The Basic Theory of Interest (2, Chapter 2, Luenberger)

- **Comparing alternatives that repeat indefinitely with NPV**
  - Consider two alternatives that are composed of cycles of cash flows that repeat indefinitely.
  - The two alternatives can be compared in two different ways
    (i) Repeat the cycles of alternatives until they terminate at the same time (i.e., evaluate the two alternatives over the least common multiplier of cycles; see discussion of Example 2.4).
    (ii) Evaluate the NPV of each alternative directly over the infinite horizon based on a recursive equation (see Example 2.8).

- **Inflation**
  - Inflation is characterized by an increase in general prices with time. That is, purchasing power declines with time.
  - Inflation can be quantified with an inflation rate f.
  - \$1 today has the same purchasing power as $(1+f)^n$ dollars n years from now.
  - That is, $(1+f)^n$ dollars n years from now are worth 1 *constant dollar* or one *real dollar* today.

- If the *real* interest rate is r_0, then the *nominal* market interest rate, r, is such that $1 + r = (1+r_0)(1+f)$, or equivalently,

$$r = r_0 + f + r_0 f .$$

- The real interest rate can be concluded from the nominal rate,

$$r_0 = \frac{r - f}{1 + f} .$$

- See Example 2.10 (text) about the effect of inflation on the feasibility of a project via the NPV method.
- In general, when applying NPV under inflation, one should understand whether future cash flows have been estimated while accounting for inflation.
- Inflated cash flows require the use of the nominal rate r, while cash flows in real (today's) dollar require using r_0.

- **Taxes and depreciation**
  - Taxes can complicate a cash flow analysis.
  - One situation where tax considerations have important implications is that involving property depreciation.
  - The annual depreciation amount is exempt from tax, which reduces tax on net revenues. (See Example 2.9.)
  - While depreciation does not lead to real cash flows, it reduces income, and accordingly tax which is a real cash flow.

- **Present value of a uniform stream of cash flows**
  - Consider a cash flow stream extending from year 1 to n such that the net cash flow at the end of years 1, 2, …, n is A.
  - Then, at an annual interest rate of r, the PV of this stream is

$$PV = \frac{A}{1+r} + \frac{A}{(1+r)^2} + \ldots + \frac{A}{(1+r)^n} = \frac{A}{1+r}\sum_{j=0}^{n-1}\left(\frac{1}{1+r}\right)^j = \frac{A}{1+r}\,\frac{1-\left(\frac{1}{1+r}\right)^n}{1-\left(\frac{1}{1+r}\right)}$$

  - That is,

$$PV = A\left[\frac{(1+r)^n - 1}{r(1+r)^n}\right] = \frac{A}{r}\left[1 - \frac{1}{(1+r)^n}\right].$$

  - E.g., you're applying for a home loan and the maximum you can pay is $1,000/month over 20 years. The bank offered you an interest rate of 6% per year (compounded monthly).
  - Then, the maximum loan you can obtain is

$$PV = \frac{1}{(0.06/12)}\left[1 - \frac{1}{(1+0.06/12)^{20\times 12}}\right] = \frac{1}{0.005}\left[1 - \frac{1}{(1.005)^{240}}\right] = \$139.581\text{ K}$$

  - The formula also works in the other direction. E.g., if you want a $150 K loan at 6% interest, paid monthly over 20 years, then your monthly payment is

$$A = rPV \Big/ \left[1 - \frac{1}{(1+r)^n}\right] = 0.005\times 150 \Big/ \left(1 - \frac{1}{1.005^{240}}\right) = \$1.075\text{ K}$$

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- **Examples**

Example 2.1 (A short stream) Consider the cash flow stream $(-2, 1, 1, 1)$ when the periods are years and the interest rate is 10%. The future value is

$$\text{FV} = -2 \times (1.1)^3 + 1 \times (1.1)^2 + 1 \times 1.1 + 1 = .648. \tag{2.1}$$

Example 2.3 (The old stream) Consider again the cash flow sequence $(-2, 1, 1, 1)$ discussed earlier. The internal rate of return is found by solving the equation

$$0 = -2 + c + c^2 + c^3.$$

The solution can be found (by trial and error) to be $c = .81$, and thus $\text{IRR} = (1/c) - 1 = .23$.

