Fixed-Income Securities (Chapter 3, Luenberger)

- **Financial instruments and securities**
  - Money is the most popular of traded commodities.
  - Interest rate is a price for money.
  - Vast assortments of bills, notes, bonds, annuities, futures contracts, and mortgages are part of the market for money.
  - These items are not real goods. They are traded based on the promises they represent.
  - These items are called *financial instruments* (products).
  - A financial instrument that has a well-developed market so that it can be traded freely and easily is called a *security*.
  - A fixed-income security is a financial instrument that promises a fixed (deterministic) income to the holder (e.g. bonds, mortgages).[1]

- **Examples of fixed-income securities**
  - Saving deposits are offered by commercial banks and savings and loan institutions.
    - ***Demand deposit*** pays interest that varies with market condition.
    - ***Time deposit account*** pays a guaranteed interest but the deposit must be maintained for a length of time (or penalties apply).

[1] Certain fixed-income securities may generate income that varies with an interest rate index or a stock price.

- A ***certificate of deposit*** (CD) is similar to a time deposit. CDs with large denominations can be sold in the market.

- ***Money market*** is the market for short-term loans (≤ 1 year) by corporations and financial intermediaries (e.g. banks).
  - A ***commercial paper*** is an unsecured loan to a corporation.
  - A ***banker's acceptance*** is a promise by a bank to pay an amount to a company at a future date. The company can sell this promise at a discount.
  - ***Eurodollar deposits*** are deposits in dollars held in a bank outside the U.S.
  - ***Eurodollar CDs*** are CDs in dollars issued by a bank outside the U.S. This allows escaping U.S. regulations.

- The U.S. government obtains loans by issuing fixed-income securities.
  - ***U.S. Treasury bills*** are issued with a fixed *maturity date* of 13, 26, or 52 weeks, and are sold at a discount from the *face value*.
  - ***U.S. Treasury notes*** are issued with maturities of 1-10 years and offer the holder *coupon payments* every six months. At maturity, the holder receives the last coupon payment and the face value.
  - *U.S.* ***Treasury bonds*** are issued with maturities > 10 years. They are similar to Treasury notes but some are *callable*.
  - *U.S.* ***Treasury strips*** are bonds where the coupon payments and the principle are issued in strips. Each strip can be traded independently as a *zero-coupon bond*.

- Bonds are also issued by other entities.
    - ***Municipal bonds*** are issued by agencies of state and local governments. Their interest income is exempt from tax.
    - ***Corporate bonds*** are issued by corporations. They vary in *quality*.
- For a homeowner a ***mortgage*** looks like the opposite of a bond.
- A homeowner will *sell* a mortgage in order to obtain cash to buy a home. In return, the homeowner makes periodic (usually monthly) payments to the mortgage holder.
- Mortgages are typically bundled into large packages and traded among financial institutions. These packages are called ***mortgage-backed securities***.
- An ***annuity*** is a contract that pays the holder (the *annuitant*) money periodically. E.g., pension benefits.
- Annuity sometimes depends on the age of the annuitant at the time the annuity is purchased.
- A ***perpetual annuity***, or *perpetuity*, is an annuity that pays fixed income periodically forever.
- Annuities are not real securities since they are not traded. But they provide investment opportunities.

- **Value formulas**
  - The present value of an annuity which pays an amount A per period for n period at an interest rate of r per period is

$$P = \frac{A}{r}\left[1 - \frac{1}{(1+r)^n}\right].$$

  - The present value of a perpetual annuity which pays an amount A per period at an interest rate of r per period is

$$P = \lim_{n\to\infty} \frac{A}{r}\left[1 - \frac{1}{(1+r)^n}\right] = \frac{A}{r}.$$

  - Amortization is the process of substituting a current payment P for periodic payments of A per period. (E.g. a car loan, home mortgage.)
  - At an interest rate of r per period, the periodic payment is

$$A = P\left[\frac{r(1+r)^n}{(1+r)^n - 1}\right].$$

- **Bond details (see also Chapter 12 in Antle's book)**
  - Bonds represent the greatest monetary values of fixed income securities.
  - A bond is an obligation by the bond issuer to pay money to the bond holder (buyer).
  - A bond pays its face value or par value at its maturity date.
  - In addition, bonds usually pay periodic coupon payments. In the U.S., coupon payments are made every 6 months.
  - The coupon amount is described in percent of face value.

- Usually coupon rates are close to the prevailing interest rate.
- A bond can be traded freely in the market place. Its price varies continuously.
- The *bid price* of a bond is the price dealers are willing to pay for a bond. The *ask price* is the price at which dealers are willing to sell the bond.

