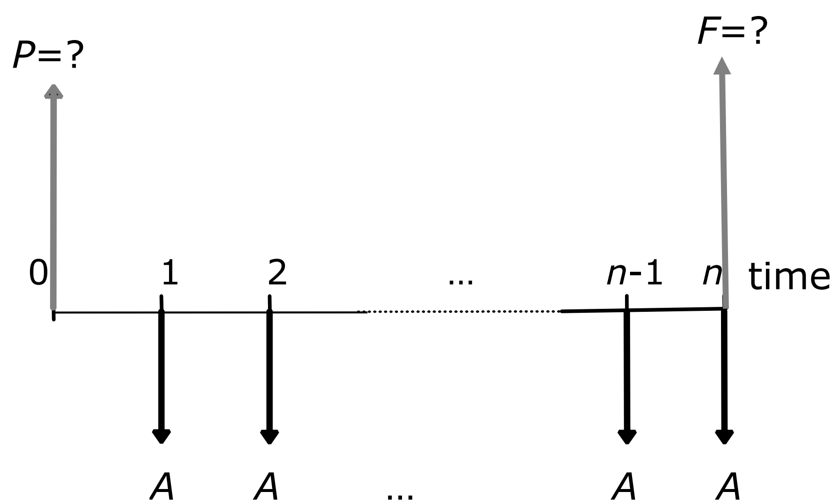
$$P = \frac{F}{(1+i)^n} = F \times (P/F, i, n),$$

    where $(P/F, i, n) = 1/(1+i)^n$ is the "*P*/*F* factor."

[1] Here and elsewhere, when the type of interest is not specified, assume it's *compound* interest.

- **Tables and Spreadsheets**
  - The P/F and F/P factors, as well as other factors, are tabulated at the end of your textbook (pp. 581 – 609).
  - You may use these tables or the formulas that we derive here.
  - Spreadsheets (i.e., Excel) has built-in function for factors (or you can easily build your own functions) for the calculating the factors. Excel is very suited for practical interest calculations.

- **Uniform Series Factors**
  - Suppose one will pay A dollars every time period for n period starting with the end of Period 1.



  - Then, this series of cash flows is now (at time 0) equivalent to

$$P = \frac{A}{1+i} + \frac{A}{(1+i)^2} + \ldots + \frac{A}{(1+i)^n} = \frac{A}{1+i}\sum_{j=0}^{n-1}\left(\frac{1}{1+i}\right)^j = \frac{A}{1+i}\frac{1-\left(\frac{1}{1+i}\right)^n}{1-\left(\frac{1}{1+i}\right)}.$$

- Upon simplification,

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = \frac{A}{i}\left[1 - \frac{1}{(1+i)^n}\right] = A \times (P/A, i, n),$$

where $(P/A, i, n) = [(1+i)^n - 1]/[i(1+i)^n]$ is the "P/A factor."

- An example of using this factor is the loan you would get if you are willing to pay *A* dollars every year for *n* years from a bank with an interest rate *i* per year.
- E.g., if you can pay $6,000 per year for five years and the bank's interest rate is 8%, the loan you'll get is approximately[2], 6,000(P/A, 8%, 5) = (6,000)(3.9927) = $23,956.
- In addition, a uniform cash flow series of *A* payments over *n* periods starting one period from now (Time 1) is now equivalent to

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = P \times (A/P, i, n) \;,$$

where $(A/P, i, n) = i(1+i)^n / [(1+i)^n - 1]$ is the "A/P factor."

- E.g., going back to the bank loan example, if you need a loan of $25,000, you make annual payments for five years equal to, *A* = $25,000(A/P, 8%, 5) = 25,000(0.25406) = $6,262.

[2] Banks typically charge fees, so you'll actually get slightly less than that. You may also have to pay insurance, e.g. on your life covering the loan in case you could not pay it anymore, which would reduce the amount you get further.

- Finally, to find the future amount, F, equivalent to the uniform series at time n (time of the last cash flow), note that

$$F = P(1+i)^n = A\left[\frac{(1+i)^n - 1}{i}\right] = A \times (F/A, i, n)\,,$$

where $(F/A, i, n) = [(1+i)^n - 1]/i$ is the "*F/A* factor."