Note on Example 2.3. This can be done in Excel using the function RATE(3,1,-2), where the first entry, 3, is the number of years, the second entry, 1, is the uniform payment, and the third entry, -2, is the initial investment. The function RATE() works for an investment with a uniform revenue only. A more general function is IRR(), which requires a guess value though. (However, for conventional cash flows, the guess value can be anything based on the fundamental theorem of IRR.) See Ex_2.3_Luenberger_2.xlsx file on the course website.

Example 2.4 (When to cut a tree) Suppose that you have the opportunity to plant trees that later can be sold for lumber. This project requires an initial outlay of money in order to purchase and plant the seedlings. No other cash flow occurs until the trees are harvested. However, you have a choice as to when to harvest: after 1 year or after 2 years. If you harvest after 1 year, you get your return quickly; but if you wait an additional year, the trees will have additional growth and the revenue generated from the sale of the trees will be greater.

We assume that the cash flow streams associated with these two alternatives are

(a) $(-1, 2)$ cut early

(b) $(-1, 0, 3)$ cut later.

We also assume that the prevailing interest rate is 10%. Then the associated net present values are

(a) $\text{NPV} = -1 + 2/1.1 = .82$

(b) $\text{NPV} = -1 + 3/(1.1)^2 = 1.48.$

Hence according to the net present value criterion, it is best to cut later

Example 2.5 (When to cut a tree, continued) Let us use the internal rate of return method to evaluate the two tree harvesting proposals considered in Example 2.4. The equations for the internal rate of return in the two cases are

(a) $-1 + 2c = 0$

(b) $-1 + 3c^2 = 0.$

As usual, $c = 1/(1 + r)$. These have the following solutions:

(a) $c = \dfrac{1}{2} = \dfrac{1}{1+r}; \qquad r = 1.0$

(b) $c = \dfrac{\sqrt{3}}{3} = \dfrac{1}{1+r}; \qquad r = \sqrt{3} - 1 \approx .7.$

In other words, for (a), cut early, the internal rate of return is 100%, whereas for (b) it is about 70%. Hence under the internal rate of return criterion, the best alternative is (a). Note that this is opposite to the conclusion obtained from the net present value criterion.

Discussion of Examples 2.4 and 2.5. So, in Ex 2.4, the NPV criterion suggests cutting later, while in Ex 2.5 the IRR criterion suggests cutting earlier. One conclusion is that the NPV and IRR don't always agree.

If you're wondering when do they agree, consider the case when the cutting cycles in Ex 2.4 repeat indefinitely. That is, there are two *strategies*, (i) cut every year - plant the tree at the beginning of a year, and cut it at the end of it, and repeat, and (ii) cut every other year, plant the tree at the beginning of a year, and cut two years later, and repeat. To compare these two strategies, note that they have a common cycle of two years. (That is, if you look at each two years separately, you'll see the same cash flows for each strategy.) For Strategy (i), the cash flows over two years are (−1,2 – 1,1) and the NPV is $PV = -1 + 1/1.1 + 2/1.1^2 = 1.562$, For Strategy (ii), the NPV is 1.48 as in Ex 2.4. So, under this cash flow repetition scenario, NPV recommends cutting every year, similar to IRR, and the two criteria agree. For IRR, the results are the same with and without repetition of cash flows (why?).

To conclude, the IRR criterion implicitly assumes that cash flows repeat indefinitely. If this is indeed the case, then the two criteria, NPV and IRR, agree. Otherwise, for one-time only projects, the two criteria may diverge. The NPV is generally the more acceptable criterion.

Example 2.8 (Machine replacement) A specialized machine essential for a company's operations costs \$10,000 and has operating costs of \$2,000 the first year. The operating cost increases by \$1,000 each year thereafter. We assume that these operating costs occur at the end of each year. The interest rate is 10%. How long should the machine be kept until it is replaced by a new identical machine? Assume that due to its specialized nature the machine has no salvage value.