- **Bond accrued interest**
  - When buying a bond, one must pay the *accrued interest* to the seller in addition to the ask price.
  - This is the interest accrued since the last coupon payment till the sale date. The accrued interest (AI) is given by

$$AI = \frac{\text{number of days since last coupon}}{\text{number of days in coupon period}} \times \text{coupon amount} .$$

- **Bond quality ratings**
  - Bonds are subject to the risk of *default* if the issuer faces financial difficulties or falls into bankruptcy.
  - To characterize this risk bonds are rated by rating organizations. U.S. treasury securities are not rated because they are considered risk-free.
  - A bond with low rating will have a lower price than a similar bond with high rating.
  - The two main rating organizations are Moody's and Standard & Poor's. Their classification is as follows:

Bond Grade	Moody's Rating	Standard & Poor's Rating	Common Bond Name
High Grade	Aaa	AAA	Investment Grade
	Aa	AA	"
Medium Grade	A	A	"
	Baa	BBB	"
Speculative Grade	Ba	BB	Junk Bond
	B	B	"
Default Danger	Caa	CCC	"
	Ca	CC	"
	C	C	"
		D	"

- **Bond yield**
  - Yield to maturity (YTM) is the IRR of the bond.
  - Specifically, for a bond with a price of *P* and a face value *F* making *m* coupon payments per year of *C*/*m* (with a total of *n* payments), the YTM is the value of λ such that

$$P = \frac{F}{(1+\lambda/m)^n} + \sum_{k=1}^{n} \frac{C/m}{[1+(\lambda/m)]^k} \ .$$

  - This formula assumes that the interest is compounded every payment period.

  - Upon simplification,

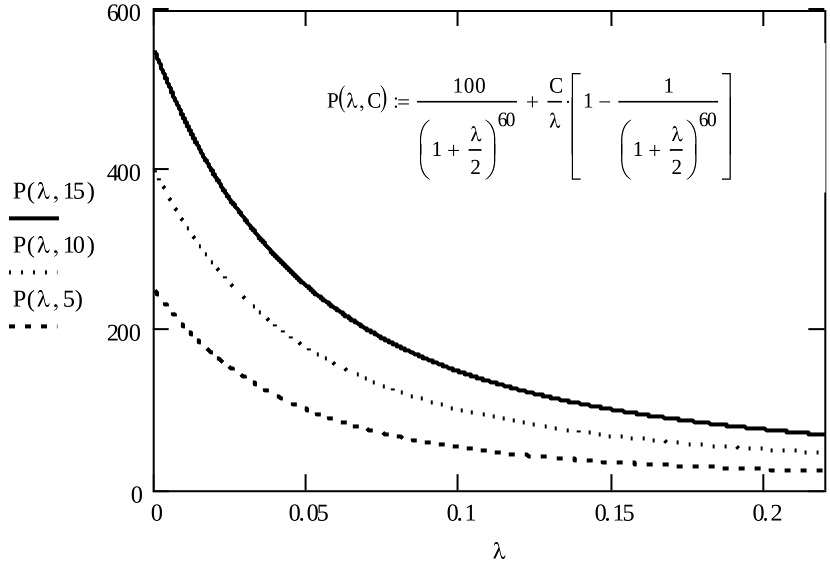
$$P = \frac{F}{[1+(\lambda/m)]^n} + \frac{C}{\lambda}\left(1 - \frac{1}{[1+(\lambda/m)]^n}\right) .$$

- **Example: Bond Price**
  - Price of a bong having 10-year (maturity), 9% (coupon, paid semiannually), 8% yield (and face value 100),

$$P := \frac{100}{\left(1+\frac{0.08}{2}\right)^{20}} + \frac{9}{0.08}\cdot\left[1 - \frac{1}{\left(1+\frac{0.08}{2}\right)^{20}}\right]$$

