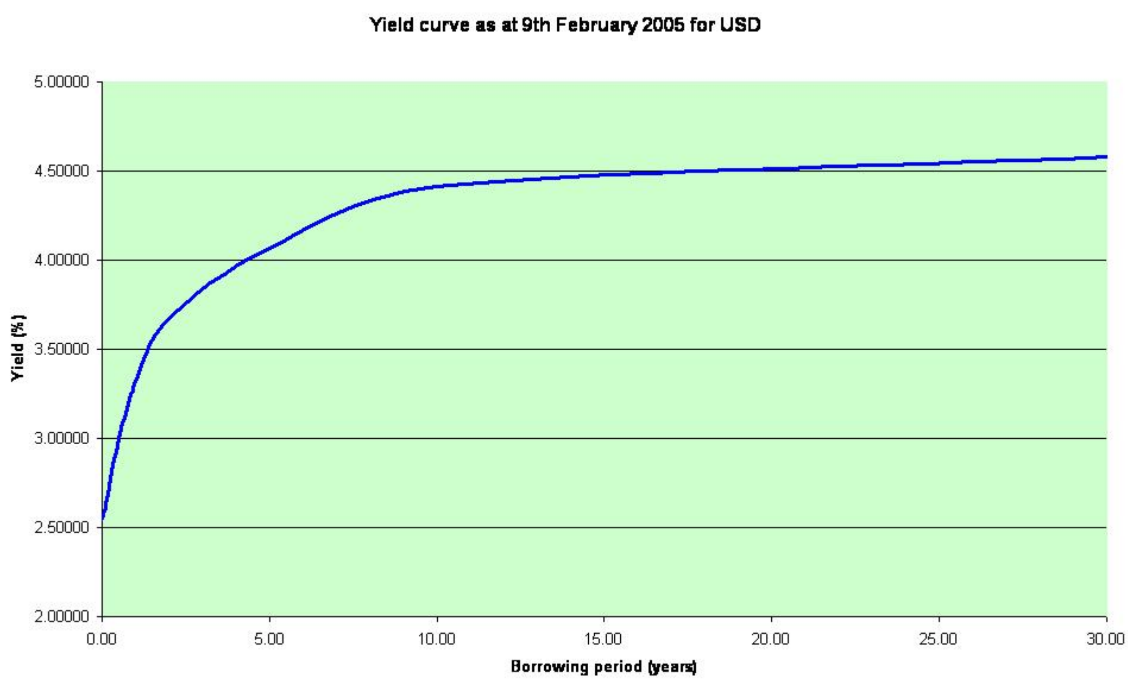
Chapter 4 The Term Structure of Interest Rate

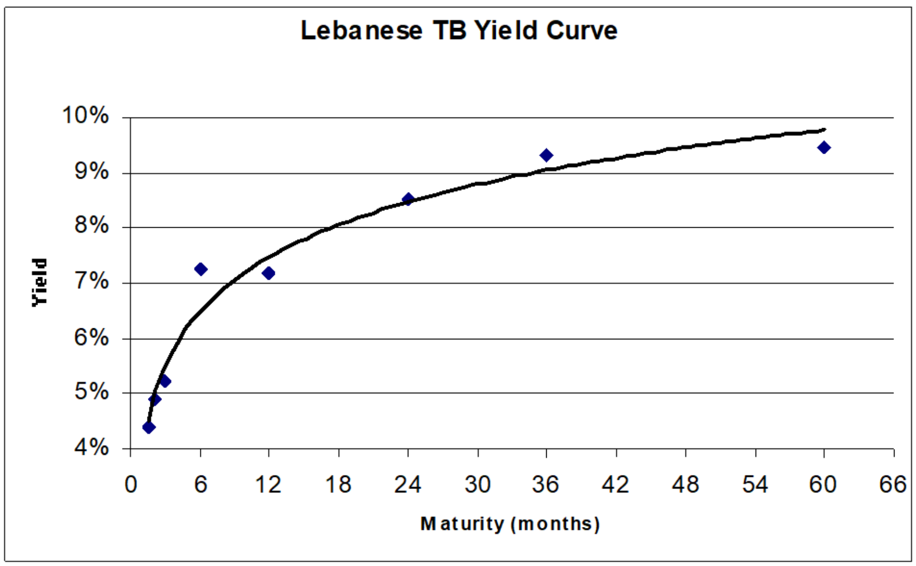
- **The yield curve**
  - Long bonds tend to offer higher yields than short bonds of the same quality.
  - The *yield curve* display yield as a function of time to maturity.
  - The yield is constructed based on yields of available bonds of a given quality class.
  - A rising yield curve is *normally shaped*. This occurs most often.
  - If long bonds happen to have lower yields than short bonds then the result is an *inverted yield curve*.