This is an example where the cash flow stream is not fixed in advance because of the unknown replacement time. We must also account for the cash flows of the replacement machines. This can be done by writing an equation having PV on *both* sides. For example, suppose that the machine is replaced every year. Then the cash flow (in thousands) is $(-10, -2)$ followed by $(0, -10, -2)$ and then $(0, 0, -10, -2)$, and so forth. However, we can write the total PV of the costs compactly as

$$\text{PV} = 10 + 2/1.1 + \text{PV}/1.1$$

because after the first machine is replaced, the stream from that point looks identical to the original one, except that this continuing stream starts 1 year later and hence must be discounted by the effect of 1 year's interest. The solution to this equation is PV = 130 or, in our original units, \$130,000.

We may do the same thing assuming 2-year replacement, then 3 years, and so forth. The general approach is based on the equation

$$\text{PV}_{\text{total}} = \text{PV}_{1\ \text{cycle}} + \left(\frac{1}{1.1}\right)^k \text{PV}_{\text{total}}$$

where k is the length of the basic cycle. This leads easily to Table 2.3.

From the table we see that the smallest present value of cost occurs when the machine is replaced after 5 years. Hence that is the best replacement policy.

TABLE 2.3
Machine Replacement

Replacement year	Present value
1	130,000
2	82,381
3	69,577
4	65,358
5	64,481
6	65,196

The total present value is found for various replacement frequencies. The best policy corresponds to the frequency having the smallest total present value.

Example 2.9 (Depreciation) Suppose a firm purchases a machine for $10,000. This machine has a useful life of 4 years and its use generates a cash flow of $3,000 each year. The machine has a salvage value of $2,000 at the end of 4 years.

The government does not allow the full cost of the machine to be reported as an expense the first year, but instead requires that the cost of the machine be depreciated over its useful life. There are several depreciation methods, each applicable under various circumstances, but for simplicity we shall assume the straight-line method. In this method a fixed portion of the cost is reported as depreciation each year. Hence corresponding to a 4-year life, one-fourth of the cost (minus the estimated salvage value) is reported as an expense deductible from revenue each year.

If we assume a combined federal and state tax rate of 43%, we obtain the cash flows, before and after tax, shown in Table 2.4. The salvage value is not taxed (since it was not depreciated). The present values for the two cash flows (at 10%) are also shown. Note that in this example tax rules convert an otherwise profitable operation into an unprofitable one.

TABLE 2.4
Cash Flows Before and After Tax

Year	Before-tax cash flow	Depreciation	Taxable income	Tax	After-tax cash flow
0	−10,000				−10,000
1	3,000	2,000	1,000	430	2,570
2	3,000	2,000	1,000	430	2,570
3	3,000	2,000	1,000	430	2,570
4	5,000	2,000	1,000	430	4,570
PV	876				−487

Example 2.10 (Inflation) Suppose that inflation is 4%, the nominal interest rate is 10%, and we have a cash flow of real (or constant) dollars as shown in the second column of Table 2.5. (It is common to estimate cash flows in constant dollars, relative to the present, because "ordinary" price increases can then be neglected in a simple estimation of cash flows.) To determine the present value in real terms we must use the real rate of interest, which from (2.5) is $r_0 = (.10 - .04)/1.04 = 5.77\%$.

5000*1.04 5000*1.04^2

TABLE 2.5
Inflation

Year	Real cash flow	PV @5.77%	Nominal cash flow	PV @10%
0	−10,000	−10,000	−10,000	−10,000
1	5,000	4,727	5,200	4,727
2	5,000	4,469	5,408	4,469
3	5,000	4,226	5,624	4,226
4	3,000	2,397	3,510	2,397
Total		5,819		5,819

Alternatively, we may convert the cash flow to actual (nominal) terms by inflating the figures using the appropriate inflation factors. Then we determine the present value using the nominal interest rate of 10%. Both methods produce the same result.

Cash flow Estimate \ Interest Rate	Real	Inflation-adjusted
Today's Dollars	:)	:(
Inflated	:(	:)