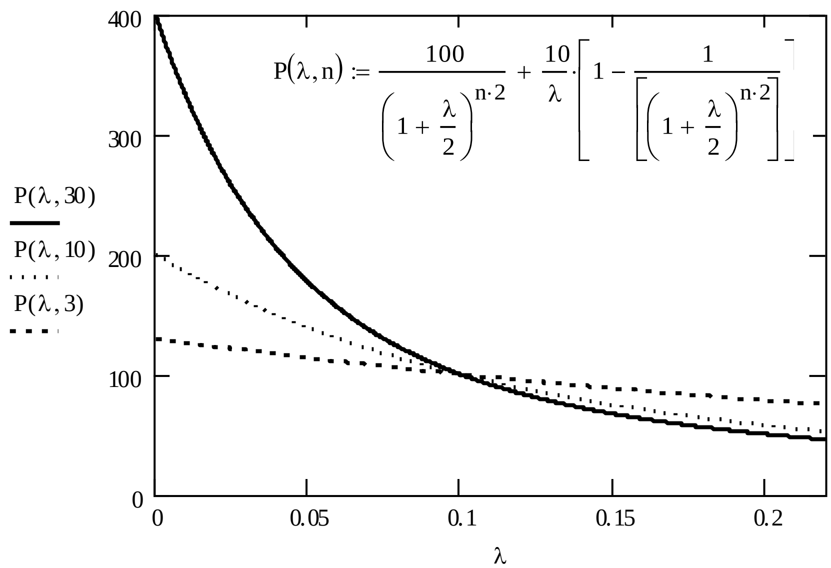
$$P = 106.8$$

- **Price-yield curves**
  - These give the price, P, as a function of yield to maturity λ .
  - Nature of price-yield curves.
    - P is decreasing in λ (also λ decreases with P).
    - At $\lambda = 0$, $P = F + n(C/m)$, this is the undiscounted price.
    - P tends to zero as λ increases.
    - The yield curve is convex (P is decreasing in λ at a decreasing rate).
    - As the coupon payment, C, increases, P increases.
    - At $\lambda = C/F$, $P = F$, for all n. This is the price of a *par bond*.
    - As time to maturity, n, increases, the price-yield curve becomes steeper.
    - *That is, long-maturity bonds are very sensitive to interest rate.*
  - The following figure shows price-yield curves for a 30-year bond having a coupon value C = 5%, 10%, and 15%.



$$P(\lambda, C) := \frac{100}{\left(1+\frac{\lambda}{2}\right)^{60}} + \frac{C}{\lambda}\cdot\left[1-\frac{1}{\left(1+\frac{\lambda}{2}\right)^{60}}\right]$$

- The following figure shows different price-yield curves for a 10% coupon bond with maturity n = 3, 10, and 30 years.



$$P(\lambda, n) := \frac{100}{\left(1+\frac{\lambda}{2}\right)^{n\cdot 2}} + \frac{10}{\lambda}\cdot\left[1-\frac{1}{\left[\left(1+\frac{\lambda}{2}\right)^{n\cdot 2}\right]}\right]$$

- **E.g., Lebanese treasury bills yield in October 2019 (BDL)**

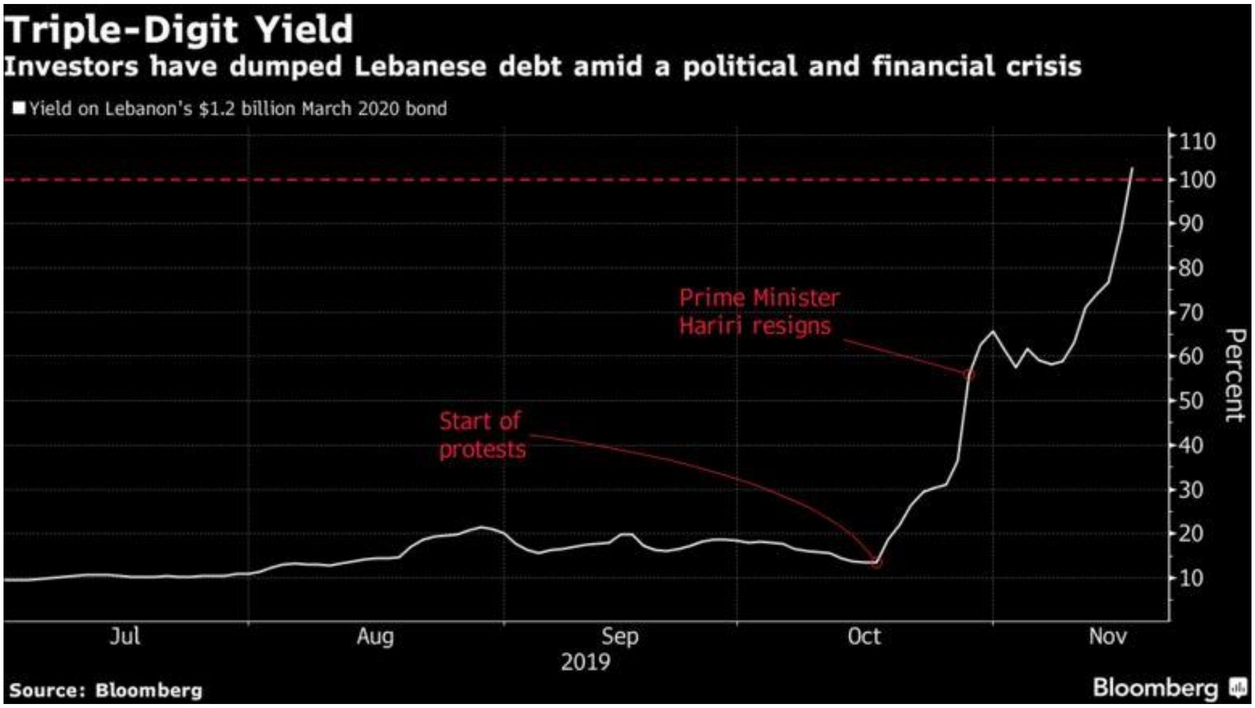
Maturity	3 months	6 months	12 months	24 months	36 months	60 months	84 months	120 months
Latest Yield								
24-Oct-19		5.85		7.00				10.00
17-Oct-19	5.30		6.50			8.00		
10-Oct-19		5.85			7.50		9.00	
03-Oct-19			6.50			8.00		