(Source: http://en.wikipedia.org/wiki/Yield_curve)



(Source: The Lebanon Brief, October 20, 2007. Blom Bank)

Money Market Rates	19/10/07	11/10/07	b.p. Change
Interbank Average†	3.50%	3.50%	0
BDL 45-day CD	4.40%	4.40%	0
BDL 60-day CD	4.89%	4.89%	0
Treasury Yields	**19/10/07**	**11/10/07**	**b.p. Change**
3-M TB yield	5.22%	5.22%	0
6-M TB yield	7.24%	7.24%	0
12-M TB yield	7.19%	7.19%	0
24-M TB coupon	8.50%	8.50%	0
36-M TB coupon	9.32%	9.32%	0
60-M TB coupon	9.45%	9.45%	0



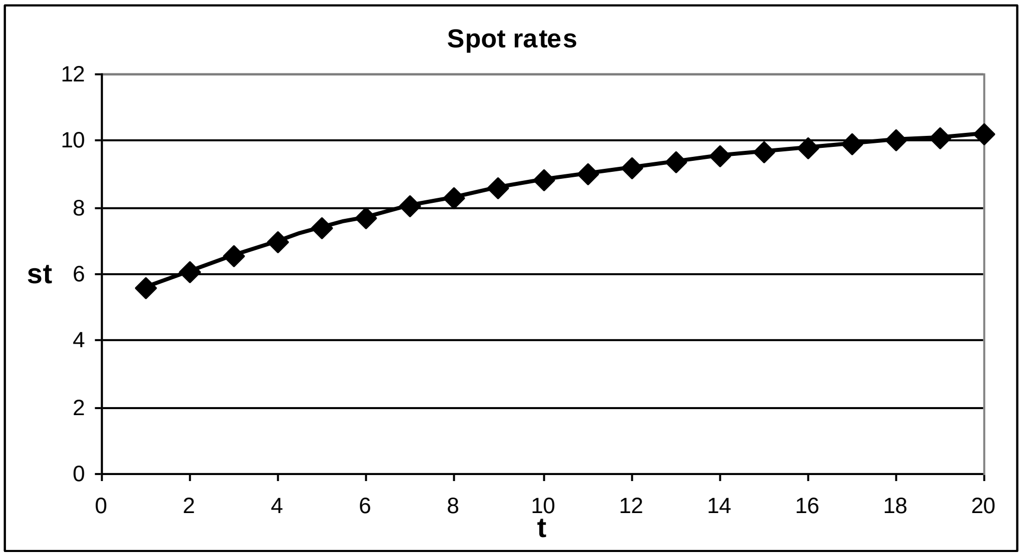
- ➢ When studying a particular bond it is useful to place it as a point in the plot of the yield curve.

- **The term structure**
  - ➢ Term structure theory is based on the observation that interest rate depends on the length of time the money is held.

- **The spot rates**
  - *Spot rates* are the basic interest rates defining the term structure.
  - The spot rate s_t is the interest rate charged for money held from present till year t.
  - For example, a 1-year deposit will grow by a factor of $(1+s_1)$. A 2-year deposit will grow by a factor of $(1+s_2)^2$.
  - In general, a t-year investment grows by a factor of $(1+s_t)^t$.
  - Compounding rules applies to spot rates. For example, under a compounding of m times per year, a t-year deposit will grow by a factor of $(1+s_t/m)^{mt}$.
  - Under continuous compounding a t-year deposit will grow by a factor of $e^{s_t t}$.
  - Discount factors and present values can then be determined in the usual way.
  - For example, with yearly compounding, the present value of a cash flow stream $\mathbf{x} = (x_0, x_1, \ldots, x_n)$ is

$$PV = \sum_{k=0}^{n} d_k x_k \text{ , where } d_k = 1/(1+s_k)^k .$$

- **Spot rates curve**
  - Spot rates can be determined from the yields of zero-coupon. bonds



  - If not enough, zero-coupon bonds are available (especially long ones), the spot rate curve can be determined from the prices of coupon-bearing bonds.
  - For example, suppose you have a 1-year zero-coupon bond and a 2-year bond paying a coupon C_2 every year. The yield of the first bond ($F_1/P_1 - 1$) gives the spot rate s_1.
  - Then, the spot rate s_2 can be determined from the equation

$$P_2 = \frac{C_2}{1+s_1} + \frac{C_2 + F_2}{(1+s_2)^2}.$$

  - Spot rates can also be found by "subtraction" of two bonds with different coupons to construct a zero-coupon bond.