- E.g., if you deposit \$2,000/year in a saving account earning 5%/year for 10 years you can withdraw at the end of the tenth year, F = \$2,000(F/A, 5%, 10) = 2,000(12.5779) = \$25,516.
- In addition, it can be shown that F dollars after n periods from now is equivalent to a uniform cash flow series of A payments over n periods starting Time 1, where $A = F(A/F,\ i,n)$.
- The factor $(A/F, i, n) = i/[(1+i)^n - 1]$ is the "*A/F* factor."
- E.g., if you want to have \$30,000 in your saving account gaining 5%/year you need to deposit A = 30,000(A/F,5%,10) = (30,000)(0.07950) = \$2,385, each year for 10 years.

- **Factors in Excel**
  - The following table form the text (P. 49) summarizes how the factors introduced thus far can be used in Excel.

Factor	To Do This	Excel Function
P/F	Find P, given F.	$= -\mathrm{PV}(i\%,n,,1)$
F/P	Find F, given P.	$= -\mathrm{FV}(i\%,n,,1)$
P/A	Find P, given A.	$= -\mathrm{PV}(i\%,n,1)$
A/P	Find A, given P.	$= -\mathrm{PMT}(i\%,n,1)$
F/A	Find F, given A.	$= -\mathrm{FV}(i\%,n,1)$
A/F	Find A, given F.	$= -\mathrm{PMT}(i\%,n,,1)$

- Note that the fourth entry, "1," can be replaced by the given *F*, *P*, or *A* value, and then the function does the whole equivalence calculation. E.g., $(P)[-\mathrm{PV}(i,n,,1)] = -\mathrm{PV}(i,n,,P)$
- Note also the double commas ",," in the P/F, F/P, and A/F factors.
- These are not typos. These are there because the PV and PMT Excel functions can take four arguments, (i,n,A,F) and (i,n,P,F), respectively.
- For example, $-\mathrm{PV}(i,n,A,F)$ finds the equivalent value at time 0 of a cash flow stream composed of uniform series of value *A* extending from Period 1 to Period *n*, and a terminal value *F*.
- This can be useful as many investments have a salvage value or a disposal cost.

- **Running amortization**
  - Amortization is the process of substituting a current payment *P* for periodic payments of *A* per period (e.g. car or home loan.)
  - One can view each amortization payment (*A*) as composed of two parts: (i) interest on running (outstanding) balance and (ii) partial repayment of principal.
  - This procedure is equivalent to re-amortizing the running balance every period over the remaining time horizon.
  - This is consistent with accounting practice.

- E.g., consider a loan of \$1,000 issued on Jan 1, 2006, to be paid back in equal monthly payments over 5 years at an interest rate of 1% per month.
- The monthly payment is

$$A = 1000\left[\frac{(0.01)(1+0.01)^{60}}{(1+0.01)^{60}-1}\right] = \$22.24\ .$$

- Then, the outstanding balance on Feb 1, 2006 is \$1,000 minus the monthly payment (\$22.24) plus the monthly interest (0.01×1,000 = \$10), which gives \$987.76.
- The (running) amortization of the \$987.76 at 1-Mar-2006 over the remaining 59 months is

$$A = 987.67\left[\frac{(0.01)(1+0.01)^{59}}{(1+0.01)^{59}-1}\right] = \$22.24\ .$$

- The (running) amortization of the \$975.39 at 1-Apr-2006 over the remaining 58 months is also \$22.24, and so on.