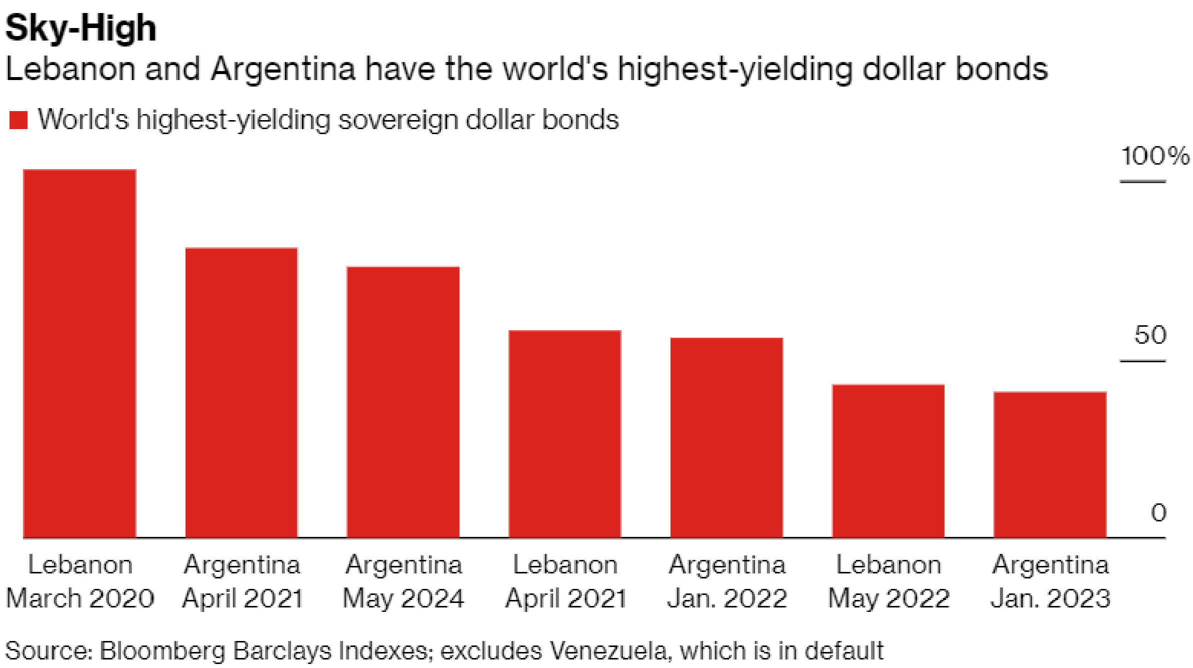
- **E.g.: U.S. treasury bills yield in November 2019 (www.treasury.gov)**

Date	1 Mo	2 Mo	3 Mo	6 Mo	1 Yr	2 Yr	3 Yr	5 Yr	7 Yr	10 Yr
11/01/19	1.58	1.58	1.52	1.55	1.53	1.56	1.55	1.55	1.63	1.73
11/04/19	1.58	1.57	1.53	1.57	1.56	1.60	1.59	1.60	1.69	1.79
11/05/19	1.56	1.57	1.56	1.58	1.62	1.63	1.63	1.66	1.77	1.86

- **E.g.: Lebanese Bonds Rating (highlighted in red, Alakhbar newspaper)**

Fitch	Moody's	S&P	
AAA	Aaa	AAA	ممتاز
+AA	Aa1	+AA	درجة عالية
AA	Aa2	AA	
-AA	Aa3	-AA	
+A	A1	+A	درجة متوسطة عليا
A	A2	A	
-A	A3	-A	
+BBB	Baa1	+BBB	درجة متوشطة منخفضة
BBB	Baa2	BBB	
-BBB	Baa3	-BBB	
+BB	Ba1	+BB	درجة غير مشجعة على الاستثمار
BB	Ba2	BB	
-BB	Ba3	-BB	
+B	B1	+B	درجة مخاطر عالية
B	B2	B	
-B	B3	**-B**	
CCC	**Caa1**	+CCC	درجة مخاطر كبيرة
	Caa2	CCC	درجة مخاطر مرتفعة جدا
	Caa3	-CCC	في حالة تخلّف عن السداد مع حظوظ متدنية بإمكانية معاودة السداد
	Ca	CC	
	c	C	
DDD		D	في حالة تخلّف عن السداد
DD			
D			