- **Examples**

Example 4.1 (Price of a 10-year bond) Using the spot rate curve of Figure 4.2, let us find the value of an 8% bond maturing in 10 years.

Normally, for bonds we would use the rates and formulas for 6-month compounding; but for this example let us assume that coupons are paid only at the end of each year, starting a year from now, and that l-year compounding is consistent with our general evaluation method. We write the cash flows together with the discount factors, take their products, and then sum, as shown in Table 4.1. The value of the bond is found to be 97.34.

Years	Spot Rate
1	5.571
2	6.088
3	6.555
4	6.978
5	7.361
6	7.707
7	8.020
8	8.304
9	8.561
10	8.793

TABLE 4.1
Bond Evaluation

1/1.05571 1/(1.06088^2) 1/(1.0802^7)

Year	1	2	3	4	5	6	7	8	9	10	Total PV
Discount	.947	.889	.827	.764	.701	.641	.583	.528	.477	.431	
Cash flow	8	8	8	8	8	8	8	8	8	108	
PV	7.58	7.11	6.61	6.11	5.61	5.12	4.66	4.22	3.82	46.50	97.34

Example 4.3 (Construction of a zero) Bond A is a 10-year bond with a 10% coupon. Its price is $P_A = 98.72$. Bond B is a 10-year bond with an 8% coupon. Its price is $P_B = 85.89$. Both bonds have the same face value, normalized to 100.

Consider a portfolio with $-.8$ unit of bond A and 1 unit of bond B. This portfolio will have a face value of 20 and a price of $P = P_B - .8P_A = 6.914$. The coupon payments cancel, so this is a zero-coupon portfolio. The 10-year spot rate s_{10} must satisfy $(1+s_{10})^{10}P = 20$. Thus $s_{10} = 11.2\%$.

- **Forward rates**
  - Forward rates are interest rates for money to be borrowed between two future dates, under terms agreed upon today.
  - E.g., suppose there are two ways of investing \$1 for 2 years

    (i) Deposit it in a 2-year bank account where it will grow to $(1+s_2)^2$ at the end of the two years.

    (ii) Deposit it in a 1-year bank account where it will grow to $(1+s_1)$ at the end of the first year, and then deposit the proceeds for one more year at a rate $f_{1,2}$ (yielding $(1+s_1)(1+f_{1,2})$ at the end of the two years).

- In this case, $f_{1,2}$ is the forward rate between years 1 and 2.
- Invoking the comparison principle implies that

$$(1+s_2)^2 = (1+s_1)(1+f_{1,2}) \Rightarrow f_{1,2} = \frac{(1+s_2)^2}{1+s_1} - 1.$$

- The use of the comparison principle can be justified through an ***arbitrage*** argument.
- Arbitrage is earning money without any risk or investing anything.
- If $(1+s_2)^2 < (1+s_1)(1+f_{1,2})$, then one can borrow \$1 for two years and invest it according to (ii) and make an arbitrage profit of $(1+s_1)(1+f_{1,2}) - (1+s_2)^2$ after two years.
- If $(1+s_2)^2 > (1+s_1)(1+f_{1,2})$, then one can borrow \$1 for one year and invest it according to (i). Then, at the end of the first year, borrow another $(1+s_1)$ dollars to pay the first loan. This will yield an arbitrage profit of $(1+s_2)^2 - (1+s_1)(1+f_{1,2})$.
- Such an arbitrage scheme cannot exist in the market because many people will jump on it leading to closing the gap.
- This arbitrage argument assumes that there are no transaction costs and that borrowing and lending rates are identical. This is a reasonable approximation.
- In general, the forward rate f_{t_1,t_2} is the *annual* interest rate charged for borrowing money between times t_1 and t_2, $t_1 < t_2$.
- Forward rates deduced from spot rates are termed *implied forward rates* to distinguish them from *market forward rates*.