Date	**Previous balance**	**Interest**	**Payment Received**	**New Balance**
1-Jan-06				\$1,000
1-Feb-06	\$1,000	\$10.00	\$22.24	\$987.76
1-Mar-06	\$987.76	\$9.88	\$22.24	\$975.39
1-Apr-06	\$975.39	\$9.75	\$22.24	\$962.90
1-May-06	\$962.90	\$9.63	\$22.24	\$950.28
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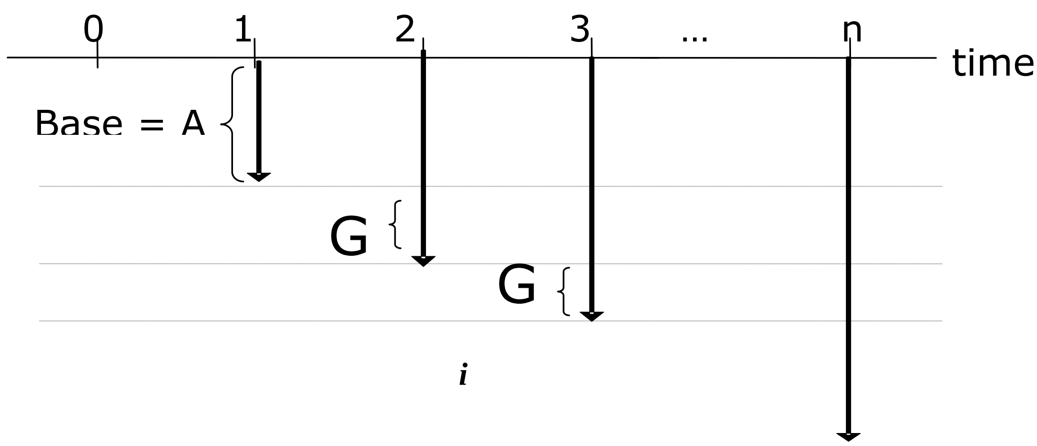
1-Dec-10	$43.83	$0.44	$22.24	$22.02
1-Jan-11	$22.02	$0.22	$22.24	$0.00

- **Arithmetic Gradient Factors**



- In some cases cash flows increase by a fixed amount in each time period starting with Period 2, e.g. for some maintenance costs.
- Starting with a cash flow of A at the end of period 1, the cash flows increase by the gradient, G, in each period. That is, the cash flows are A, $A + G$, $A + 2G$, …, $A + nG$, in Periods 1, 2, …, n.
- Then, at time 0, this series of cash flows is equivalent to

$$P = P_A + P_G ,$$

where P_A is the equivalent at time 0 of the series with uniform cash flows A per time period, $P_A = A(P/A, i, n)$.

- P_G is the equivalent at time zero of the arithmetic cash flow series with gradient G, i.e., the series having G, $2G$, …, $(n–1)G$ cash flows at the end of Periods 2, 3, …, n.
- P_G is evaluated as follows

$$P_G = \frac{G}{(1+i)^2} + \frac{2G}{(1+i)^3} + \ldots + \frac{(n-2)G}{(1+i)^{n-1}} + \frac{(n-1)G}{(1+i)^n}$$
$$= \frac{G}{1+i}\sum_{j=1}^{n-1}\frac{j}{(1+i)^j} = \frac{G}{1+i}\left(\frac{(1+i)-[1+i+(n-1)i]/(1+i)^{n-1}}{i^2}\right)$$
$$= \frac{G}{i}\left(\frac{(1+i)^n-(1+ni)}{i(1+i)^n}\right) = \frac{G}{i}\left(\frac{(1+i)^n-1}{i(1+i)^n} - \frac{n}{(1+i)^{n-1}}\right) = G(P/G,i,n).$$

- The factor $(P/G,i,n) = \frac{G}{i}\left[\frac{(1+i)^n-1}{i(1+i)^n} - \frac{n}{(1+i)^n}\right]$ is the "P/G factor."
- In the above we have used the fact, that for j and m integers

$$\sum_{j=1}^{m} j/(1+y)^j = \frac{(1+y)-(1+y+my)/(1+y)^m}{y^2}.$$ [3]

- E.g., if a machine annual maintenance costs follow the arithmetic gradient series, and the machine has a useful life of n years, then the present worth of the machine maintenance can be estimated as above.
- Finally, one can define “F/G” and “A/G” factors, as