Source: https://www.bloomberg.com/news/articles/2019-11-20/yields-past-100-just-put-lebanon-debt-into-venezuela-territory

Fixed-Income Securities (2) (Chapter 3, Luenberger)

- **Duration**
  - Duration gives a direct measure of bond price sensitivity to interest rate.
  - Generally, for a cash flow stream with cash flows at times t_0, t_1, …, t_n, the duration is given by

$$D = \frac{\sum_{k=1}^{n} PV(t_k)t_k}{P},$$

where $PV(t_k)$ is the present value of the cash flow at time k and $P = \sum_{k=1}^{n} PV(t_k)$ is the present value of the whole stream.

  - Duration is a weighted average of cash flow times. When cash flows are all nonnegative $t_0 \le D \le t_n$.

  - For a financial instrument that generates a cash flow stream of m payments per year over a total of n periods, and a payment C_k in period k the duration is given by

$$D = \frac{\sum_{k=1}^{n} (k/m)C_k /[1+(\lambda/m)]^k}{\sum_{k=1}^{n} C_k /[1+(\lambda/m)]^k},$$

where λ is the yield to maturity (interest rate).

  - This duration is called *Macaulay duration*.

- For a bond with a price of P and a face value F making m coupon payments per year of C/m (with a total of n payments), and YTM λ , $C_k = C/m$ for $k = 1, \ldots, n - 1$, and $C_n = C/m + F$.
- Define the coupon rate per period as $c = (C/m)/F$, and the yield per period as $y = \lambda/m$. Then, the Macaulay duration of the bond is

$$D = \frac{\sum_{k=1}^{n}(k/m)(cF)/(1+y)^k + (n/m)F/(1+y)^n}{F/(1+y)^n + [(cmF)/(my)]\left[1-1/(1+y)^n\right]}$$

$$= \frac{(c/m)\sum_{k=1}^{n}k/(1+y)^k + (n/m)/(1+y)^n}{1/(1+y)^n + (c/y)\left[1-1/(1+y)^n\right]}$$

$$= \frac{cy(1+y)^n\sum_{k=1}^{n}k/(1+y)^k + ny}{my + cm\left[(1+y)^n - 1\right]}.$$

- It can be shown that

$$\sum_{k=1}^{n}k/(1+y)^k = \frac{(1+y)-(1+y+ny)/(1+y)^n}{y^2}.$$

- Upon simplification, the duration of the bond is

$$D = \frac{1+y}{my} - \frac{1+y+n(c-y)}{my + cm\left[(1+y)^n - 1\right]}.$$

- If the bond is at par, $c = y$,

$$D = \frac{1+y}{my}\left[1 - \frac{1}{(1+y)^n}\right].$$

- **Duration Estimation Example**

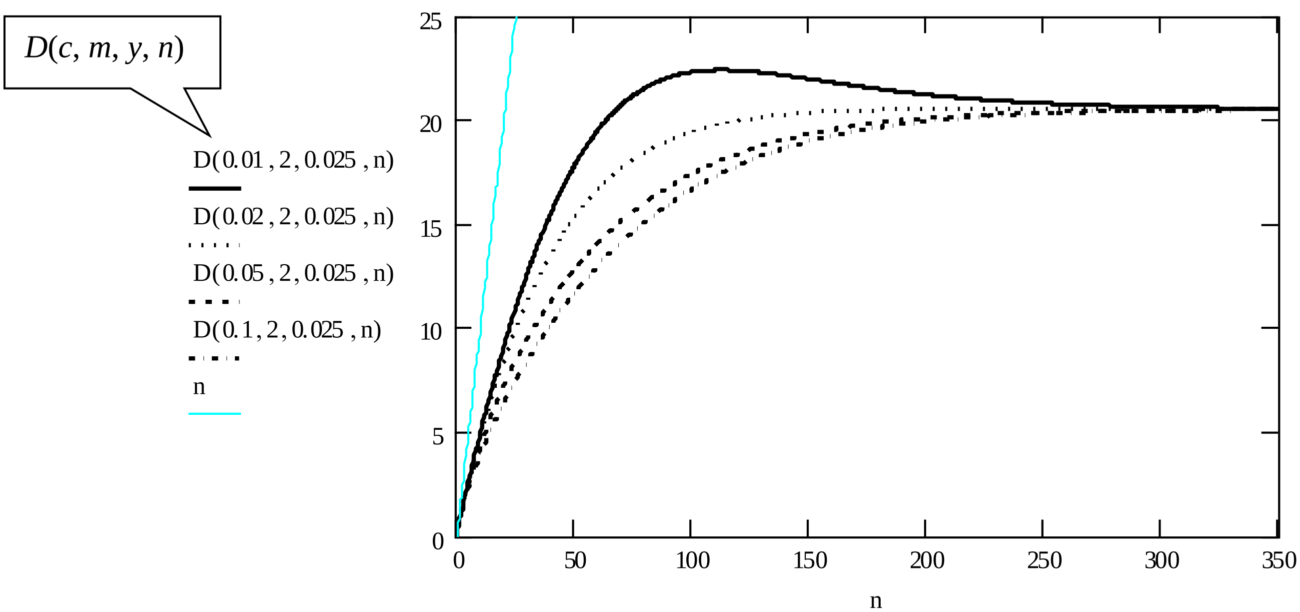
Example 3.7 (Duration of a 30-year par bond) Consider the 10%, 30-year bond represented in Figure 3.3. Let us assume that it is at par; that is, the yield is 10%. At par, $c = y$, and (3.3) reduces to