- The implied forward rate between year i and year j satisfies $(1+s_j)^j = (1+s_i)^i(1+f_{i,j})^{j-i}$, which implies that

$$f_{i,j} = \left[\frac{(1+s_j)^j}{(1+s_i)^i}\right]^{1/(j-i)} - 1.$$

- For m period-per-year compounding, the implied forward rate (per year) between periods i and j satisfies $(1+s_j/m)^j = (1+s_i/m)^i(1+f_{i,j}/m)^{j-i}$, which implies that

$$f_{i,j} = m\left[\frac{(1+s_j/m)^j}{(1+s_i/m)^i}\right]^{1/(j-i)} - m.$$

- Under continuous compounding, the implied forward rate (per year) between times t_1 and t_2 satisfies $e^{s_{t_2}t_2} = e^{s_{t_1}t_1}e^{f_{t_1,t_2}(t_2-t_1)}$, which implies that

$$f_{t_1,t_2} = \frac{s_{t_2}t_2 - s_{t_1}t_1}{t_2 - t_1}.$$

- Note that (at any compounding) the spot rate at time t can be seen as the forward rate between time 0 and t, $f_{0,t} = s_t$.

- **Term structure explanations**
  - The spot rate curve is almost never flat but usually slopes upward.
  - Why is this curve not just flat at a common interest rate?
  - There three standard explanations for this: *Expectation theory*, *liquidity preference*, and *market segmentation*.
  - We adopt the expectation theory explanation.

- **Expectation theory**
  - This theory explains the shape of the spot rate curve based on expectations of what rates will be in the future.
  - E.g., the theory argues that most people in the market believe that the 1-year rate next year will be higher than the current 1-year rate.
  - The *expectation hypothesis* expresses this expectation in terms of forward rates.
  - E.g., according to this hypothesis, the forward rate, $f_{1,2}$, is *exactly* equal to market expectation of what the 1-year rate will be next year, s_1'. That is, $s_1' = f_{1,2}$.
  - More generally, the hypothesis is $s_{n-1}' = f_{1,n}$.
  - The main weakness of expectation theory is that it implies that spot rates always increase, which is not always true.

- **Expectation dynamics**
  - The expectation hypothesis leads to useful tools.
  - Spot rate forecasts: Under the expectation hypothesis, the k-year spot rate i years from now is

$$s_k^{(i)} = f_{i,k+i} .$$

- Specifically, if the current spot rates are $s_0 = f_{0,1}$, $s_2 = f_{0,2}$, …, $s_n = f_{0,n}$, then forecasts for spot rates for years 1 to $n-1$ are

Year, *i*	$s_1^{(i)}$	$s_2^{(i)}$	….	$s_{n-2}^{(i)}$	$s_{n-1}^{(i)}$	$s_n^{(i)}$
0	$f_{0,1}$	$f_{0,2}$	…	$f_{0,n-2}$	$f_{0,n-1}$	$f_{0,n}$
1	$f_{1,2}$	$f_{1,3}$	…	$f_{1,n-1}$	$f_{1,n}$	
2	$f_{2,3}$	$f_{2,4}$	…	$f_{2,n-2}$		
⋮	⋮	⋮				
$n-2$	$f_{n-2,n-1}$	$f_{n-2,n}$				
$n-1$	$f_{n-1,n}$					

- The discount factor between years i and j is

$$d_{i,j} = \left(\frac{1}{1+f_{i,j}} \right)^{j-i} .$$

- Note that $d_{i,k} = d_{i,j}\, d_{j,k}$.

- **Short rates**
  - Short rates are the forward rates spanning a single time period. The short rate at year k is

$$r_k = f_{k,k+1} .$$

  - Short rates are as fundamental as spot rates because a complete set of short rates fully defines a term structure.
  - The spot rate can be obtained from short rates as follows.

$$(1+s_k)^k = (1+r_0)(1+r_1)\ldots(1+r_{k-1})$$
$$\Rightarrow s_k = [(1+r_0)(1+r_1)\ldots(1+r_{k-1})]^{1/k} - 1.$$

- Similarly, the forward rates can be obtained from the short rates as follows.

$$(1+f_{i,j})^{j-i} = (1+r_i)(1+r_{i+1})\ldots(1+r_{j-1})$$
$$\Rightarrow f_{i,j} = [(1+r_i)(1+r_{i+1})\ldots(1+r_{j-1})]^{1/j-i} - 1.$$

