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The Basic Theory of Interest (1, Chapter 2, Luenberger)

- **Interest concept: Review**
  - Recall that interest is the manifestation of time value of money.
  - Under a compound interest rule, an investment earns interest on interest. Specifically, P dollars invested for n years at an interest rate of r per year will have a total value of

$$F = P(1+r)^n \ .$$

  - An amount F received n years from now is equivalent to having $P = F/(1 + r)^n$ now, where P is called the *present value* or the *discounted value* of F.
  - The *future value*, n years from now, of an amount P one has today is $F = P(1 + r)^n$.
  - The present value of a cash flow stream $\mathbf{x} = (x_0, x_1, \ldots, x_n)$ is

$$PV = x_0 + \frac{x_1}{1+r} + \frac{x_2}{(1+r)^2} + \ldots + \frac{x_n}{(1+r)^n} = \sum_{j=0}^{n} \frac{x_j}{(1+r)^j} \ .$$

  - The future value stream $\mathbf{x} = (x_0, x_1, \ldots, x_n)$ is

$$FV = x_0(1+r)^n + x_1(1+r)^{n-1} + \ldots + x_n = \sum_{j=0}^{n} x_j(1+r)^{n-j} \ .$$

- **Simple interest**
  - Under a simple interest rule, P dollars invested for n years at a rate of r per year will accumulate interest of Pr every year.
  - Then, the total value of the investment after n years is

$$F = P(1 + nr) .$$

  - Note that the investment value grows *linearly* with time.

- **The seven-ten rule and the rule of 72**
  - Money invested at 10% (7%) per year, under compounding, doubles in approximately 7 (10) years.
  - More generally, money invested at an interest rate of i % per year will double in approximately $72/i$ years.
  - For example, money invested at 12% doubles in approximately 72/12 = 6 years.

- **Compounding at intervals less than 1 year**
  - Interest may be compounded more frequently than one year. E.g., every quarter, every month, or even every day.
  - In these situations, it is common to quote the interest rate on a yearly basis and then to apply the appropriate proportion of the interest rate over the compounding period.
  - For example, a *nominal* interest rate of 8% per year compounded quarterly means that the *effective* interest rate per quarter is 8% / 4 = 2%.

- However, the effective interest rate per year is $(1 + 0.08/4)^4 - 1 = 0.0824$, i.e., 8.24% .
- Generally, if the nominal interest rate is r per year and interest is compounded at m equally spaced epochs per year, then the effective interest rate per period is r/m and an amount P invested for k periods grows to

$$F = P(1 + r/m)^k \,.$$

- The effective interest rate per year, r', is found by noting that the future value of an amount P after one year is

$$P(1+r') = P(1+r/m)^m \,.$$

- Then,

$$r' = (1+r/m)^m - 1 \,.$$

- **Continuous compounding**
  - If the compounding period length becomes very small, i.e., $m \to \infty$, the effective interest rate per year is

$$r' = \lim_{m\to\infty}(1+r/m)^m - 1 = e^r - 1 \,.$$

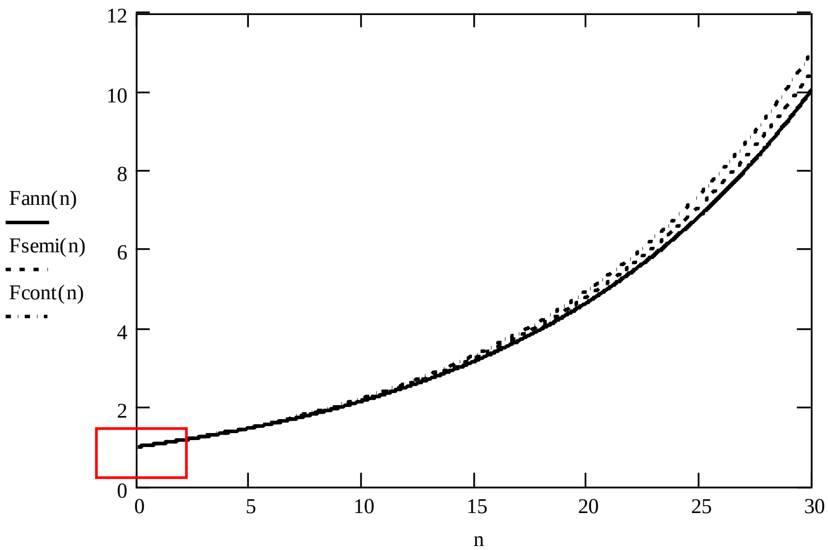
  - If an amount P is invested for t years under a continuous compounding of r per year then, letting $k = mt$ with $m \to \infty$, implies that the value of the investment after t years is

$$F = \lim_{m\to\infty} P(1+r/m)^k = \lim_{m\to\infty} P(1+r/m)^{mt} = Pe^{rt} \,.$$

  - Note that P is growing exponentially.

- **Example: $1 invested at a nominal rate of 8%**

year	***m* = 1**	***m* = 2**	***m* = 4**	***m* = 12**	***m* = ∞**
1	1.080	1.082	1.082	1.083	1.083
2	1.166	1.170	1.172	1.173	1.174
3	1.260	1.265	1.268	1.270	1.271
4	1.360	1.369	1.373	1.376	1.377
5	1.469	1.480	1.486	1.490	1.492
6	1.587	1.601	1.608	1.614	1.616
7	1.714	1.732	1.741	1.747	1.751
8	1.851	1.873	1.885	1.892	1.896
9	1.999	2.026	2.040	2.050	2.054
10	2.159	2.191	2.208	2.220	2.226



- **Constant ideal bank**
  - ➢ This is a bank that charges the same interest rate for borrowing and lending with no transaction fees.
  - ➢ Moreover, the interest rate is the same regardless of the length of time the money is held.
  - ➢ In this chapter, we assume that we have a constant ideal bank situation. We relax this assumption later.