$$D = \frac{1+y}{my}\left[1 - \frac{1}{(1+y)^n}\right].$$

Hence,

$$D = \frac{1.05}{.1}\left[1 - \frac{1}{(1.05)^{60}}\right] = 9.938.$$

- **Properties of bond duration**



- ➢ Duration is always less than time to maturity.
- ➢ As time to maturity gets large, duration tends to a finite limit.
- ➢ Duration is not too sensitive to coupon rate.
- ➢ Long durations are achieved with long maturities and low coupon rates.

- **Duration and sensitivity**
  - The present value of the cash flow at time k is

$$PV_k = \frac{C_k}{[1+(\lambda/m)]^k}.$$

Calculus Moment
u(x) = (1+x/m)^(-k)
u'(x) = (1/m)(-k)(1+x/m)^(-k-1)
= -(k/m)(1+x/m)^(-1)u(x)

Then, $\dfrac{dPV_k}{d\lambda} = \dfrac{-(k/m)C_k}{[1+(\lambda/m)]^{k+1}} = -\dfrac{(k/m)}{[1+(\lambda/m)]}PV_k$

  - Recall that the price of the instrument is $P = \sum_{k=1}^{n} PV_k$. Then,

$$\frac{dP}{d\lambda} = \sum_{k=1}^{n}\frac{dPV_k}{d\lambda} = -\sum_{k=1}^{n}\frac{(k/m)}{1+(\lambda/m)}PV_k = -\frac{1}{1+(\lambda/m)}DP.$$

  - Therefore,

$$\frac{1}{P}\frac{dP}{d\lambda} = -D_M ,$$

where $D_M \equiv D / [1 + (\lambda/m)]$, is the *modified duration*.

  - That is, D_M measures the relative change of P as λ changes.
  - For a small change of $\Delta\lambda$, the relative price change is

$$\frac{\Delta P}{P} \approx -D_M \Delta\lambda .$$

- **Duration of a portfolio**
  - Consider a portfolio having m_b bonds. Let PV_k^j be the present value of the cash flow at time k from Bond j, having price P_j. Suppose there is a total of n time periods.
  - Then, the price of the portfolio is

$$P = \sum_{j=1}^{m_b}\sum_{k=1}^{n} PV_k^j = \sum_{j=1}^{m_b} P_j ,$$

- The duration of the portfolio is

$$D = \frac{\sum_{j=1}^{m_b}\sum_{k=1}^{n} t_k PV_k^j}{P} = \frac{\sum_{j=1}^{m_b} P_j D_j}{P} = \sum_{j=1}^{m_b} w_j D_j \ ,$$

 where D_j is the duration of bond j and $w_j \equiv P_j / P$ is the weight of bond j.

- **Immunization**
  - Immunization is the process of structuring a bond portfolio to protect against interest rate risk.
  - Specifically, suppose that a series of future obligations is to be met from investment in a bond portfolio.
  - Matching the present value of the obligations with the present value of the portfolio allows meeting the obligations if the yield (interest rate) does not change.
  - (In this chapter, we assume all bonds have the same yield.)
  - However, if the yield changes, then the present values may not match anymore.
  - Immunization approximately solves this problem by matching both present value and durations.
  - Immunization implies that the present values of the obligations and the bond portfolio will respond identically (to the first order) to yield change.
  - Immunization is widely used in practice.