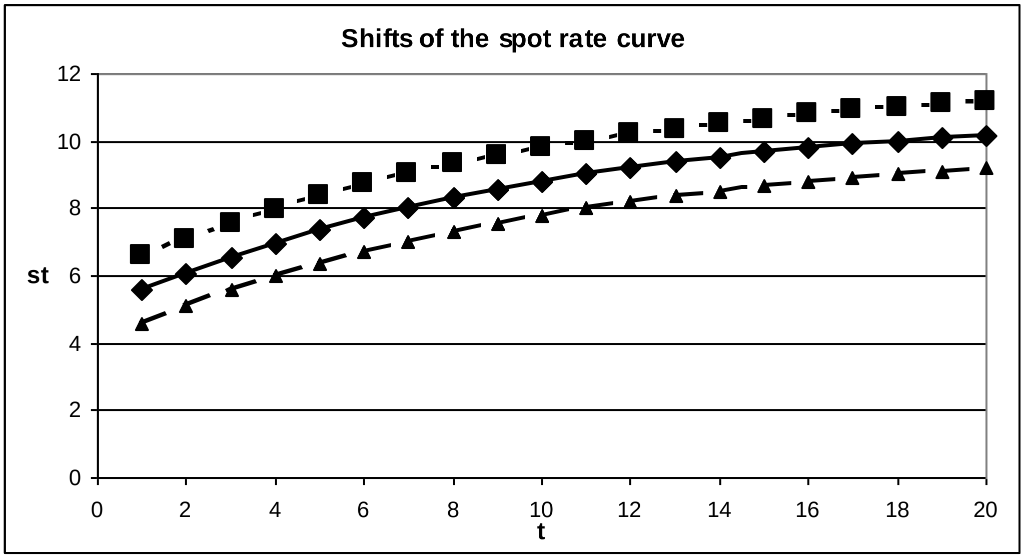
- A useful feature of short rates (under the expectation hypothesis) is that they do not change from year to year. (spot rates do change.)
- If the short rates now are r_0, r_1, …, r_n, then the short rates next year are r_1, …, r_n .

- **Examples**

> **Example 4.5 (A simple forecast)** Let us take as given the spot rate curve shown in the first row of the table. The second row is then the forecast of next year's spot rate curve under expectations dynamics. This row is found using equation (4.1).
>
> | | s_1 | s_2 | s_3 | s_4 | s_5 | s_6 | s_7 |
> |---|---|---|---|---|---|---|---|
> | Current | 6.00 | 6.45 | 6.80 | 7.10 | 7.36 | 7.56 | 7.77 |
> | Forecast | 6.90 | 7.20 | 7.47 | 7.70 | 7.88 | 8.06 | |
>
> The first two entries in the second row were computed as follows:
>
> $$f_{1,2} = \frac{(1.0645)^2}{1.06} - 1 = .069$$
> $$f_{1,3} = \left[\frac{(1.068)^3}{1.06}\right]^{1/2} - 1 = .072.$$

To see how short rates work, we will go together over re-generating Table 4.2 in the text in Excel. The example is available on the course website.

- **Duration under the term structure**
  - Under the term structure, duration is defined as sensitivity to linear shifts in the spot rate curve.
  - Specifically, the duration of a bond is the sensitivity of the bond value relative to λ when the spot rates shift from s_1, s_2, …, s_n, to $s_1 + \lambda$, $s_2 + \lambda$, …, $s_n + \lambda$.



  - Under continuous compounding, the *Fisher-Weil duration* of a cash flow stream with cash flows x_{t_i} at time t_i, $i = 1, \ldots, n$ is

$$D_{FW} = \frac{1}{PV}\sum_{i=0}^{n} t_i x_{t_i} e^{-s_{t_i} t_i} ,$$

where $PV = \sum_{i=0}^{n} x_{t_i} e^{-s_{t_i} t_i}$.

  - Let $P(\lambda)$ be the value (price) of the stream when the spot rate curve shifts by λ. Then,

$$P(\lambda) = \sum_{i=0}^{n} x_{t_i} e^{-(s_{t_i}+\lambda)t_i} \; .$$

(Observe that with no shift this value is $P(0) = PV$.)

- Upon differentiation,

$$\frac{dP(\lambda)}{d\lambda} = \sum_{i=0}^{n} -x_{t_i} t_i e^{-(s_{t_i}+\lambda)t_i} \; .$$

- Then,

$$\frac{1}{P(0)}\left[\frac{dP(\lambda)}{d\lambda}\right]_{\lambda=0} = -D_{FW} \; .$$