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- **Main theorem on present value**

The cash flow streams $\mathbf{x} = (x_0, x_1, \ldots, x_n)$ and $\mathbf{y} = (y_0, y_1, \ldots, y_n)$ *are equivalent for a constant ideal bank with interest rate r if and only if the present value of the two streams are equal.*

- **Internal rate of return**
  - For an investment involving one single payment P and returning F at a future time the rate of return is
$$r = F / P - 1 .$$
  - Alternatively, r is given by the solution to
$$-P + F / (1+r) = 0 .$$
  - That is, r is the solution to the equation $PV(r) = 0$.
  - With more sophisticated cash flow streams involving several payments and receipts, solving the same equation, $PV(r) = 0$, defines *the internal rate of return* (IRR) of the stream.
  - Formally, the IRR of a cash flow stream $\mathbf{x} = (x_0, x_1, \ldots, x_n)$ is a number r satisfying the equation
$$x_0 + \frac{x_1}{1+r} + \frac{x_2}{(1+r)^2} + \ldots + \frac{x_n}{(1+r)^n} = 0 .$$
  - Defining $c = 1/(1+r)$ (that is $r = 1/c - 1$) then the IRR can be obtained by solving the polynomial equation
$$x_0 + x_1 c + x_2 c^2 + \ldots + x_n c^n = 0 . \quad (1)$$

- The IRR is an important measure especially that it has the nice property of not depending on the prevailing *external* market interest rate.
- One issue with the IRR is that equation (1) may have multiple solutions. In this case, it becomes unclear which solution is the true IRR of the cash flow stream at hand.
- Fortunately, for one of the most common forms of investments, involving an initial payment followed by many receipts[1], (1) has a unique > 0 root which gives a true IRR.
- The following theorem proves this.

- **Main theorem on internal rate of return**

Suppose the cash flow stream $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *has* $x_0 < 0$ and $x_k \geq 0$ for $k = 1, 2, \ldots, n$, *with at least one* $x_j > 0$, $j \in \{1, 2, \ldots, n\}$. *Then, there is a unique root to the equation*

$$x_0 + x_1 c + x_2 c^n + \ldots + x_n c^n = 0 .$$

Furthermore if $\sum_{k=0}^{n} x_k > 0$, *then the corresponding internal rate of* return $r = 1/c - 1$ *is positive.*

Proof. Define $f(c) = x_0 + x_1 c + x_2 c^n + \ldots + x_n c^n$. Then, $f(0) = x_0 < 0$ and $f(\infty) = \infty$. This implies that the equation $f(c) = 0$ has at least one positive root. It can also be seen that $f(c)$ is strictly increasing in c. This proves the first part of the theorem. To prove

[1] This is called a "conventional" cash flow stream.

the second part note that $f(1) = x_0 + x_1 + \ldots + x_n = \sum_{k=0}^{n} x_k$. This implies that if $\sum_{k=0}^{n} x_k > 0$ then $f(1) > 0$, and the unique root of $f(c)$ is less than 1. Hence, the corresponding $r = 1/c - 1 > 0$. $\blacksquare$

- **Evaluation criteria**
  - Several criteria are used in comparing cash flow streams.
  - The objective of the comparison is selecting the stream which is most desirable from an economic perspective.
  - The two most important criteria are based on present value and internal rate of return.

- **Net present value**
  - Under net present value (NPV) criteria, alternatives are ranked based on their present value (the larger the better).
  - The prefix "net" indicates that present values of costs and payments are considered.
  - To be worthy of consideration a cash flow stream must have a NPV > 0.
  - All present values are evaluated with an interest rate based on the firm's *cost of capital*, called *minimum attractive rate of return* (MARR).

- **IRR criterion**
  - Under IRR criteria, alternatives are ranked based on their internal rate of return (the larger the better).
  - To be worthy of consideration a cash flow stream must have an IRR greater than the interest rate (determined based on the firm cost of capital).
- **Discussion of criteria**
  - NPV and IRR criteria do not always give the same answer especially if alternatives have different life spans.
  - In certain situations, e.g., if cash flows occur in repetitive cycles, then the two criteria can lead to similar conclusions.
  - The advantages of NPV criteria are ease of computation and "linearity" (meaning that the NPV of sum of cash flows streams is equal to the sum of the streams NPVs).
  - The main disadvantage of NPV criteria is that it requires the estimation of an "external" interest rate based on cost of capital which is not always easy to do.
  - The main advantage of IRR criteria is that IRR depends only on the cash flow stream under consideration (and not on external factors such as the market interest rate).
  - The main disadvantages of IRR criteria are difficulty of computation and the ambiguity associated with several possible roots of the IRR equation.

- **Which criteria to use?**
  - It is widely agreed that NPV is the best criterion (if applied prudently).
  - But NPV is not "the whole story."

- **NPV, IRR, and Economic Feasibility Analysis**
  - Utilizing NPV or IRR one can perform a so-called economic feasibility analysis whereby different alternatives are compared based on their cash flows.
  - The key thing to keep in mind about this analysis is that it is *forward looking* and requires *estimation* of future cash flows, and the MARR, which is often the hard part.
  - The estimation part is under-studied in textbooks and requires a mixture of experience and economic data.