- **Immunization Example**

Example 3.10 (The X Corporation) The X Corporation has an obligation to pay $1 million in 10 years It wishes to invest money now that will be sufficient to meet this obligation.

The purchase of a single zero-coupon bond would provide one solution; but such zeros are not always available in the required maturities. We assume that none are available for this example. Instead the X Corporation is planning to select from the three corporate bonds shown in Table 3.7. (Note that in this table, and throughout this example, prices are expressed in ordinary decimal form, not in 32nd's.)

These bonds all have the same yield of 9%, and this rate is used in all calculations. The X Corporation first considers using bonds 2 and 3 to construct its portfolio. As a first step it calculates the durations and finds $D_2 = 6.54$ and $D_3 = 9.61$, respectively. This is a serious problem! The duration of the obligation is obviously 10 years, and there is no way to attain that with a weighted average of D_2 and D_3 using positive weights. A bond with a longer duration is required. Therefore the X Corporation decides to use bonds 1 and 2. It is found that $D_1 = 11.44$. (Note that, consistent with the discussion on the qualitative nature of durations, it is quite difficult to obtain a long duration when the yield is 9%—a long maturity and a low coupon are required.) Fortunately $D_1 > 10$, and hence bonds 1 and 2 will work.

Next the present value of the obligation is computed at 9% interest. This is PV = $414,643. The immunized portfolio is found by solving the two equations

$$V_1 + V_2 = \text{PV}$$

$$D_1 V_1 + D_2 V_2 = 10\,\text{PV}$$

for the amounts of money V_1 and V_2 to be invested in the two bonds. The first equation states that the total value of the portfolio must equal the total present value of the obligation. The second states that the duration of the portfolio must equal the duration (10 years) of the obligation. (This relation is best seen by dividing through by PV.) The solution to these equations is $V_1 = \$292,788.73$ and $V_2 = \$121,854.27$. The number of bonds to be purchased is then found by dividing each value by the respective bond price. (We assume a face value of $100.) These numbers are then rounded to integers to define the portfolio.

The results are shown in Table 3.8. Note that, except for rounding error, the present value of the portfolio does indeed equal that of the obligation. Furthermore, at different yields (8% and 10% are shown) the value of the portfolio is still approximately equal to that of the obligation. In fact, due to the structure of the price–yield

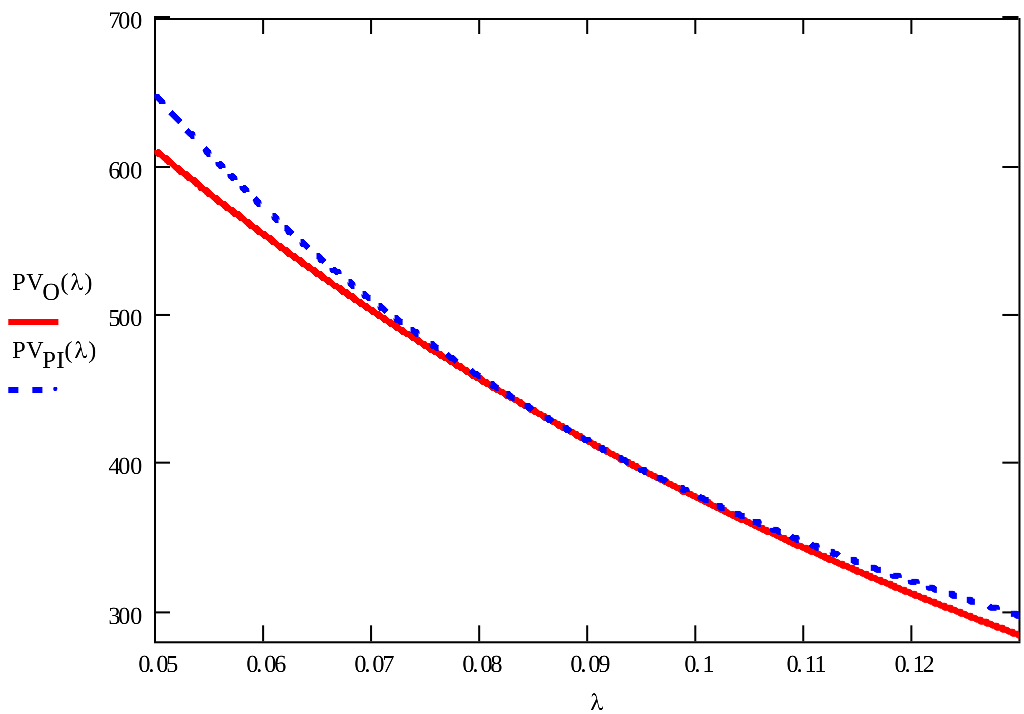
TABLE 3.7
Bond Choices

	Rate	Maturity	Price	Yield
Bond 1	6%	30 yr	69.04	9.00%
Bond 2	11%	10 yr	113.01	9.00%
Bond 3	9%	20 yr	100.00	9.00%