- Under discrete compounding, with *m* period-per-year compounding,

$$P(\lambda) = \sum_{k=0}^{n} \frac{x_k}{[1+(s_k+\lambda)/m]^k} \; .$$

- Then,

$$\left[\frac{dP(\lambda)}{d\lambda}\right]_{\lambda=0} = -\sum_{k=0}^{n} \frac{(k/m)x_k}{[1+(s_k+\lambda)/m]^{k+1}}\Bigg|_{\lambda=0}$$

$$= -\sum_{k=0}^{n} \frac{(k/m)x_k}{[1+s_k/m]^{k+1}} \, .$$

- Then, the *quasi-modified duration* is then defined as

$$D_Q = \frac{1}{P(0)}\left[\frac{dP(\lambda)}{d\lambda}\right]_{\lambda=0} = \frac{\sum_{k=0}^{n}(k/m)x_k(1+s_k/m)^{-(k+1)}}{\sum_{k=0}^{n} x_k(1+s_k/m)^{-k}} \, .$$

- **Immunization Idea and Example**
  - Immunization can now be done similar to Chapter 3 but by matching D_{FW} (or D_Q) instead of D_M.
  - This can be done for structuring a portfolio of bonds with different yields.

Example 4.8 (A million dollar obligation) Suppose that we have a \$1 million obligation payable at the end of 5 years, and we wish to invest enough money today to meet this future obligation. We wish to do this in a way that provides a measure of protection against interest rate risk. To solve this problem, we first determine the current spot rate curve. A hypothetical spot rate curve s_k is shown as the column labeled spot in Table 4.4.

We use a yearly compounding convention in this example in order to save space in the table. We decide to invest in two bonds described as follows: B_1 is a 12-year 6% bond with price 65.95, and B_2 is a 5-year 10% bond with price 101.66. The prices of these bonds are consistent with the spot rates; and the details of the price calculation are given in Table 4.4. The cash flows are multiplied by the discount factors (column d), and the results are listed and summed in columns headed PV_1 and PV_2 for the two bonds.

We decide to immunize against a parallel shift in the spot rate curve. We calculate $dP/d\lambda$, denoted by $-PV'$ in Table 4.4, by multiplying each cash flow by t and by $(1+s_t)^{-(t+1)}$ and then summing these. The quasi-modified duration is then the quotient of these two numbers; that is, it equals $-(1/P)dP/d\lambda$. The quasi-modified duration of bond 1 is, accordingly, $466/65.95 = 7.07$.

We also find the present value of the obligation to be \$627,903.01 and the corresponding quasi-modified duration is $5/(1+s_5) = 4.56$.

To determine the appropriate portfolio we let x_1 and x_2 denote the number of units of bonds 1 and 2, respectively, in the portfolio (assuming, for simplicity, face values of \$100). We then solve the two equations[3]

$$P_1 x_1 + P_2 x_2 = \mathrm{PV}$$

$$P_1 D_1 x_1 + P_2 D_2 x_2 = \mathrm{PV} \times D$$

where the D's are the quasi-modified durations. This leads to $x_1 = 2{,}208.17$ and $x_2 = 4{,}744.03$. We round the solutions to determine the portfolio. The results are shown in the first column of Table 4.5, where it is clear that, to within rounding error, the present value condition is met.

1 / 1.0767 6*0.925

6*0.925^2 = 5.57*0.925

TABLE 4.4
Worksheet for Immunization Problem

Year	Spot	d	B_1	PV_1	$-PV'_1$	B_2	PV_2	$-PV'_2$
1	7.67	.929	6	5.57	5.18	10	9.29	8.63
2	8.27	.853	6	5.12	9.45	10	8.53	15.76
3	8.81	.776	6	4.66	12.84	10	7.76	21.40
4	9.31	.700	6	4.20	15.38	10	7.00	25.63
5	9.75	.628	6	3.77	17.17	110	69.08	314.73
6	10.16	.560	6	3.36	18.29			
7	10.52	.496	6	2.98	18.87			
8	10.85	.439	6	2.63	18.99			
9	11.15	.386	6	2.32	18.76			
10	11.42	.339	6	2.03	18.26			
11	11.67	.297	6	1.78	17.55			
12	11.89	.260	106	27.53	295.26			
Total				65.95	466.00		101.66	386.15

TABLE 4.5
IMMUNIZATION RESULTS

	Lambda		
	0	1%	−1%
Bond 1			
Shares	2,208.00	2,208.00	2,208.00
Price	65.94	51.00	70.84
Value	145,602.14	135,805.94	156,420.00
Bond 2			
Shares	4,744.00	4,744.00	4,744.00
Price	101.65	97.89	105.62
Value	482,248.51	464,392.47	501,042.18
Obligation value	627,903.01	600,063.63	657,306.77
Bonds minus obligation	−$52.37	$134.78	$155.40