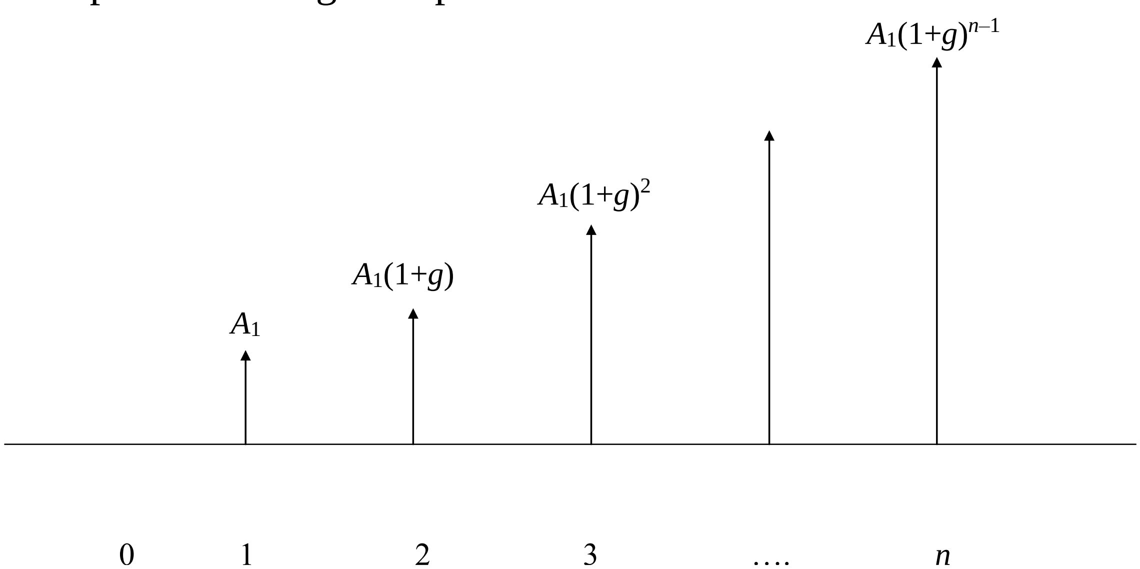
  (F/G, i, n) = (P/G, i, n)(F/P, i, n),

  (A/G, i, n) = (P/G, i, n)(A/P, i, n).
- See text for details on these factors.

[3] The place called “library” has this in some thick old books.

- There is no Excel function equivalent to the P/G factor.
- Once can place the cash flows, A, $A + G$, …, $A + (n-1)G$ of the series in n Excel cells, and the use the general function NPV(i, first_cell, last_cell) to get the equivalent present value.
- Once the equivalent present value is determined Excel functions corresponding to F/P and A/P can be used to estimate equivalent single future value and uniform series.

- **Geometric Gradient Factors**
  - Suppose now that in a series of a cash flows the amounts increase (or decrease) by a fixed amount $(1+g)$, in each time period starting with period 2.



  - These are useful models for a company financial growth or decline, as they allow valuating the company's present value.
  - These models are used by venture capitalists to value start-ups and by investment bankers to value mature companies.

- The equivalent present value of the geometric series is

$$P_g = \frac{A_1}{1+i} + \frac{A_1(1+g)}{(1+i)^2} + \ldots + \frac{A_1(1+g)^{n-1}}{(1+i)^n} = \frac{A_1}{(1+i)} \sum_{j=0}^{n-1} \left(\frac{1+g}{1+i} \right)^j .$$

- It follows that

$$P_g = \begin{cases} A_1 \left\{ \dfrac{1-[(1+g)/(1+i)]^n}{i-g} \right\}, & \text{if } i \neq g . \\ \dfrac{nA_1}{1+i}, & \text{if } i = g . \end{cases}$$

- There are no factors tabulated for the geometric gradient series. One can use the formula for P_g and then apply other F/P and A/P factors as needed.
- There is also no specific Excel functions for a geometric gradient. One can proceed utilizing the NPV function as explained for the arithmetic gradient case.

- **Summary of terminology**

F/P factor: Compound Amount Factor

P/F factor: Present Worth Factor

P/A factor : Uniform-Series Present Worth Factor

A/P factor: Capital Recovery Factor

A/F factor: Sinking Fund Factor

F/A factor: Uniform-Series Compound Amount Factor

P/G factor: Arithmetic gradient present worth factor

A/G factor: Arithmetic gradient uniform-series factor