TABLE 3.8
Immunization Results

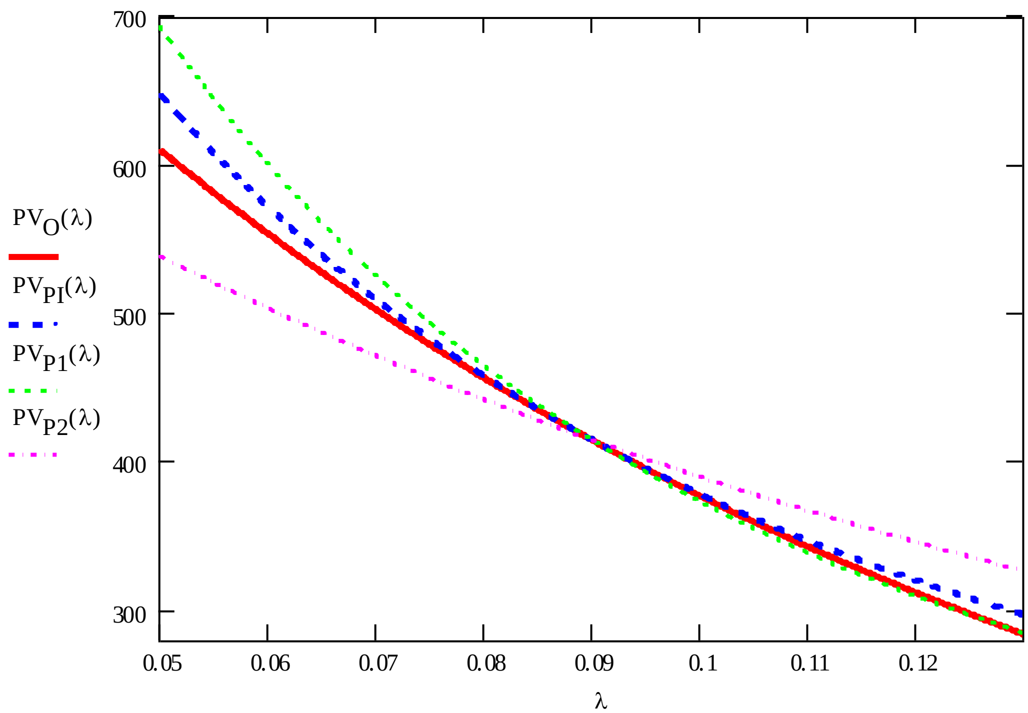
	Percent yield		
	9.0	8.0	10.0
Bond 1			
Price	69.04	77.38	62.14
Shares	4,241.00	4,241.00	4,241.00
Value	292,798.64	328,168.58	263,535.74
Bond 2			
Price	113.01	120.39	106.23
Shares	1,078.00	1,078.00	1,078.00
Value	121,824.78	129,780.42	114,515.94
Obligation Value	414,642.86	456,386.95	376,889.48
Surplus	−19.44	1,562.05	1,162.20

- **Graphical Illustration of Immunization (based on Example 3.10)**
  - The following figure gives the present value of the obligation (PV_O) and of the immunized portfolio (PV_{PI}) as a function of the interest rate λ for Example 3.10.



  - Note that the immunized portfolio has the same value and the same slope as the obligation at the initial yield (interest rate) value λ = 0.09.
  - As a result, as λ changes by small amounts around 0.09, the values of the obligation and the portfolio remain close.
  - To see the benefit of immunization, consider two alternate portfolios: Portfolio 1 having Bond 1 only with value PV_{P1} and portfolio 2 with Bond 2 only with value PV_{P2} .

- The number of Bonds *i* , *i* =1,2, in Portfolio *i* is selected so that $PV_O = PV_{Pi}$ at the initial yield λ = 0.09.
- (Recall that the immunized portfolio has both Bonds 1 and 2 with weights that match the obligation value and duration.)
- The following figure show how PV_{P1} and PV_{P2} vary as λ changes on the same graph as PV_O and PV_{PI} .



- Unlike the immunized portfolio, Portfolios 1 and 2 values do not follows the value of the obligation closely as λ varies.

- **Issues with immunization**
  - If the yield changes, then the portfolio will not be immunized at the new rate.
  - It is therefore desirable to *rebalance* the portfolio.

- Assuming equal yields is problematic as long-maturity bonds usually have higher yields than short-maturity bonds.
- In addition, it is unlikely that yield on all bonds will change by the same amount.
- Chapter 4 considers of bonds with different yields.